Advances in Dynamical Systems and Applications (ADSA).
ISSN 0973-5321, Volume 16, Number 2, (2021) pp. 485-498

https://www.ripublication.com/adsa.htm

A Frequency-Domain Analysis and Design of a Ball and Plate System Controller

Khalid Lefrouni
Mohammed V University in Rabat, Morocco.

Saoudi Taibi
Mohammed V University in Rabat, Morocco.

Abstract

In this article, we have developed a state feedback controller to regulate the position of the ball on the plate. First, we determine the linear model of the ball and plate system based on retarded functional differential equations (RFDE), and secondly, according to the geometric approach and considering the feedback delay in the closed-loop, we integrated a controller based on a state feedback of the system. Finally, we tested the effectiveness of our methodology through simulations.

Keywords: Frequency-Domain, Ball-plate system, State-feedback control, Delay systems.

Introduction

In this article, we are interested in the balancing problem in the ball-plate system. In particular, we focus on the determination of the set of gain values of the state feedback control law allowing to regulate the position of the ball. This result can be used subsequently to meet the other requirements of the specification.

The ball and plate structure is an extension of the ball and beam system. Because of its simple configuration and easy-to-implement functions, it has become a popular target device for controller realization. For the purpose of tracking control, an approximate input and output linearization method based on imprecise models is introduced [2]. By using the touch screen as a position sensor with a conventional PID controller, actual experiments can be carried out [1]. In addition, a ball and plate system for teaching was developed [4,5]. Many researchers have established an online learning fuzzy control algorithm based on this teaching equipment [6,7]. By using a CCD camera instead of a touch screen device to obtain the position of the ball, a fuzzy control scheme is also proposed [8,9, 10]. In [12] a large mechanical ball and plate system was built to execute a state feedback control using a camera to obtain the position of the ball.

In this article, we also use a state feedback controller, however, our difference from previous studies is the use of geometric considerations (see [3,11]) taking into account feedback delay. The rest of this article is structured as follows. In Section 2, we introduced the mathematical model of the ball and plate system. Then, in Section 3, we reveal the main results of this paper, including the formulation of the controller to ensure system stability. In Section 4, we verified the results obtained in the previous sections through simulation. Finally, we make a conclusion in Section 5.

Mathematical Model of Ball and Plate System

In this section we will determine the state space model of the ball and plate system. For this, using the Lagrangian equation (1) we will determine the equation that describes the dynamics of the system and which will be linearized afterwards.

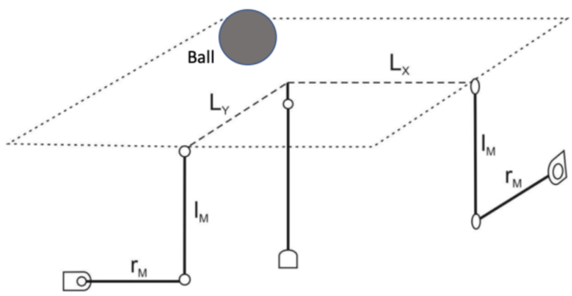

Fig.1. Schematic diagram of the ball-plate system

According to Lagrange's method, the equation that describes the dynamics of the ball and plate system is described by :

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \tag{1}$$

With

$$L = E_{k,T} + E_{k,R} - E_p \tag{2}$$

$E_{k,T}$, $E_{k,R}$ and E_p represent respectively the kinetic energy of translation, the kinetic energy of rotation and the potential energy of the ball.

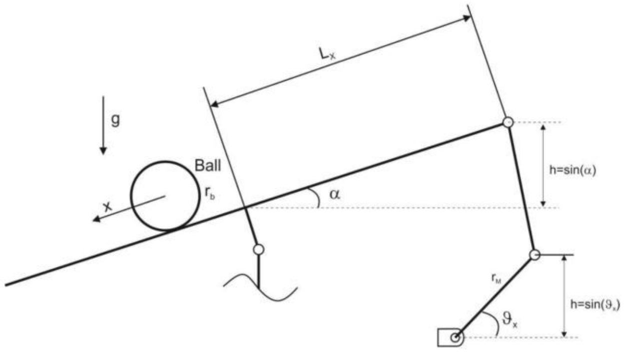

Fig.2. Lateral view of the ball-plate system

Based on the different parameters of the system (see Figure 1 and Figure 2), we have :

$$E_{k,T} = \frac{1}{2}\, m_b v_b^2 \; = \frac{1}{2}\, m_b(\dot{x}_b^2 + \dot{y}_b^2) \tag{3}$$

$$E_{k,R} = \frac{1}{2}\, J_b \omega_b^2 \; = \frac{1}{2}\, J_b \frac{(\dot{x}_b^2+\dot{y}_b^2)}{r_b^2} \tag{4}$$

$$E_p \; = \; -m_b g x_b \sin(\alpha) - m_b g y_b \sin(\beta) \tag{5}$$

After calculation, we thus have the differential equation which describes the dynamics of the system along the x-axis :

$$\ddot{x}_b = \frac{m_b g r_b^2}{m_b r_b^2 + J_b} \sin(\alpha) \tag{6}$$

The equation linking angle α and angle ϑ_x is as follows (see Figure 2) :

$$\sin(\vartheta_x)\, r_M = sin(\alpha) L_X = h \tag{7}$$

So from (6) and (7) we have :

$$\ddot{x}_b = \frac{m_b g r_b^2\, r_M}{(m_b r_b^2 + J_b)\, L_X} \sin(\vartheta_x) \tag{8}$$

Similarly, proceeding in the same way, the differential equation that describes the dynamics of the system along the y-axis is as follows :

$$\ddot{y}_b = \frac{m_b g r_b^2\, r_M}{(m_b r_b^2 + J_b)\, L_Y} \sin(\vartheta_y) \tag{9}$$

Considering small variation of ϑ_x and ϑ_y we have :

$$\sin(\vartheta_x) \approx \vartheta_x \,, \quad \sin(\vartheta_y) \approx \vartheta_y$$

Equations (13) and (13) become thus :

$$\ddot{x}_b = G_x \, \vartheta_x \tag{10}$$

$$\ddot{y}_b = G_y \, \vartheta_y \tag{11}$$

with

$$G_x = \frac{m_b g r_b^2 \, r_M}{(m_b r_b^2 + J_b) \, L_X} \quad \text{and} \quad G_y = \frac{m_b g r_b^2 \, r_M}{(m_b r_b^2 + J_b) \, L_Y}$$

Remarks: Considering the similarity of the x-axis and y-axis models, we will start the controller synthesis based on the x-axis model and infer the results of the y-axis at the end.

Therefore, considering the feedback delay τ_x, The state-space model along the x-axis can be defined by :

$$\begin{cases} \dot{x}(t) \; = \; A\,x(t) \; + \; B\,u(t-\tau_x) \\ y(t) \; = \; C\,x(t) \end{cases} \tag{12}$$

Where

$$x(t) = \; [x_b(t) \quad \dot{x}_b(t)]^{\,T}, u(t) = \; \vartheta_x(t)$$

and

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ G_x \end{bmatrix}, \qquad C = [1 \quad 0]$$

Ball And Plate Controller : Geometric Considerations

Based on a geometric approach, we will develop in this section a state feedback control law acting on the packet loss rate and allowing the stabilization of the system (12).

On the basis of a geometrical analysis, we will develop a state feedback controller acting on the angles ϑ_x and ϑ_y and providing stabilization of the closed-loop system (12).

A. State Feedback Controller

The state feedback controller is defined by the equation :

$$u(t) \; = \; -K_x \, x(t) \tag{13}$$

Where $K_x = [k_{x1} \quad k_{x2}]$ represents the state feedback gain, $x(t)$ is the internal state of the system $x(t) = [x_b(t) \quad \dot{x}_b(t)]^T$, i.e. the position and velocity of the ball, and $u(t)$ is the control signal.

Using the Laplace transform, equations (17) and (18) are transformed into :

$$\begin{cases} s\,x(s) = A\,x(s) + B\,u(s)e^{-\tau_x s} \\ y(s) = C\,x(s) \\ u(s) = -K_x\ x(s) \end{cases} \tag{14}$$

Consequently, the characteristic equation of the system defined by (12) is expressed as:

$$H(s, k_{x1}, k_{x2}, \tau_x) = \det(sI_2 - (A - BK_x\, e^{-\tau_x s}))=0 \tag{15}$$

which can be written as :

$$H(s, k_{x1}, k_{x2}, \tau_x) = Q(s) + P(s)e^{-\tau_x s} \tag{16}$$

with

$$Q(s) = s^2\ ,\ P(s) = G_x(sk_{x2} + k_{x1}) \tag{17}$$

To study the stability of the system, we will first consider a zero delay, equation (16) becomes :

$$H(s, k_{x1}, k_{x2}, 0) = s^2 + G_x(sk_{x2} + k_{x1}) = 0 \tag{18}$$

Using Routh's criterion, we have found the conditions :

$$k_{x1} > 0,\ k_{x2} > 0 \tag{19}$$

B. Analysis in the (k1, k2) plane

Let's now consider the real case, corresponding to a non-zero feedback delay.

In order to determine the conditions that guarantee the stability of the system (12), we will first consider the case of a system at the limit of stability, which corresponds to the existence of a pure imaginary root of equation (16), so we have :

$$H(j\omega, k_{x1}, k_{x2}, \tau_x^*) = 0 \tag{20}$$

This is equivalent to :

$$-\omega^2 + G_x(j\omega k_{x2} + k_{x1})e^{-j\omega\,\tau_x^*} = 0$$

By separating the real and imaginary parts, after simplification, we have the following two equations:

$$k_{x1} = \frac{\omega^2 \cos(\omega\tau_x^*)}{G_x} \tag{21}$$

$$k_{x2} = \frac{\omega\, sin(\omega\tau_x^*)}{G_x} \tag{22}$$

Equations (21) and (22) define the set of values of k_{x1} and k_{x2} as a function of ω, thus, by varying ω, we find the crossing curves [3]. Thus, we have found a second condition on the stability of the system (12) in the plane (k_{x1}, k_{x2}). To illustrate these results, we plotted the crossing curves for different values of the delay τ_x (see Figure 3, Figure 4 and Figure 5).

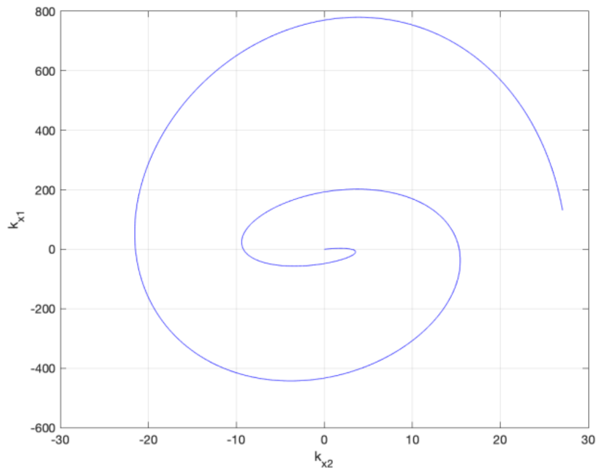

Fig.3. Crossing curves in the (k_{x1}, k_{x2}) space with $\tau_x = 0.4$

We are now going to determine the direction of the crossing [3], which is defined by the sign of $R_2 I_1 - R_1 I_2$, with :

$$R_1 + jI_1 = -\frac{1}{s}\frac{\partial H(s, k_{x1}, k_{x2}, \tau_x^*)}{\partial k_{x2}}\bigg|_{s=j\omega}$$

$$R_2 + jI_2 = -\frac{1}{s}\frac{\partial H(s, k_{x1}, k_{x2}, \tau_x^*)}{\partial k_{x1}}\bigg|_{s=j\omega}$$

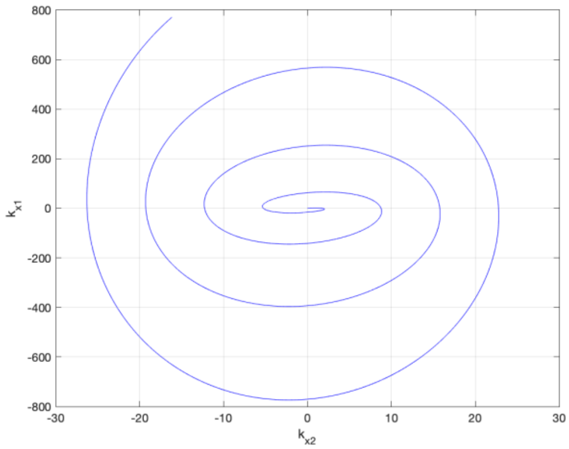

Fig.4. Crossing curves in the (k_{x1}, k_{x2}) space with $\tau_x = 0.7$

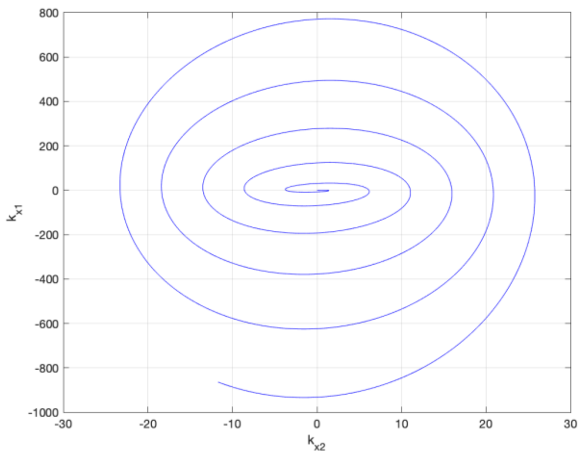

Fig.5. Crossing curves in the (k_{x1}, k_{x2}) space with $\tau_x = 1$

After calculation of the derivatives we find the real and imaginary parts, then, after simplification we have :

$$R_2 I_1 - R_1 I_2 = \frac{{G_x}^2}{\omega} > 0 \tag{23}$$

This means that the direction of the crossing is to the right, i.e. when ω varies in the direction of the positive pulsations, a solution of (12) (k_{x1}^*, k_{x2}^*) will cross the imaginary axis from right to left.

C. Stability region of closed-loop system

We will now use all the results found previously to determine the region of the stability of the closed-loop system (12) in the plane (k_{x1}, k_{x2}). Thus, we have proved that the conditions (19) must be checked, and that the direction of crossing is to the right. Thus, considering a feedback delay along the x-axis equal to 0.7s , we find the region of stability shown in Figure 6.

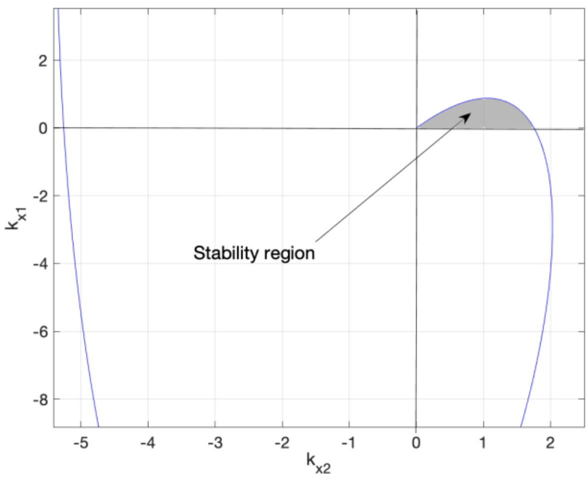

Fig.6. Stability region in the (k_{x1}, k_{x2}) space with $\tau_x = 0.7$

This region of stability defines the set of $K_x^* = [k_{x1}^* \quad k_{x2}^*]$ gain values for which the characteristic equation $H(s, k_{x1}^*, k_{x2}^*, \tau_x) = 0$ is Hurwitz for any feedback delay τ_x such as $\tau_x < \tau_x^*$.

It is therefore important to know this critical value of delay τ_x^* , for this, let us consider the equation (20) for a selected gain K_x^* and for a root which is at the limit of stability, i.e. . $s = j\omega$, we have :

$$Q(j\omega) + P(j\omega)e^{-j\omega\tau_x^*} = 0 \tag{24}$$

With

$$Q(j\omega) = -\omega^2 , \quad P(j\omega) = G_x(j\omega\, k_{x2} + k_{x1})$$

Equation (24) becomes

$$\cos(\omega\, \tau_x^*) - \mathrm{j}\sin(\omega\, \tau_x^*) = \frac{k_{x1}^*\, \omega^2 - j\omega^3\, k_{x2}^*}{G_x((k_{x1}^*)^2 + (\omega k_{x2}^*)^2)}$$

By equality of the real parts, we find the critical feedback delay :

$$\tau_x^* = \frac{1}{\omega} Arccos\left[\frac{k_{x1}^* \omega^2}{G_x\left((k_{x1}^*)^2+(\omega k_{x2}^*)^2\right)}\right] \tag{25}$$

D. Results of the closed-loop system along the y-axis

The control signal $\boldsymbol{u(t)}$ ensuring the stability of the closed-loop system along the y-axis is defined by the equation :

$$u(t) = -K_y \, x(t) \tag{26}$$

Where the state feedback gain and the state vector are respectively defined by $K_y = [k_{y1} \quad k_{y2}]$ and $x(t) = [y_b(t) \quad \dot{y}_b(t)]^T$

By considering the system parameters along the y-axis and following the same steps illustrated in the previous sections, we have determined the stability region of the closed-loop system along the y-axis. Thus, considering a feedback delay along the y-axis equal to 0.9s , we find the region of stability shown in Figure 7.

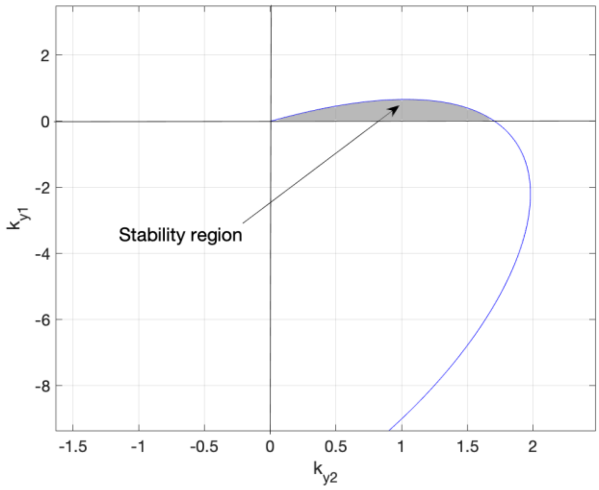

Fig.7. Stability region in (k_{y1}, k_{y2}) plane with $\tau_y^* = 0.9$

The critical feedback delay is also defined by :

$$\tau_y^* = \frac{1}{\omega} Arccos\left[\frac{k_{y1}^* \omega^2}{G_y\left((k_{y1}^*)^2+(\omega k_{y2}^*)^2\right)}\right] \tag{27}$$

Which means that the state feedback gain $K_y^* = \left[k_{y1}^* \quad k_{y2}^*\right]$ asymptotically stabilizes the evolution of the ball along the y-axis for any delay $\tau_y < \tau_y^*$.

Simulation Results

In order to test the results found, we will consider the following system parameters:
$L_X = 0.134\,m$, $L_Y = 0.168\,m$, $r_M = 0.0245\,m$, $r_b = 0.02\,m$,
$J_b = 0.0000416\,kg * m^2$, $m_b = 0.26\,kg$, $\tau_x = 0.7s$, $\tau_y = 0.9$s .

In order to test if our approach actually allows to control the position of the ball on the plate. In our case, the objective is that the ball must be positioned at $(x = 0, y = 0)$.

To do this, we will choose several gain values $K_x^* = [k_{x1}^* \;\; k_{x2}^*]$ and $K_y^* = [k_{y1}^* \;\; k_{y2}^*]$ within the regions shown in Figure 6 and Figure 7 respectively. Then, based on equations (25) and (27), we will determine the critical delays. The results are shown in Table 1 and Table 2.

TABLE I. Critical delay and state feedback gain for each pair selected $(k_{x1}^*,\, k_{x2}^*)$

Case	$(\boldsymbol{k_{x1}^*},\, \boldsymbol{k_{x2}^*})$	$\boldsymbol{K_x^*}$	$\boldsymbol{\tau_x^*}$
1	(0.44, 0.54)	[0.44 0.54]	0.9183
2	(0.64, 0.92)	[0.64 0.92]	0.8187
3	(0.45, 1.22)	[0.45 1.22]	0.8385

TABLE II. Critical delay and state feedback gain for each pair selected $(k_{y1}^*,\, k_{y2}^*)$

Case	$(\boldsymbol{k_{y1}^*},\, \boldsymbol{k_{y2}^*})$	$\boldsymbol{K_y^*}$	$\boldsymbol{\tau_y^*}$
4	(0.34, 0.52)	[0.34 0.52]	1.1587
5	(0.53, 0.95)	[0.53 0.95]	1.0069
6	(0.34, 1.33)	[0.34 1.33]	1.0042

In order to test the efficiency of this approach we will consider different values of gains along the x and y axes, and to visualize the evolution in time of the ball position $(x_b\,, y_b)$, we have plotted the different curves in space $(x_b\,, y_b, t)$.

Figures Figure 8, Figure 9, Figure 10 and Figure 11 illustrate respectively the temporal evolution of the ball position in the configurations (case 1, case 4), (case 2, case 4), (case 2, case 6) and (case 3, case 5).

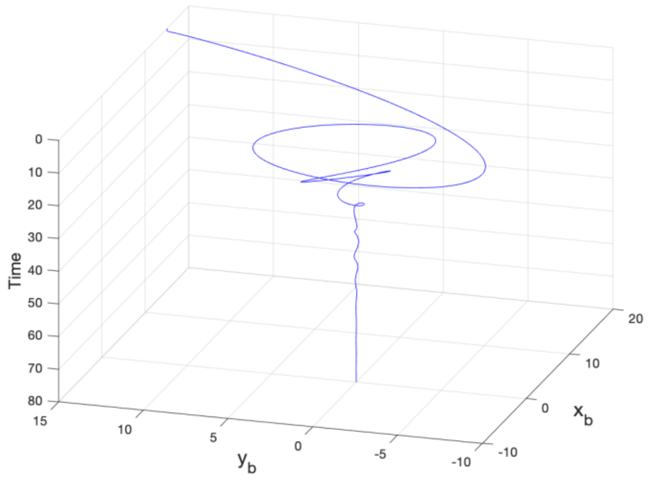

Fig.8. Evolution of the ball position with the configuration (case 1, case 4)

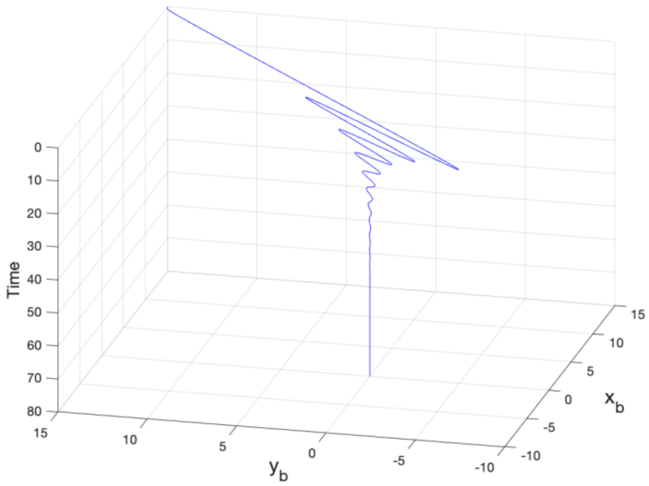

Fig.9. Evolution of the ball position with the configuration (case 2, case 4)

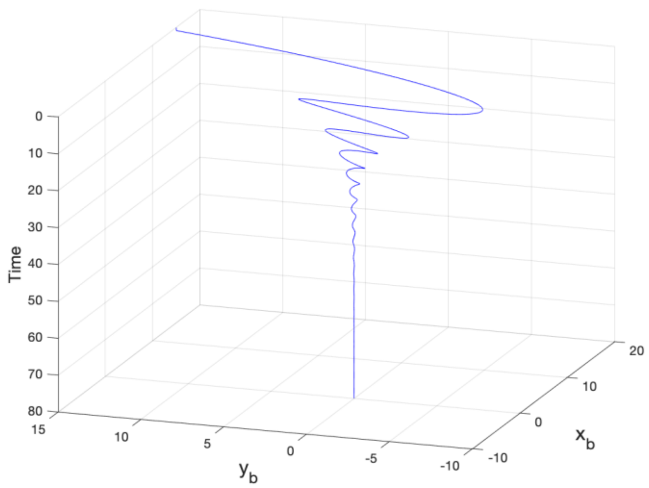

Fig.10. Evolution of the ball position with the configuration (case 2, case 6)

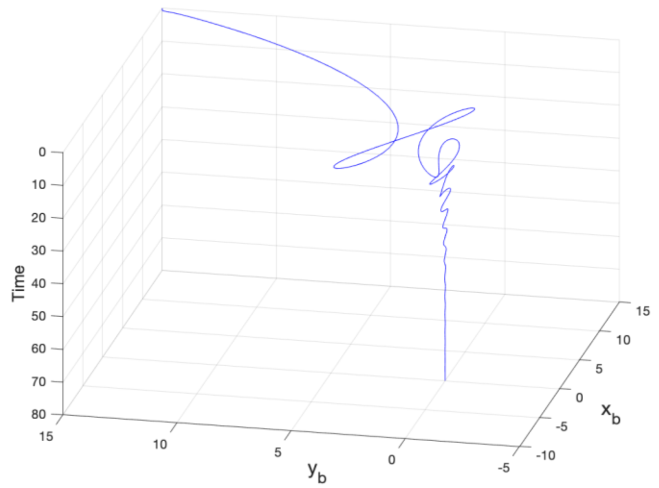

Fig.11. Evolution of the ball position with the configuration (case 3, case 5)

In analysing the results, it can be seen that all the chosen state feedback gains can ensure the stability of the system. However, we also find that inappropriate selection can lead to large oscillations (see Figure 8 and Figure 10) and shorten the intervals of permitted delays (see Table 1 and Table 2).

Therefore, we recommend that an algorithm must be developed to ensure the desired evolution of the ball, taking into account the response time and the desired overshoot.

It should also be noted that the controller we have developed (13) is based on the system state $x(t)$, thus the efficiency of this state feedback controller is dependent on the exact measurement of the system state. Measuring the state is not always accessible in practice, so it is necessary to set up an observer allowing the estimation of the system's state.

In our case, taking into consideration the feedback delays, the most convenient observer is the Luenberger observer (see Figure 12) defined by the following equations :

$$\begin{cases} \dot{\hat{x}}(t) = A\,\hat{x}(t) + B\,u(t-\tau_{1x}) + L_x[y(t-\tau_{2x}) - \hat{y}(t-\tau_{2x})] \\ \hat{y}(t) = C\,\hat{x}(t) \end{cases}$$

Where $\hat{x}(t)$ is an estimation of the system state and L_x is the gain of the observer.

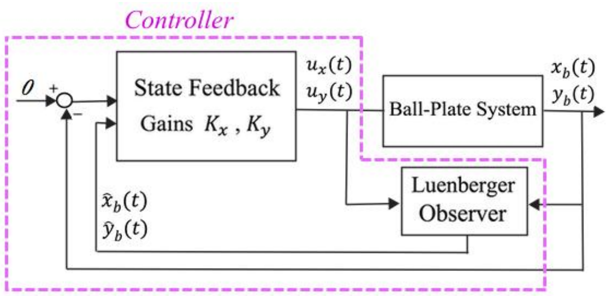

Fig.12. The block diagram of the ball and plate controller

Conclusion

In this article, we proposed a frequency method allowing the state feedback control of the ball position. Thus, as shown in section 4, all the selected gains ensure the regulation of the ball position. Moreover, this flexibility in the choice of controller parameters allows the adaptation of the controller with the desired requirements such as response time or first overshoot. However, it should be noted that in order to ensure the effectiveness of these results, it is necessary to be able to measure accurately the state of the system. Therefore, in future work, we will increase the controller performance by implementing the Luenberger observer.

References

[1] Shorya Awtar, C. Bernard, N. Boklund, A. Master, D. Ueda, Kevin Craig, Mechatronic design of a ball-on-plate balancing system, Volume 12, Issue 2, 2002, Pages 217-228, ISSN 0957-4158.

[2] Hongwei Liu and Yanyang Liang, "Trajectory tracking sliding mode control of ball and plate system," 2010 2nd International Asia Conference on Informatics in Control, Automation and Robotics (CAR 2010), Wuhan, 2010, pp. 142-145, doi: 10.1109/CAR.2010.5456649.

[3] Khalid Lefrouni and Rachid Ellaia. Frequency-domain analysis and design of a tcp flow controller using the aqm mechanism. In The International Symposium on Ubiquitous Networ- king, Proceedings of the UNet'15, pages 242–253, 2015.

[4] Ernesto Fabregas, Sebastián Dormido-Canto, Sebastián Dormido, Virtual and Remote Laboratory with the Ball and Plate System, IFAC-PapersOnLine, Volume 50, Issue 1, Pages 9132-9137, ISSN 2405-8963, 2017.

[5] Carlos Parga, Wen Yu, Xiaoou Li, A low-cost ball and plate system for advanced control education, The International Journal of Electrical Engineering & Education, Volume: 52 issue: 4, page(s): 370-384, October 1, 2015

[6] Rad, A.B. & Chan, P.T. & Lo, Wai & Mok, C.K.. An online learning fuzzy controller. Industrial Electronics, IEEE Transactions on. 50. 1016 - 1021. 10.1109/TIE.2003.817690, 2003.

[7] Mohajerin, Nima & Menhaj, Mohammad & Doustmohammadi, Ali. A Reinforcement Learning Fuzzy Controller for the Ball and Plate system. 2010 IEEE World Congress on Computational Intelligence, WCCI 2010. 1-8. 10.1109/FUZZY.2010.5584025, 2010.

[8] Bai, Ming & Lu, Huiqiu & Su, Jintao & Tian, Yantao. Motion Control of Ball and Plate System Using Supervisory Fuzzy Controller. 2. 8127 - 8131. 10.1109/WCICA.2006.1713557, 2006.

[9] Negash A., Singh N.P., Position Control and Tracking of Ball and Plate System Using Fuzzy Sliding Mode Controller. In: Abraham A., Krömer P., Snasel V. (eds) Afro-European Conference for Industrial Advancement. Advances in Intelligent Systems and Computing, vol 334. Springer, Cham. 2015.

[10] Xingzhe Fan, Naiyao Zhang, Shujie Teng, Trajectory planning and tracking of ball and plate system using hierarchical fuzzy control scheme, Fuzzy Sets and Systems, Volume 144, Issue 2, Pages 297-312, ISSN 0165-0114, 2004.

[11] K. Lefrouni and R. Ellaia, State-feedback control in TCP network: Geometric Approach, International Review of Automatic Control (IREACO), 8, 127-133, 2015.

[12] Acosta, Gustavo & López, Juan & Jiménez-Builes, Jovani. Control of a ball-and-plate system using a State-feedback controller. Ingeniare. 28. 6-15. 10.4067/S0718-33052020000100006, 2020.