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Uniqueness Theorems Concerning Homogeneous Differential Polynomials of L-functions and Weakly Weighted Sharing

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Abstract

In this paper, we prove some uniqueness results when a polynomial and a homogeneous differential polynomial of an L-function weakly share a rational function. Our results improve and generalize some earlier results due to Mandal, Datta [10].

Key words and phrases: L-function, meromorphic function, uniqueness, weakly weighted sharing, homogeneous differential polynomial.

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1. INTRODUCTION

In 1992 a model for L-functions is introduced by Selberg. The study of value distributions of L-functions is mainly concerned with the set $\{z \in \mathbb{C} : L(z) = a\}$ where $a \in \mathbb{C}$.

A meromorphic function L is said to be an L-function in the Selberg class if it satisfy the following properties.

(i) $L(z)$ can be expressed as a Dirchlet series $L(z) = \sum_{m=1}^{\infty} a(m)/m^z$.

(ii) $\mid a(m) \mid = O(m^{\epsilon})$, for any $\epsilon > 0$.

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(iii) There exists a nonnegative integer n such that $(z-1)^n L(z)$ becomes an entire function of finite order.

(iv) Every L-function satisfies the functional equation

$$\lambda_L(z) = \omega\overline{\lambda_L(1-\overline{z})},$$

where

$$\lambda_L(z) = L(z)A^z \prod_{j=1}^{n} \Gamma(\eta_j z + \nu_j)$$

with positive real numbers A, η_j and complex numbers ν_j, ω with $Re(\nu_j) \geq 0$ and $\mid \omega \mid= 1$.

(v) $L(z)$ satisfies $L(z) = \prod_p L_p(z)$, where $L_p(z) = exp(\sum_{n=1}^{\infty} b(p^n)/p^{nz})$ with $b(p^n) = O(p^{n\theta})$ for some $\theta < 1/2$ and p denotes prime number.

If L satisfies (i) - (iv) then we say that L is an L-function in the extended Selberg class. In this paper, by an L-function we mean an L-function in the extended Selberg class with $a(1) = 1$. Here we use the standard notations and definitions of the value distribution theory [4].

2. PRELIMINARIES

Let ξ and ψ be a meromorphic function defined in the complex plane $\mathbb{C}$. If $\xi - \sigma$ and $\psi - \sigma$ have same set of zeros ignoring(counting) multiplicities, then we say that ξ and ψ share σ IM (CM). If $\frac{1}{\xi}$ and $\frac{1}{\psi}$ share 0 CM (IM), we say that ξ and ψ share ∞ CM (IM). We denote by $S(r, \xi)$ any quantity satisfying $S(r, \xi) = o(T(r, \xi))$ as $r \longrightarrow \infty$, outside a possible exceptional set of finite linear measure. A meromorphic function ρ is said to be a small function of ξ if $T(r, \rho) = S(r, \xi)$. Let $\alpha \in \mathbb{C} \cup \{\infty\}$ and n be a positive integer. We denote by $E_{n)}(\alpha; \xi)(\overline{E}_{n)}(\alpha; \xi))$ the set of all zeros of $\xi - \alpha$ with multiplicities not exceeding n, where zeros are counted according to their multiplicities(ignoring multiplicities). We denote by $S(\xi)$ the set of all the small functions of ξ.

With the help of CM sharing Steuding [13] proved the following theorem in 2007.

Theorem 2.1. *[13] Let L and H be two L-functions with $a(1) = 1$ and $d \neq \infty$ be a complex number. If L and H share d CM, then $L \equiv H$.*

Definition 2.1. *[5, 6] Let ξ and χ be two meromorphic functions defined in the complex plane and m be an integer (≥ 0) or infinity . For $d \in \mathbb{C} \cup \{\infty\}$ we denote by $E_m(d; \xi)$*

the set of all zeros of $\xi - d$ where an zero of multiplicity t is counted t times if $t \leq m$ and $m+1$ times if $t > m$. If $E_m(d;\xi) = E_m(d;\chi)$, we say that ξ, χ share the value d with weight m. We write ξ, χ share (d,m) to mean that ξ, χ share the value d with weight m.

Definition 2.2. *[10] Let ξ be a meromorphic function defined in the complex plane and ρ be a small function of ξ. Then we denote by $E_{m)}(\rho;\xi)$, $\overline{E}_{m)}(\rho;\xi)$ and $E_m(\rho;\xi)$ the sets $E_{m)}(0;\xi-\rho)$, $\overline{E}_{m)}(0;\xi-\rho)$ and $E_m(0;\xi-\rho)$ respectively.*

With the help of weighted sharing Wu and Hu [14] proved the following uniqueness result in 2015.

Theorem 2.2. *[14] Let L and H be two L-functions, and let $\alpha, \beta \in \mathbb{C}$ be two distinct values. Take two positive integers m_1, m_2 with $m_1 m_2 > 1$. If $E_{m_1}(\alpha, L) = E_{m_1}(\alpha, H)$, and $E_{m_2}(\alpha, L) = E_{m_2}(\alpha, H)$, then $L \equiv H$.*

In 2018 Hao and Chen [3] proved the following uniqueness theorem considering weighted sharing .

Theorem 2.3. *[3] Let L be an L-function and F be a meromorphic function defined in the complex plane $\mathbb{C}$ with finitely many poles. Let $\alpha_1, \alpha_2 \in \mathbb{C}$ be distinct and m_1,m_2 be positive integers such that $m_1 m_2 > 1$. If $E_{m_j}(\alpha_j, F) = E_{m_j}(\alpha_j, L)$, $j = 1,2$, then $L \equiv F$.*

Considering weighted sharing in 2020 Datta and Mandal [2] proved the following uniqueness theorem.

Theorem 2.4. *[2] Let ξ be a nonconstant meromorphic function and L be a nonconstant L-function. If $E_0(0;\xi) = E_0(0;L)$, $E_1(1;\xi) = E_1(1;L)$ and $N(r;0;\xi) + N(r;1;\xi) = S(r;\xi)$ then either $L \equiv \xi$ or $T(r;L) = N(r;0;L| \leq 2) + S(r;L)$ and $T(r;\xi) = N(r;0;L'| \leq 1) + S(r;L)$.*

With the help of small function sharing in 2020 Mandal and Datta [10] proved the following theorem.

Theorem 2.5. *[10] Let L be a nonconstant L-function and ρ be a small function of L such that $\rho \not\equiv 0, \infty$. If $\overline{E}_{4)}(\rho;L) = \overline{E}_{4)}(\rho;(L^m)^{(k)})$, $E_{2)}(\rho;L) = E_{2)}(\rho;(L^m)^{(k)})$ and*

$$2N_{2+k}(r,0;L^m) \leq (\sigma + o(1))T(r,L), \tag{2.1}$$

where $m \geq 1$, $k \geq 1$ are integers and $0 < \sigma < 1$, then $L \equiv (L^m)^{(k)}$.

Definition 2.3. *Let ξ, ψ and χ be nonconstant meromorphic functions. We denote by $N_E(r,\chi;\xi,\psi)$ the counting function of all common zeros of $\xi-\chi$ and $\psi-\chi$ with same multiplicities. We denote by $\overline{N}_E(r,\chi;\xi,\psi)$ the corresponding reduced counting function.*

Definition 2.4. *Let ξ, ψ be nonconstant meromorphic functions and χ be a meromorphic function. We denote by $N_0(r,\chi;\xi,\psi)$ the counting function of all common zeros of $\xi-\chi$ and $\psi-\chi$. We denote by $\overline{N}_0(r,\chi;\xi,\psi)$ the corresponding reduced counting function.*

Definition 2.5. *[9] Let ξ, ψ be nonconstant meromorphic functions and $\rho\in S(\xi)\cap S(\psi)$. If*

$$\overline{N}(r,\rho;\xi)+\overline{N}(r,\rho;\psi)-2\overline{N}_E(r,\rho;\xi,\psi)=S(r,\xi)+S(r,\psi),$$

we say that ξ and ψ share ρ "CM". If

$$\overline{N}(r,\rho;\xi)+\overline{N}(r,\rho;\psi)-2\overline{N}_0(r,\rho;\xi,\psi)=S(r,\xi)+S(r,\psi),$$

we say that ξ and ψ share ρ "IM".

Definition 2.6. *Let ξ, ψ be nonconstant meromorphic functions and χ be a meromorphic function. We say that ξ and ψ share χ "CM" ("IM") if $\xi-\chi$ and $\psi-\chi$ share 0 "CM" ("IM").*

Definition 2.7. *[8]. Let ξ be a meromorphic function defined in the complex plane. Let n be a positive integer and $\alpha\in\mathbb{C}\cup\{\infty\}$. By $N(r,\alpha;\xi\mid\leq n)$ we denote the counting function of the α points of ξ with multiplicity $\leq n$ and by $\overline{N}(r,\alpha;\xi\mid\leq n)$ the reduced counting function. Also by $N(r,\alpha;\xi\mid\geq n)$ we denote the counting function of the α points of ξ with multiplicity $\geq n$ and by $\overline{N}(r,\alpha;\xi\mid\geq n)$ the reduced counting function. We define*

$$N_n(r,\alpha;\xi)=\overline{N}(r,\alpha;\xi)+\overline{N}(r,\alpha;\xi\mid\geq 2)+\cdots+\overline{N}(r,\alpha;\xi\mid\geq n).$$

Definition 2.8. *[8]. Let ξ and ψ be two meromorphic functions defined in the complex plane. Then we denote by $N(r,\psi;\xi\mid\leq m)$, $\overline{N}(r,\psi;\xi\mid\leq m)$, $N(r,\psi;\xi\mid\geq m)$, $\overline{N}(r,\psi;\xi\mid\geq m)$, $N_m(r,\psi;\xi)$ etc. the counting functions $N(r,0;\xi-\psi\mid\leq m)$, $\overline{N}(r,0;\xi-\psi\mid\leq m)$, $N(r,0;\xi-\psi\mid\geq m)$, $\overline{N}(r,0;\xi-\psi\mid\geq m)$, $N_m(r,0;\xi-\psi)$ etc. respectively.*

Definition 2.9. *Let two nonconstant meromorphic functions ξ and ψ share a value α "IM" and m be a positive integer or ∞. We denote by $\overline{N}_E(r,\alpha;\xi,\psi\mid\leq m)$*

($\overline{N}_E(r,\alpha;\xi,\psi| \geq m)$) *the counting function of the α-points of ξ and ψ with multiplicities not greater than m(not less than m) and the multiplicities with respect to ξ is equal to the multiplicities with respect to ψ, where each α-point is counted once only.*

Definition 2.10. *Let two nonconstant meromorphic functions ξ and ψ share a value α "IM" and m be a positive integer or ∞. We denote by $\overline{N}_0(r,\alpha;\xi,\psi| \geq m)$ the counting function of the common α-points of ξ and ψ with multiplicities not less than m, where each α-point is counted once only.*

Definition 2.11. *Let two nonconstant meromorphic functions ξ, ψ share a meromorphic functions χ "IM". By $\overline{N}_E(r,\chi;\xi,\psi| \leq m)$ and $\overline{N}_0(r,\chi;\xi,\psi| \geq m)$ we denote the counting functions $\overline{N}_E(r,0;\xi-\chi,\psi-\chi| \leq m)$ and $\overline{N}_0(r,0;\xi-\chi,\psi-\chi| \geq m)$ respectively.*

Definition 2.12. *[9] Let $\rho \in S(\xi) \cap S(\psi)$ and two nonconstant meromorphic functions ξ, ψ share ρ "IM". If m is a positive integer or ∞ and*

$$\overline{N}(r,\rho;\xi| \leq m) - \overline{N}_E(r,\rho;\xi,\psi| \leq m) = S(r,\xi)$$

$$\overline{N}(r,\rho;\psi| \leq m) - \overline{N}_E(r,\rho;\xi,\psi| \leq m) = S(r,\psi)$$

$$\overline{N}(r,\rho;\xi| \geq m+1) - \overline{N}_0(r,\rho;\xi,\psi| \geq m+1) = S(r,\xi)$$

$$\overline{N}(r,\rho;\psi| \geq m+1) - \overline{N}_0(r,\rho;\xi,\psi| \geq m+1) = S(r,\psi)$$

or $m = 0$ and

$$\overline{N}(r,\rho;\xi) - \overline{N}_0(r,\rho;\xi,\psi) = S(r,\xi)$$

$$\overline{N}(r,\rho;\psi) - \overline{N}_0(r,\rho;\xi,\psi) = S(r,\psi),$$

then we say ξ and ψ weakly share ρ with weight m. We write ξ and ψ share "(ρ,m)" to mean that ξ and ψ weakly share ρ with weight m.

Definition 2.13. *Let two nonconstant meromorphic functions ξ, ψ share a meromorphic functions χ "IM". Also let m be a positive integer or ∞. We say that ξ, ψ share "(χ,m)" if $\xi-\chi$, $\psi-\chi$ share "$(0,m)$".*

Definition 2.14. *[1] Let ξ be a meromorphic function, t_{ij} $(i = 0, 1, 2, ...n,\ j = 1, 2, ..., m)$ be nonnegative integers and $\rho_j \in S(\xi)$ such that $\rho_j \not\equiv 0$ for $j = 1, 2, ..., m$. We define the differential polynomial $P(\xi)$ of ξ by $P(\xi) = \sum_{j=1}^{m} M_j(\xi)$, where $M_j(\xi) = \rho_j \prod_{i=0}^{n} (\xi^{(i)})^{t_{ij}}$. The numbers $\overline{d}(P) = \max_{1\le j\le m} \sum_{i=0}^{n} t_{ij}$ and $\underline{d}(P) = \min_{1\le j\le m} \sum_{i=0}^{n} t_{ij}$ are called degree and lower degree of $P(\xi)$ respectively. If $\overline{d}(P) = \underline{d}(P) = d$ (say), then we say that $P(\xi)$ is a homogeneous differential polynomial of degree d generated by ξ. We define Q by $Q = \max_{1\le j\le m} \sum_{i=0}^{n} it_{ij}$.*

Definition 2.15. *[7] Let ξ be a meromorphic function and k be a positive integer. We denote by $N_{\otimes}(r, 0; \xi^{(k)})$ $(\overline{N}_{\otimes}(r, 0; \xi^{(k)}))$ the counting function (reduced counting function) of those zeros of $\xi^{(k)}$ which are not the zeros of $\xi(\xi - 1)$.*

Definition 2.16. *[11] Let two nonconstant meromorphic functions ξ and ψ share a value α "IM". We denote by $\overline{N}(r, \alpha; \xi| > \psi)$ the counting function of the α-points of ξ and ψ with multiplicities with respect to ξ is greater than the multiplicities with respect to ψ, where each α-point is counted once only.*

Definition 2.17. *Let two nonconstant meromorphic functions ξ and ψ share a value α "IM" and k be a positive integer. We denote by $\overline{N}(r, \alpha; \xi, \psi|\xi > \psi = k)$ the counting function of the common α-points of ξ and ψ with multiplicities with respect to ξ is greater than the multiplicities with respect to ψ and multiplicities with respect to ψ is equal to k, where each α-point is counted once only.*

Now the following questions come naturally.

Question 2.1. *Is it possible to consider rational function sharing in place of small function sharing in theorem 2.5?*

Question 2.2. *Is it possible to consider polynomial of L and homogeneous differential polynomial generated by L in place of L and $(L^m)^{(k)}$ respectively in theorem 2.5?*

3. MAIN RESULTS

Let L be a nonconstant L-function and $a_i, b_j \in S(L)$, $i = 0, 1, 2, ...t$, $j = 0, 1, 2, ..., s$. Henceforth we denote by $R(z)$ the function $R(z) = \frac{\sum_{i=0}^{t} a_i z^i}{\sum_{j=0}^{s} b_j z^j}$, where $a_t \not\equiv 0$ and $b_s \not\equiv 0$. Also we denote by $P(L)$ a homogeneous differential polynomial of degree d generated by L as defined in definition 2.14.

Using the concept of weakly weighted sharing we try to solve Questions 2.1, 2.2 and prove the following theorems.

Theorem 3.1. *Let L be a nonconstant L-function and $p(z)$ be a polynomial of degree $\lambda \geq 1$ with $p(0) = 0$. Let $P(L)$ be a homogeneous differential polynomial of degree d generated by L. If $p(L)$ and $P(L)$ share "$(R(z), l)$" and one of the following holds*

(i) l=0 and

$$2\overline{N}(r, 0; p(L)) + N_2(r, 0; p(L)) + 2N_{2+n}(r, 0 : L) < (\lambda + o(1))T(r, L) \quad (3.1)$$

(ii) l=1 and

$$\frac{1}{2}\overline{N}(r, 0; p(L)) + N_2(r, 0; p(L)) + N_{2+n}(r, 0 : L) < (\lambda + o(1))T(r, L), \quad (3.2)$$

then $p(L) \equiv P(L)$.

4. LEMMAS

In this section we present some necessary lemmas.

Henceforth we denote by Ω the function defined by

$$\Omega = (\frac{\Phi''}{\Phi'} - \frac{2\Phi'}{\Phi - 1}) - (\frac{\Psi''}{\Psi'} - \frac{2\Psi'}{\Psi - 1})$$

Lemma 4.1. *[13]. Let L be an L-function with degree q. Then $T(r, L) = \frac{q}{\pi} r \log r + O(r)$.*

Lemma 4.2. *[10]. Let L be an L-function. Then $N(r, \infty; L) = S(r, L) = O(\log r)$.*

Lemma 4.3. *[16]. Let $\xi(z) = \frac{\alpha_0+\alpha_1 z+.....+\alpha_t z^t}{\beta_0+\beta_1 z+.....+\beta_s z^s}$ be a nonconstant rational function defined in the complex plane $\mathbb{C}$, where $\alpha_0, \alpha_1, ..., \alpha_t(\neq 0)$ and $\beta_0, \beta_1, ..., \beta_s(\neq 0)$ are complex constants. Then $T(r, \xi) = max\{t, s\} \log r + O(1)$.*

Lemma 4.4. *Let ξ be a nonconstant meromorphic function defined in the complex plane and $P(\xi)$ be a homogeneous differential polynomial of degree d generated by ξ defined as in definition 2.14. If $P(\xi) \not\equiv 0$ then for any positive integer k*

(i) $N_k(r, 0; P(\xi)) \leq N_{k+n}(r, 0; \xi) + T(r, P(\xi)) - dT(r, \xi) + S(r, \xi)$

(ii) $N_k(r, 0; P(\xi)) \leq N_{k+n}(r, 0; \xi) + Q\overline{N}(r, \infty; \xi) + S(r, \xi)$.

Proof. Using first fundamental theorem we have

$$\begin{aligned}
N_k(r,0;P(\xi)) &\leq N(r,0;P(\xi)) - \sum_{l=k}^{\infty} \overline{N}(r,0;P(\xi)| \geq l) \\
&= T(r,P(\xi)) - m(r,0;P(\xi)) - \sum_{l=k}^{\infty} \overline{N}(r,0;P(\xi)| \geq l) + O(1) \\
&\leq T(r,P(\xi)) + m(r,\infty;\frac{P(\xi)}{\xi^d}) - m(r,0;\xi^d) - \sum_{l=k}^{\infty} \overline{N}(r,0;P(\xi)| \geq l) + O(1) \\
&\leq T(r,P(\xi)) - dT(r,\xi) + N(r,0;\xi^d) - \sum_{l=k}^{\infty} \overline{N}(r,0;P(\xi)| \geq l) + S(r,\xi) \\
&\leq T(r,P(\xi)) - dT(r,\xi) + N_{(k+n)d}(r,0;\xi^d) + \sum_{l=(k+n+1)d}^{\infty} \overline{N}(r,0;\xi^d| \geq l) \\
&- \sum_{l=k}^{\infty} \overline{N}(r,0;P(\xi)| \geq l) + S(r,\xi) \\
&\leq T(r,P(\xi)) - dT(r,\xi) + N_{k+n}(r,0;\xi) + S(r,\xi)). \qquad (4.1)
\end{aligned}$$

This proves (i).
Now

$$\begin{aligned}
T(r,P(\xi)) &= N(r,\infty;P(\xi)) + m(r,\infty;P(\xi)) \\
&\leq N(r,\infty;P(\xi)) + m(r,\infty;\xi^d) + m(r,\infty;\frac{P(\xi)}{\xi^d}) \\
&\leq N(r,\infty;P(\xi)) + dm(r,\infty;\xi) + S(r,\xi) \\
&\leq dN(r,\infty;\xi) + Q\overline{N}(r,\infty;\xi) + dm(r,\infty;\xi) + S(r,\xi) \\
&\leq dT(r,\xi) + Q\overline{N}(r,\infty;\xi) + S(r,\xi). \qquad (4.2)
\end{aligned}$$

From (4.1) and (4.2) we have

$$N_k(r,0;P(\xi)) \leq N_{k+n}(r,0;\xi) + Q\overline{N}(r,\infty;\xi) + S(r,\xi).$$

This proves (ii).
This completes the proof. □

Lemma 4.5. *[12] Let ξ be a nonconstant meromorphic function and let $\Phi(\xi) = \frac{\sum_{i=0}^{t}\alpha_i\xi^i}{\sum_{j=0}^{s}\beta_j\xi^j}$ be irreducible rational function in ξ with coefficients α_i and β_j, $i = 0,1,2,..,t$, $j = 0,1,2,...,s$ where $\alpha_t \neq 0$ and $\beta_s \neq 0$. Then $T(r,\Phi(\xi)) = \max\{t,s\}T(r,\xi) + S(r,\xi)$.*

Lemma 4.6. *Let ξ be a nonconstant meromorphic function and $\mu_i, \nu_j \in S(\xi)$, $i = 0,1,2,...t$, $j = 0,1,2,...,s$. Also let $H(\xi) = \frac{\sum_{i=0}^{t}\mu_i\xi^i}{\sum_{j=0}^{s}\nu_j\xi^j}$, where $\mu_t \not\equiv 0$ and $\nu_s \not\equiv 0$. Then $T(r,H(\xi)) = \max\{t,s\}T(r,\xi) + S(r,\xi)$.*

Proof. Since $\mu_i, \nu_j \in S(\xi)$, $i = 0, 1, 2, .., t$, $j = 0, 1, 2, ..., s$, therefore $T(r, \mu_i) = S(r, \xi)$, $i = 0, 1, ..., t$ and $T(r, \nu_j) = S(r, \xi)$, $j = 0, 1, ..., s$. Hence the result follows by lemma 4.5 □

Lemma 4.7. *Let L be a nonconstant L-function. Then $T(r, R(z)) = S(r, L)$.*

Proof. By lemma 4.1, lemma 4.3 and lemma 4.6 we get the required result. □

Lemma 4.8. *[1] Let $P(\xi)$ be a homogeneous differential polynomial of degree d generated by a nonconstant meromorphic function ξ as defined in definition 2.14. Then*

$$N(r, 0; \frac{P(\xi)}{f^d}) \leq Q(\overline{N}(r, 0; \xi) + \overline{N}(r, \infty; \xi)) + S(r, \xi).$$

Lemma 4.9. *[9] Let l be a nonnegative integer and two nonconstant meromorphic functions Φ and Ψ share "$(1, l)$". If $\Omega \not\equiv 0$ and $2 \leq l \leq \infty$, then*

$$T(r, \Phi) \leq N_2(r, \infty; \Phi) + N_2(r, 0; \Phi) + N_2(r, \infty; \Psi) + N_2(r, 0; \Psi) + S(r, \Phi) + S(r, \Psi)$$

and

$$T(r, \Psi) \leq N_2(r, \infty; \Phi) + N_2(r, 0; \Phi) + N_2(r, \infty; \Psi) + N_2(r, 0; \Psi) + S(r, \Phi) + S(r, \Psi).$$

Lemma 4.10. *[15] Let two nonconstant meromorphic functions Φ and Ψ share "$(1, 0)$", then*

(i) $\overline{N}(r, 1; \Phi| > \Psi) + 2\overline{N}(r, 1; \Psi| > \Phi) + N(r, 1; \Phi, \Psi| \geq 2) - \overline{N}(r, 1, \Phi, \Psi|\Phi > \Psi = 1)$
$- \overline{N}(r, 1, \Psi, \Phi|\Psi > \Phi = 1) \leq N(r, 1; \Psi) - \overline{N}(r, 1; \Psi)$.

(ii) $\overline{N}(r, 1; \Phi| > \Psi) \leq \overline{N}(r, \infty; \Phi) + \overline{N}(r, 0; \Phi) + S(r, \Phi)$.

(iii) $\overline{N}(r, 1; \Phi, \Psi|\Phi > \Psi = 1) \leq \overline{N}(r, 0; \Phi) + \overline{N}(r, \infty; \Phi) - \overline{N}_{\otimes}(r, 0; \Phi^{(1)}) + S(r, \Phi)$.

(iv) $\overline{N}(r, 1; \Psi, \Phi|\Psi > \Phi = 1) \leq \overline{N}(r, 0; \Psi) + \overline{N}(r, \infty; \Psi) - \overline{N}_{\otimes}(r, 0; \Psi^{(1)}) + S(r, \Psi)$.

Lemma 4.11. *[15] Let two nonconstant meromorphic functions Φ and Ψ share "$(1, 1)$", then*

(i) $2\overline{N}(r, 1; \Phi| > \Psi) + 2\overline{N}(r, 1; \Psi| > \Phi) + N(r, 1; \Phi, \Psi| \geq 2) - \overline{N}(r, 1, \Phi, \Psi|\Phi > \Psi = 2)$
$\leq N(r, 1; \Psi) - \overline{N}(r, 1; \Psi)$.

(ii) $\overline{N}(r,1;\Phi,\Psi|\Phi > \Psi = 2) \leq \frac{1}{2}\overline{N}(r,0;\Phi) + \frac{1}{2}\overline{N}(r,\infty;\Phi) - \frac{1}{2}\overline{N}_{\otimes}(r,0;\Phi^{(1)}) + S(r,\Phi)$.

Lemma 4.12. *Let L be a nonconstant L-function and $p(z)$ be a polynomial of degree $\lambda \geq 1$ with $p(0) = 0$. Let $P(L)$ be a homogeneous differential polynomial of degree d generated by L. Let $\Phi(z) = \frac{(p(L(z))}{R(z)}$ and $\Psi(z) = \frac{P(L(z))}{R(z)}$. If $p(L)$ and $P(L)$ share "$(R(z),0)$" and $\Omega \not\equiv 0$, then*

$$2\overline{N}(r,0;p(L)) + N_2(r,0;p(L)) + 2N_{2+n}(r,0:L) \geq (\lambda + o(1))T(r,L).$$

Proof. Clearly Φ and Ψ share "$(1,0)$" except for the zeros and poles of $R(z)$.

By lemma 4.7 we have $R(z)$ is a small function of L.
Hence

$$\begin{aligned} N(r,\infty;\Omega) &\leq \overline{N}(r,\infty;\Phi) + \overline{N}(r,0;\Phi| \geq 2) + \overline{N}(r,0;\Psi| \geq 2) \\ &+ \overline{N}(r,1;\Phi| \geq \Psi) + \overline{N}(r,1;\Psi| \geq \Phi) + \overline{N}_{\otimes}(r,0;\Phi^{(1)}) \\ &+ \overline{N}_{\otimes}(r,0;\Psi^{(1)}) + S(r,\Phi) + S(r,\Psi) \end{aligned} \tag{4.3}$$

and

$$\overline{N}_E(r,1;\Phi,\Psi| \leq 1) \leq N(r,\infty;\Omega) + S(r,\Phi). \tag{4.4}$$

Using (4.3), (4.4) and lemma 4.10 we have

$$\begin{aligned} \overline{N}(r,1;\Phi) + \overline{N}(r,1;\Psi) &\leq \overline{N}_E(r,1;\Phi,\Psi| \leq 1) + \overline{N}(r,1;\Phi| > \Psi) + \overline{N}(r,1;\Psi| > \Phi) \\ &+ \overline{N}_E(r,1;\Phi,\Psi| \geq 2) + \overline{N}(r,1;\Psi) + S(r,\Phi) + S(r,\Psi) \\ &\leq \overline{N}_E(r,1;\Phi,\Psi| \leq 1) + N(r,1;\Psi) - \overline{N}(r,1;\Psi| > \Phi) \\ &+ \overline{N}(r,1,\Phi,\Psi|\Phi > \Psi = 1) + \overline{N}(r,1,\Psi,\Phi|\Psi > \Phi = 1) \\ &+ S(r,\Phi) + S(r,\Psi) \\ &\leq \overline{N}(r,\infty;\Phi) + \overline{N}(r,0;\Phi| \geq 2) + \overline{N}(r,0;\Psi| \geq 2) \\ &+ \overline{N}(r,1;\Phi| \geq \Psi) + \overline{N}_{\otimes}(r,0;\Phi^{(1)}) \\ &+ \overline{N}_{\otimes}(r,0;\Psi^{(1)}) + N(r,1;\Psi) + \overline{N}(r,1,\Phi,\Psi|\Phi > \Psi = 1) \\ &+ \overline{N}(r,1,\Psi,\Phi|\Psi > \Phi = 1) + S(r,\Phi) + S(r,\Psi). \end{aligned} \tag{4.5}$$

Using Nevanlinna second fundamental theorem we get from (4.5), lemma 4.2, lemma

4.4, lemma 4.7 and lemma 4.10

$$\begin{aligned}
T(r,\Phi)+T(r,\Psi) &\le \overline{N}(r,\infty;\Phi)+\overline{N}(r,\infty;\Psi)+\overline{N}(r,0;\Phi)+\overline{N}(r,0;\Psi)+\overline{N}(r,1;\Phi)\\
&+ \overline{N}(r,1;\Psi)-\overline{N}_{\otimes}(r,0;\Phi^{(1)})-\overline{N}_{\otimes}(r,0;\Psi^{(1)})+S(r,\Phi)+S(r,\Psi)\\
&\le N(r,1;\Psi)+4\overline{N}(r,\infty;\Phi)+2\overline{N}(r,\infty;\Psi)+3\overline{N}(r,0;\Phi)\\
&+ 2\overline{N}(r,0;\Psi)+\overline{N}(r,0;\Phi|\ge 2)+\overline{N}(r,0;\Psi|\ge 2)+S(r,\Phi)+S(r,\Psi)\\
&\le N(r,1;\Psi)+4\overline{N}(r,\infty;L)+2\overline{N}(r,\infty;L)+3\overline{N}(r,0;\Phi)\\
&+ 2\overline{N}(r,0;\Psi)+\overline{N}(r,0;\Phi|\ge 2)+\overline{N}(r,0;\Psi|\ge 2)+S(r,L)\\
&\le T(r,\Psi)+2\overline{N}(r,0;p(L))+N_2(r,0;p(L))+\overline{N}(r,0;P(L))\\
&+ N_2(r,0;P(L))+S(r,L)\\
&\le T(r,\Psi)+2\overline{N}(r,0;p(L))+N_2(r,0;p(L))+2N_{2+n}(r,0;L)\\
&+ 2Q\overline{N}(r,\infty;L)+S(r,L)\\
&\le T(r,\Psi)+2\overline{N}(r,0;p(L))+N_2(r,0;p(L))\\
&+ 2N_{2+n}(r,0;L)+S(r,L). \qquad (4.6)
\end{aligned}$$

Using lemma 4.7 we have from (4.6)

$$2\overline{N}(r,0;p(L))+N_2(r,0;p(L))+2N_{2+n}(r,0;L)\ge(\lambda+o(1))T(r,L).$$

This completes the proof of the lemma. □

Lemma 4.13. *Let* L *be a nonconstant L-function and* $p(z)$ *be a polynomial of degree* $\lambda\ge 1$ *with* $p(0)=0$*. Let* $P(L)$ *be a homogeneous differential polynomial of degree* d *generated by* L*. Let* $\Phi(z)=\frac{(p(L(z))}{R(z)}$ *and* $\Psi(z)=\frac{P(L(z))}{R(z)}$*. If* $p(L)$ *and* $P(L)$ *share "*$(R(z),1)$*" and* $\Omega\not\equiv 0$*, then*

$$\frac{1}{2}\overline{N}(r,0;p(L))+N_2(r,0;p(L))+N_{2+n}(r,0:L)\ge(\lambda+o(1))T(r,L).$$

Proof. Clearly Φ and Ψ share "$(1,1)$" except for the zeros and poles of $R(z)$.
By lemma 4.7 we have $R(z)$ is a small function of L.
Hence using Lemma 4.2 and Lemma 4.7 we get

$$\begin{aligned}
N(r,\infty;\Omega) &\le \overline{N}(r,\infty;\Phi)+\overline{N}(r,0;\Phi|\ge 2)+\overline{N}(r,0;\Psi|\ge 2)\\
&+ \overline{N}_{\otimes}(r,0;\Phi^{(1)})+\overline{N}_{\otimes}(r,0;\Psi^{(1)})+S(r,\Phi)+S(r,\Psi)\\
&\le \overline{N}(r,\infty;L)+\overline{N}(r,0;\Phi|\ge 2)+\overline{N}(r,0;\Psi|\ge 2)\\
&+ \overline{N}_{\otimes}(r,0;\Phi^{(1)})+\overline{N}_{\otimes}(r,0;\Psi^{(1)})+S(r,L)\\
&\le \overline{N}(r,0;\Phi|\ge 2)+\overline{N}(r,0;\Psi|\ge 2)+\overline{N}_{\otimes}(r,0;\Phi^{(1)})\\
&+ \overline{N}_{\otimes}(r,0;\Psi^{(1)})+S(r,L) \qquad (4.7)
\end{aligned}$$

and

$$N(r,1;\Phi|=1) \le N(r,0;\Omega) + S(r,\Phi) \le N(r,\infty;\Omega) + S(r,\Phi). \tag{4.8}$$

Using (4.7), (4.8) and Lemma 4.2, Lemma 4.7 and Lemma 4.11 we have

$$\begin{aligned}
\overline{N}(r,1;\Phi)+\overline{N}(r,1;\Psi) &\le \overline{N}_E(r,1;\Phi,\Psi|\le 1)+\overline{N}(r,1;\Phi|>\Psi)+\overline{N}(r,1;\Psi|>\Phi)\\
&+ \overline{N}_E(r,1;\Phi,\Psi|\ge 2)+\overline{N}(r,1;\Psi)+S(r,\Phi)+S(r,\Psi)\\
&\le \overline{N}_E(r,1;\Phi,\Psi|\le 1)+N(r,1;\Psi)-\overline{N}(r,1;\Phi|>\Psi)\\
&- \overline{N}(r,1;\Psi|>\Phi)+\overline{N}(r,1,\Phi,\Psi|\Phi>\Psi=2)+S(r,\Phi)+S(r,\Psi)\\
&\le \overline{N}_E(r,1;\Phi,\Psi|\le 1)+N(r,1;\Psi)-\overline{N}(r,1;\Phi|>\Psi)\\
&- \overline{N}(r,1;\Psi|>\Phi)+\frac{1}{2}\overline{N}(r,0,\Phi)+\frac{1}{2}\overline{N}(r,\infty,\Phi)\\
&- \frac{1}{2}\overline{N}_{\otimes}(r,0;\Phi^{(1)})+S(r,\Phi)+S(r,\Psi)\\
&\le \overline{N}(r,\infty;\Phi)+\overline{N}(r,0;\Phi|\ge 2)+\overline{N}(r,0;\Psi|\ge 2)-\overline{N}(r,1;\Phi|\ge\Psi)\\
&- \overline{N}(r,1;\Psi|\ge\Phi)+\overline{N}_{\otimes}(r,0;\Phi^{(1)})+\overline{N}_{\otimes}(r,0;\Psi^{(1)})+N(r,1;\Psi)\\
&+ \frac{1}{2}\overline{N}(r,0,\Phi)+\frac{1}{2}\overline{N}(r,\infty,\Psi)-\frac{1}{2}\overline{N}_{\otimes}(r,0;\Phi^{(1)})+S(r,\Phi)+S(r,\Psi)\\
&\le \overline{N}(r,0;\Phi|\ge 2)+\overline{N}(r,0;\Psi|\ge 2)-\overline{N}(r,1;\Phi|\ge\Psi)\\
&- \overline{N}(r,1;\Psi|\ge\Phi)+\overline{N}_{\otimes}(r,0;\Phi^{(1)})+\overline{N}_{\otimes}(r,0;\Psi^{(1)})\\
&+ N(r,1;\Psi)+\frac{1}{2}\overline{N}(r,0,\Phi)-\frac{1}{2}\overline{N}_{\otimes}(r,0;\Phi^{(1)})+S(r,L).
\end{aligned} \tag{4.9}$$

Using Nevanlinna second fundamental theorem we get from (4.9), lemma 4.2, lemma 4.4, lemma 4.7 and lemma 4.11

$$\begin{aligned}
T(r,\Phi)+T(r,\Psi) &\le \overline{N}(r,\infty;\Phi)+\overline{N}(r,\infty;\Psi)+\overline{N}(r,0;\Phi)\\
&+ \overline{N}(r,0;\Psi)+\overline{N}(r,1;\Phi)+\overline{N}(r,1;\Psi)\\
&- \overline{N}_{\otimes}(r,0;\Phi^{(1)})-\overline{N}_{\otimes}(r,0;\Psi^{(1)})+S(r,\Phi)+S(r,\Psi)\\
&\le N(r,1;\Psi)+\overline{N}(r,\infty;\Phi)+\overline{N}(r,\infty;\Psi)+\frac{3}{2}\overline{N}(r,0;\Phi)\\
&+ \overline{N}(r,0;\Psi)+\overline{N}(r,0;\Phi|\ge 2)+\overline{N}(r,0;\Psi|\ge 2)+S(r,L)\\
&\le T(r,\Psi)+\overline{N}(r,\infty;L)+\overline{N}(r,\infty;L)+\frac{3}{2}\overline{N}(r,0;p(L))\\
&+ \overline{N}(r,0;P(L))+\overline{N}(r,0;p(L)|\ge 2)+\overline{N}(r,0;P(L)|\ge 2)+S(r,L)\\
&\le T(r,\Psi)+\frac{1}{2}\overline{N}(r,0;p(L))+N_2(r,0;p(L))+N_2(r,0;P(L))+S(r,L)\\
&\le T(r,\Psi)+\frac{1}{2}\overline{N}(r,0;p(L))+N_2(r,0;p(L))+N_{2+n}(r,0;L)\\
&+ Q\overline{N}(r,\infty;L)+S(r,L)\\
&\le T(r,\Psi)+\frac{1}{2}\overline{N}(r,0;p(L))+N_2(r,0;p(L))\\
&+ N_{2+n}(r,0;L)+S(r,L).
\end{aligned} \tag{4.10}$$

Using lemma 4.7 we have from (4.10)

$$\frac{1}{2}\overline{N}(r,0;p(L)) + N_2(r,0;p(L)) + N_{2+n}(r,0;L) \geq (\lambda + o(1))T(r,L).$$

This completes the proof of the lemma. □

5. PROOF OF THE MAIN RESULT

Proof of Theorem 3.1

Let $\Phi(z) = \frac{(p(L(z))}{R(z)}$ and $\Psi(z) = \frac{P(L(z))}{R(z)}$.

By lemma 4.7 we have $T(r, R(z)) = S(r, L)$. Now we have to consider the following two cases

Case 1 Let $\Omega \not\equiv 0$.

Then by Lemma 4.12 and Lemma 4.13 we arrive at a contradiction.

Case 2 Let $\Omega \equiv 0$.

Hence

$$(\frac{\Phi''}{\Phi'} - \frac{2\Phi'}{\Phi - 1}) - (\frac{\Psi''}{\Psi'} - \frac{2\Psi'}{\Psi - 1}) = 0. \tag{5.1}$$

Integrating (5.1) we get

$$\Psi = \frac{(D+1)\Phi + (C-D-1)}{D\Phi + (C-D)}, \tag{5.2}$$

where $C \neq 0$ and D are constants.

Now we have to consider the following three subcases.

Subcase 2.1 Let $D = 0$. Then from (5.2) we have

$$\Psi = \frac{\Phi + C - 1}{C}. \tag{5.3}$$

If $C \neq 1$, then from (5.3) we get

$$\overline{N}(r, 1-C, \Phi) = \overline{N}(r, 0; \Psi). \tag{5.4}$$

Using lemma 4.2, lemma 4.4, (5.4) we get by Nevanlinna second fundamental theorem

$$\begin{aligned}
T(r,p(L)) &= T(r,\Phi) + S(r,L) \\
&\leq \overline{N}(r,\infty;\Phi) + \overline{N}(r,0;\Phi) + \overline{N}(r,1-C;\Phi) + S(r,L) \\
&\leq \overline{N}(r,\infty;L) + \overline{N}(r,0;\Phi) + \overline{N}(r,0;\Psi) + S(r,L) \\
&\leq \overline{N}(r,0;P(L)) + \overline{N}(r,0;p(L)) + S(r,L) \\
&\leq N_{1+n}(r,0;L) + Q\overline{N}(r,\infty;L) + \overline{N}(r,0;p(L)) + S(r,L) \\
&\leq N_{2+n}(r,0;L) + N_2(r,0;p(L) + S(r,L).
\end{aligned} \tag{5.5}$$

From (5.5) we get $N_2(r,0;p(L))+N_{2+n}(r,0;L) \geq (\lambda+o(1))T(r,L)$, which contradicts (3.2). Hence $C=1$ and therefore $p(L) \equiv P(L)$.

Subcase 2.2 Let $D=-1$. Then from (5.2) we have

$$\Psi = \frac{C}{C+1-\Phi}. \tag{5.6}$$

If $C \neq -1$, then using lemma 4.2 we get from (5.6)

$$\overline{N}(r,1+C,\Phi) = \overline{N}(r,\infty;\Psi) = \overline{N}(r,\infty;L) + S(r,L) = S(r,L). \tag{5.7}$$

Using lemma 4.2, lemma 4.4, (5.7) we get by Nevanlinna second fundamental theorem

$$\begin{aligned} T(r,p(L)) &= T(r,\Phi)+S(r,L) \\ &\leq \overline{N}(r,\infty;\Phi)+\overline{N}(r,0;\Phi)+\overline{N}(r,1+C;\Phi)+S(r,L) \\ &\leq \overline{N}(r,\infty;L)+\overline{N}(r,0;\Phi)+\overline{N}(r,\infty;\Phi)+S(r,L) \\ &\leq \overline{N}(r,0;p(L))+\overline{N}(r,\infty;L)+S(r,L) \\ &\leq \overline{N}(r,0;p(L))+S(r,L). \end{aligned} \tag{5.8}$$

From (5.8)we get $\overline{N}(r,0;p(L)) \geq (\lambda+o(1))T(r,L)$, which contradicts (3.2).
If $C=-1$, then

$$\Phi\Psi = 1. \tag{5.9}$$

From (5.9) we have

$$p(L)P(L) = R^2(z). \tag{5.10}$$

From (5.10) we have

$$\overline{N}(r,\infty;L)+\overline{N}(r,0;L) = S(r,L). \tag{5.11}$$

Using lemma 4.8 and (5.11) we get $N(r,\infty;\frac{P(L)}{L^d}) = S(r,L)$ and hence

$$T(r,\frac{P(L)}{L^d}) = N(r,\infty;\frac{P(L)}{L^d})+m(r,\infty;\frac{P(L)}{L^d}) = S(r,L). \tag{5.12}$$

Using lemma 4.6 and (5.12) we have

$$\begin{aligned} (d+\lambda)T(r,L) &\leq T(r,\frac{R^2(z)}{d^{\lambda+d}})+O(1) \\ &\leq T(r,(1+\frac{\rho_{\lambda-1}}{L}+...+\frac{\rho_1}{L^{\lambda-1}})\frac{P(L)}{L^d})+O(1) \\ &\leq (\lambda-1)T(r,L)+T(r,\frac{P(L)}{L^d})+S(r,L) \\ &\leq (\lambda-1)T(r,L)+S(r,L). \end{aligned} \tag{5.13}$$

From (5.13) we get $T(r, L) = S(r, L)$, which is a contradiction.

Subcase 2.3 Let $D \neq 0, -1$.
If $C - D \neq 1$, the from (5.2) we get $\overline{N}(r, \frac{-C+D+1}{D+1}; \Phi) = \overline{N}(r, 0; \Psi)$. Now proceeding as in Subcase 2.1 we arrive at a contradiction.
If $C - D = 1$, then by (5.2) we get $\overline{N}(r, \frac{-1}{D}; \Phi) = \overline{N}(r, \infty; \Psi)$. Now proceeding as in Subcase 2.2 we arrive at a contradiction.
This completes the proof.

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