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# Solutions of the Thomas Equation Using the Pure Lie Symmetries Approach and Manifolds

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**ABSTRACT**

In this contribution, we use differentiable topological manifolds to determine solutions of the Thomas equation. First we transform the equation from partial differential form to ordinary differential form, using Sophus Lie' symmetry group theoretical methods. We then apply our differentiable topological manifolds approach to it.

**Keywords:** Thomas equation, Lie symmetry group theoretical methods, Differential equations, Differentiable topological manifolds.

## 1 Introduction

The Thomas equation is a nonlinear second-order partial differential equation (PDE) given by

$$u_{xt} + \alpha u_x + \ \beta u_t + \delta u_x u_t = 0. \tag{1}$$

Here $u$ depends on $t$ and $x$, while the coefficients $\alpha, \beta$ and $\sigma$ are constant parameters. It is a subject in the physical sciences, particularly in the study of chemical processes; a model proposed by Henry Thomas [1].

As indicated in the abstract, we partly use Sophus Lie' symmetry group theoretical methods to simplify the equation. Several authors have used this technique. They include Sakovich [2] Stephani [3], Ouheden [4] , Al-Ghafri [5], Yan [6], Wei at al [7], and many others The problems most encountered is that they ended up with integrals they could not resolve analytically, subsequently resorting to special cases of the coefficients, consequently compromising the results. Our differentiable topological methods approach is designed to circumvent this impasse.

We introduce Lie's methods in Section 2, a theory developed by the Norwegian Mathematician Sophus Lie (1842-1899), first introduced through his now famous 1881 paper [8]. It has since snowballed, now followed and practiced by many scholars. The Russian Mathematician, Lev Vasilyevich Ovsyannikov (1919-2014) is among those who revived the theory in the 1950s. His work includes [9], [10], [11], [12] and [13]. Many more followed in his footsteps, that includes Nail Ibragimov and Gazizov, see [14].

The approach as used in Section 2, consists of a systematic procedure for the determination of continuous symmetry transformations of a system of the nonlinear PDE (1). This procedure is replicated in many texts, see Olver [15], Bluman and Kumei [16], and Ibragimov [17].

In Section 3, we briefly outline the differentiable topological methods approach, a procedure first proposed by the third author [18].

Section 4 is on the application of the technique discussed in Section 3 to the Thomas equation. That is, the Thomas equation that has been transformed into an ordinary differential equation using methods discussed in Section 2.

## 2 The pure Lie approach

The application of Lie's theory to a differential involves first applying a symmetry generator to it, resulting in what is known as the determining equation, which in turn leads to what are known as Lie symmetries, then to invariants, then the ordinary differential equation.

The infinitesimal symmetry generator for a PDE with one dependent variable and two independent variables, is given by

$$X = \xi \frac{\partial}{\partial x} + \tau \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial u}. \tag{2}$$

The Thomas equation is second-order PDE, for this we require an extension of the generator to the second-order form

$$X^{[2]} = X + \zeta^1 \frac{\partial}{\partial u_x} + \zeta^2 \frac{\partial}{\partial u_t} + \zeta^{12} \frac{\partial}{\partial u_{xt}} + \zeta^{11} \frac{\partial}{\partial u_{xx}} + \zeta^{22} \frac{\partial}{\partial u_{tt}}. \tag{3}$$

This depends on the operators of total differentiation. That is,

$$D_x = \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{xx} \frac{\partial}{\partial u_x} + u_{xt} \frac{\partial}{\partial u_t} + \cdots \tag{4}$$

$$D_t = \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{tt} \frac{\partial}{\partial u_t} + u_{xt} \frac{\partial}{\partial u_x} + \cdots \tag{5}$$

These lead to the prolongations

$$\zeta^1 = \tau_x + u_x(\tau_u - \xi_x) - u_t\eta_x - (u_x)^2\xi_u - u_x u_t \eta_u, \tag{6}$$

$$\zeta^2 = \tau_t + u_t(\tau_u - \eta_t) - u_x\xi_t - (u_t)^2\eta_u - u_x u_t \xi_u, \tag{7}$$

$$\zeta^{12} = \tau_{xt} + u_x(\tau_{tu} - \xi_{xt}) + u_t(\tau_{xu} - \eta_{xt}) - (u_x)^2\xi_{tu} + u_x u_t(\tau_{uu} - \xi_{ux} - \eta_{tu}) - (u_t)^2\eta_{ux} - (u_x)^2 u_t \xi_{uu} - (u_t)^2 u_x \eta_{uu} - 2u_x u_{xt}\xi_u - 2u_t u_{xt}\eta_u - u_{xx}\xi_t - u_{tt}\eta_x - u_t u_{xx}\xi_u - u_x u_{tt}\eta_u + u_{xt}(\tau_u - \xi_x - \eta_t). \tag{8}$$

### 2.1 Application of the symmetry generator

The application of the generator, such as the one we just presented above, to an expression $F = 0$, leads to the invariance condition

$$X^{[2]}F \,\Big|_{F=0} = 0. \tag{9}$$

Here $F = u_{xt} + \alpha u_x + \beta u_t + \delta u_x u_t$. That is, it is equation (1). Expressing it in the following way,will prove usefull:

$$u_{xt} = -\alpha u_x - \beta u_t - \delta u_x u_t \,. \tag{10}$$

This expression assists in simplifying the invariance condition further. After substituting (2) and (3) into (9), we obtain

$$(\beta + \delta u_x)\zeta^1 + (\alpha + \beta u_t)\zeta^2 + \zeta^{12} = 0. \tag{11}$$

Putting (6), (7), (8) and (10) in (11) we have:

$$\beta g_t + \beta u f_t + \beta u_t(f - \tau_t) - \beta u_x\xi_t + \delta u_x g_t + \delta u_x u f_t + \delta u_x\, u_t(f - \tau_t) - \delta u_x u_x \xi_t + \alpha g_x + \alpha u f_x + \alpha u_x(f - \xi_x) - \alpha u_t\tau_x + \beta u_t g_x + \beta u_t u f_x + \beta u_t u_x(f - \xi_x) - \beta u_t u_t \tau_x + g_{xt} + u f_{xt} - \alpha u_x(f - \xi_x - \tau_t) - \beta u_t(f - \xi_x - \tau_t) - \delta u_x u_t(f - \xi_x - \tau_t) + u_x(f_t - \xi_{xt}) + u_t(f_x - \tau_{xt}) - u_{tt}\tau_x - u_{xx}\xi_t = 0, \tag{11}$$

known among Symmetry Analysts as the determining equation.

#### 2.1.1 The monomials

Simplifying (12), we obtain the following equations. That is, the monomials.

$$(1)^0:\ \alpha\tau_x + \beta\tau_t + \tau_{xt} = 0, \tag{12}$$

$$u_x:\ \tau_{tu} - \xi_{xt} + \delta\tau_t - \beta\xi_t + \alpha\eta_t = 0, \quad (13)$$

$$u_t:\ \tau_{xu} - \eta_{xt} + \delta\tau_x - \alpha\eta_x + \beta\xi_x = 0, \quad (14)$$

$$u_x u_t:\ \tau_{uu} - \xi_{ux} - \eta_{tu} + \beta\xi_u + \alpha\eta_u + \delta\tau_u = 0, \quad (15)$$

$$(u_x)^2: -\xi_{tu} - \delta\xi_t + \alpha\xi_u = 0, \quad (16)$$

$$(u_t)^2: -\eta_{xu} - \delta\eta_x + \beta\eta_u = 0, \quad (17)$$

$$u_x(u_t)^2:\ \eta_{uu} = 0, \quad (18)$$

$$u_t(u_x)^2: -\xi_{uu} = 0, \quad (19)$$

$$u_{xx}: -\xi_t = 0, \quad (20)$$

$$u_{tt}: -\eta_x = 0, \quad (21)$$

$$u_t u_{xx}:\ -\xi_u = 0, \quad (22)$$

$$u_x u_{tt}: -\eta_u = 0. \quad (24)$$

From equation (21) and (23) we have

$$\xi = \xi(x), \quad (24)$$

and equations (22) and (24) give

$$\eta = \eta(t). \quad (25)$$

From equation (16):

$$\tau_{uu} + \delta\tau_u = 0. \quad (26)$$

After some calculations on (27), we get

$$\tau = N(x,t) + M(x,t)e^{-\delta u}, \tag{27}$$

where $N$ and $M$ are arbitrary functions of $x$ and $t$. From (28), we get

$$\tau_t = N_t + M_t e^{-\delta u}, \tag{28}$$

$$\tau_x = N_x + M_x e^{-\delta u}, \tag{29}$$

$$\tau_{tu} = -\delta M_t e^{-\delta u}, \tag{30}$$

$$\tau_{xu} = -\delta M_x e^{-\delta u}. \tag{31}$$

Using equations (29) and (31) in equation (14), we get

$$-\delta M_t e^{-\delta u} + \delta N_t + \delta M_t e^{-\delta u} + \alpha\eta_t = 0. \tag{32}$$

We then calculate the following

$$\alpha\eta_t = -\delta N_t, \tag{33}$$

$$\eta_t = -\frac{\delta N_t}{\alpha}. \tag{34}$$

Substituting (30) and (32) in (15), we get

$$-\delta M_x e^{-\delta u} + \delta N_x + \delta M_x e^{-\delta u} + \beta\xi_x = 0. \tag{35}$$

This simplifies to

$$\beta\xi_x = -\delta N_x, \tag{36}$$

$$\xi_x = -\frac{\delta N_x}{\beta}. \tag{37}$$

From (29) and (30), we get

$$N_{xt} = 0, \tag{38}$$

$$N(x,t) = px + zt + A_1, \tag{39}$$

where $p, z$ and $A_1$ are arbitrary constants. From (40), we obtain

$$N_x = p, \tag{40}$$

and

$$N_t = z. \tag{41}$$

Putting (41) in (38):

$$\xi_x = -\frac{\delta p}{\beta}, \tag{42}$$

$$\xi = -\frac{\delta p}{\beta}x + A_2, \tag{43}$$

where $A_2$ is an arbitrary constant. Now putting (42) in (35):

$$\eta_t = -\frac{\delta z}{\alpha}. \tag{44}$$

It integrates into

$$\eta = -\frac{\delta z}{\alpha}t + A_3, \tag{45}$$

where $A_3$ is an arbitrary constant. From (13) we have

$$\alpha p + \beta z = 0. \tag{46}$$

It then follows then from (47) that

$$\frac{p}{\beta} = -\frac{z}{\alpha}. \tag{47}$$

This can be simplified to

$$\frac{p}{\beta} = A_4, \tag{48}$$

where $A_4$ is an arbitrary constant. It then implies that

$$\frac{z}{\alpha} = -A_4. \tag{49}$$

Substituting (49) in (44) gives

$$\xi = -\delta A_4 t + A_2. \tag{50}$$

Then putting (50) in (46) yields

$$\eta = \delta A_4 t + A_3. \tag{51}$$

From (48), we then get

$$z = -\frac{\alpha p}{\beta}. \tag{52}$$

Now putting (53) in (40) suggests

$$N(x,t) = px - \frac{\alpha p}{\beta} t + A_1. \tag{53}$$

This can be expressed tersely in the form

$$N(x,t) = \frac{p}{\beta}(\beta x - \alpha t) + A_1. \tag{54}$$

We simplify (55) as

$$N(x,t) = A_4(\beta x - \alpha t) + A_1. \tag{55}$$

Subsequently, we get the defining equation

$$\tau = M(x,t)e^{-\delta u} + A_4(\beta x - \alpha t) + A_1. \tag{56}$$

### 2.1.2 The symmetries

The defining equations determined in the previous subsection, lead to the symmetries

$$X_1 = \frac{\partial}{\partial t}, \tag{57}$$

$$X_2 = \frac{\partial}{\partial x}, \tag{58}$$

$$X_3 = \frac{\partial}{\partial u}, \tag{59}$$

$$X_4 = -\delta x \frac{\partial}{\partial x} + \delta t \frac{\partial}{\partial t} + (\beta x - \alpha t)\frac{\partial}{\partial u}. \tag{60}$$

## 2.2 Invariant solutions

Solutions obtained through Lie symmetry group theoretical methods are referred to as invariant solutions, or group invariant solutions. This follows from the invariance conditions discussed earlier. We establish the solutions that follow from the full symmetry $X_4$, here using the expression

$$\frac{dx}{\xi(x,t,u)} = \frac{dt}{\tau(x,t,u)} = \frac{du}{\eta(x,t,u)}. \tag{61}$$

### 2.2.1 The solutions through the symmetry $X_4$

The symmetry $X_4$, leads to the characteristic equation

$$\frac{dx}{-\delta x} = \frac{dt}{\delta t} = \frac{du}{(\beta x - \alpha t)}, \tag{62}$$

which can be separated into two equations, leading to the result in (66), expressible in the form given in (67):

$$\frac{dx}{-\delta x} = \frac{dt}{\delta t}, \tag{63}$$

$$\frac{dx}{x} = -\frac{dt}{t}. \tag{64}$$

$$\ln x = -\ln t + \ln y, \tag{65}$$

$$\ln x + \ln t = \ln y. \tag{66}$$

The first invariant that follows is

$$y = xt, \tag{67}$$

or

$$t = \frac{y}{x}. \tag{68}$$

Substituting this into (63) gives

$$-\frac{dx}{\delta x} = \frac{du}{\left(\beta x - \alpha \frac{y}{x}\right)}, \tag{69}$$

$$\frac{1}{x}\left(\beta x - \frac{\alpha y}{x}\right) dx = -\delta du, \tag{70}$$

$$\beta x + \frac{\alpha y}{x} = -\delta u + v(y). \tag{71}$$

The second invariant is

$$v(y) = \beta x + \frac{\alpha y}{x} + \delta u. \tag{72}$$

From (73) we can write $u$ in terms of $v$. That is,

$$\mathrm{u} = \frac{1}{\delta} v - \frac{\beta}{\delta} x - \frac{\alpha y}{\delta x}. \tag{73}$$

Substituting (69) in (74)

$$\mathrm{u} = \frac{1}{\delta} v - \frac{\beta}{\delta} x - \frac{\alpha}{\delta} t. \tag{74}$$

We can now find $u_x, u_t$, and $u_{xt}$ from (75):

$$u_x = \frac{t}{\delta} v' - \frac{\beta}{\delta}, \tag{75}$$

$$u_t = \frac{x}{\delta} v' - \frac{\alpha}{\delta}, \tag{76}$$

$$u_{xt} = \frac{xt}{\delta} v'' + \frac{1}{\delta} v'. \tag{77}$$

Substituting (68) into (78) leads to

$$u_{xt} = \frac{y}{\delta} v'' + \frac{1}{\delta} v'. \tag{78}$$

Substituting (76), (77) and (79) in (1) finally yields the ordinary differential equation

$$\frac{y}{\delta} v'' + \frac{1}{\delta} v' + \frac{\alpha t}{\delta} v' - \frac{\alpha\beta}{\delta} + \frac{\beta x}{\delta} v' - \frac{\alpha\beta}{\delta} + \delta\left(\frac{t}{\delta} v' - \frac{\beta}{\delta}\right)\left(\frac{x}{\delta} v' - \frac{\alpha}{\delta}\right) = 0, \tag{79}$$

$$\frac{y}{\delta} v'' + \frac{1}{\delta} v' + \frac{\alpha t}{\delta} v' - \frac{\alpha\beta}{\delta} + \frac{\beta x}{\delta} v' - \frac{\alpha\beta}{\delta} + \frac{xt}{\delta}(v')^2 - \frac{\alpha t}{\delta} v' - \frac{\beta x}{\delta} v' + \frac{\alpha\beta}{\delta} = 0. \tag{80}$$

Thus we have taken the original PDE and transformed it to an ODE. That is,

$$\frac{y}{\delta} v'' + \frac{1}{\delta} v' + \frac{xt}{\delta}(v')^2 - \frac{\alpha\beta}{\delta} = 0. \tag{81}$$

After substituting (68) in (82), we get

$$y v'' + v' + y(v')^2 - \alpha\beta = 0. \tag{82}$$

Since $\alpha\beta$ is constant we write the ODE in the form

$$yv'' + v' + y(v')^2 - \phi = 0. \tag{83}$$

## 3 The differentiable topological manifolds approach

The differentiable topological manifolds approach that is briefly discussed here, borrows heavily from the method variation of parameters, a technique commonly used for solving second-order non-homogeneous linear ODEs

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x), \tag{84}$$

where the coefficients $a, b$ and $c$ are constant parameters, for a given $f(x)$, with $y = y(x)$.

### 3.1 The basics of the variation of parameters method

The usual first steps involve solving the homogeneous case. That is the case

$$f(x) = 0, \tag{85}$$

so that

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0, \tag{86}$$

from which it is found that,

$$y_c = C_1y_1 + C_2y_2 \tag{87}$$

known as the complementary solution. The constants $C_1$ and $C_2$ are the parameters that need to be varied, hence the title *the method of variation of parameters*. That is, at some stage we will have

$$v_i = C_i \qquad i = 1,2. \tag{88}$$

Odd as it seems, this is how the method of variation of parameters proceeds. This leads to the particular solution

$$y_p = v_1y_1 + v_2y_2 \tag{89}$$

so that the general solution is given as:

$$y = y_C + y_p. \tag{90}$$

We take two assumptions to the next subsection and beyond. The assumption giving rise to (87) will be interpreted as describing points within quotient spaces, leading to

(124). The second assumption, the one leading to (89), relates this space to the entire differentiable topological manifolds, where it is located. It leads to (97) and (98) which generate (130). As indicated earlier, this procedure developed by the third author of this contribution. For ease of references, we rephrase the procedure here.

### 3.2 Differentiable Topological Manifold

We start with a topological space $M = (X, J_X)$, a Hausdorff topology. That is, a set $X$ with topology $J_{X.}$ For it to be a differentiable topological manifold, or simply a differentiable manifold, we require an atlas $A$ in addition. Then we have $DM = (X, J_X, A)$.

We now consider two points $p \in U_p$ and $q \in U_q$, such that the sets $U_\rho$ and $U_q$ are elements of the same manifold. We can then build the sub-topologies $(U_\rho, J_X|_{U_\rho})$ and $\left(U_q, J_X|_{U_q}\right)$. That is, t (the topology of $X$) is restricted to $U_\rho$ and $U_q$. A mapping $\psi_\rho$, if it exists, then the space $(U_\rho, J_X|_{U_\rho})$ into the Euclidean space $(\mathbb{R}^N, \mathcal{L}_{\mathbb{R}^N}|_{\psi_\rho}(U_\rho)$. Similarly, $\psi_q$ maps $\left(U_q, J_X|_{U_q}\right)$ into the Euclidian space $(\mathbb{R}^N, \mathcal{L}_{\mathbb{R}^N}|_{\psi_q}(U_q)$.

If these mapping are homeomorphisms, then a set $A$, with

$$A = \{(U_\rho, \psi_\rho), (U_q, \psi_q)\} \tag{91}$$

is called an atlas, with $\psi_\rho, \psi_q$ called coordinates.

Our interest is in one of the chats mapping equivalence classes. That is,

$$A = \{([U_\rho], [\psi_\rho]), (U_q, \psi_q)\} \tag{92}$$

Similarly, for mapping manifolds in derivatives of ψ, we get the atlases

$$A^{(i)} = ([U_\rho], [\psi_\rho^i]), (U_q, \psi_q^i) \tag{93}$$

#### 3.2.1 Transmission mapping

The mapping from $\left(\mathbb{R}^N, J_{\mathbb{R}}|_{[\psi U_\rho]}\right)$, to $\left(\mathbb{R}^N, J_{\mathbb{R}}|_{[\psi U_q]}\right)$ having stepped down from $\mathbb{R}^N$ to $\mathbb{R}$, is given by

$$\psi_\rho\left(\psi_q^{-1}\left(\psi_q([U_\rho])\right)\right), \tag{94}$$

and it is called a transition mapping. Its inverse is:

$$\psi_q\left(\psi_\rho^{-1}\left(\psi_\rho([U_q])\right)\right) \tag{95}$$

We are interested in the case where $[U_\rho]$ and $[U_q]$ overlap, such that there is a point $x$ in the neighbourhood of both p and $q$ such that

$$\big[\psi[x]\big] = \psi(x). \tag{96}$$

The transmission mappings in derivative spaces lead to

$$\frac{d^n[\psi[x]]}{dx^n} = \frac{d^n\psi(x)}{dx^n} \tag{97}$$

For $n = 1, 2, 3, \ldots$.

### 3.2.2 Tangent spaces

As indicated earlier, tangent spaces assist in establishing a function $f$, which allows for the results to be projected onto a metric space. A tangent space is a set.

$$T\rho = \big\{V_{\gamma,\rho}|\gamma\colon \mathbb{R} \to X\big\}, \tag{98}$$

such that

$$V_{\gamma\rho}f = (f\,.\gamma^{-1})[\gamma(\tau_0)], \tag{99}$$

where $\epsilon C^\infty(X), V_{\gamma\rho}\colon C^\infty(M) \to \mathbb{R}, \gamma(\tau_0) = \rho$ . The tangent space $T\rho$ has the basis vectors $\big\{\partial X^i\big\}$. Any vector then can be represented in terms of it, so that

$$X = \xi^i \frac{\partial}{\partial x^i}|_\rho, \tag{100}$$

where $X \epsilon T_\rho X = T_\rho M$.

### 3.2.3 Cotangent spaces

A tangent space is a vector space, and where there is one there should also be a co-vector space, hence the cotangent space. It is the set of all the maps in the tangent space to $\mathbb{R}$. That is,

$$w\colon T_\rho X \to \mathbb{R}, \tag{101}$$

with $w$ being an element of the cotangent space. The symbol $(df)_\rho$ represents a co-vector acting on mapping $f$at $\rho$. A cotangent space, therefore is

$$T\rho^* = \big\{(df)_\rho\big|f \in C^\infty(X)\big\}, \tag{102}$$

and it is a vector space and is the dual of $T\rho$. The basis vectors of a cotangent space require that

$$\left(dw^j\right)_\rho \left(\frac{\partial}{\partial x^i}\right)|_\rho = \delta_i^j, \tag{103}$$

so that

$$(T\rho^*) = \{\frac{\partial}{\partial x^i}\}|_\rho \tag{104}$$

Therefore, an element $w$ of $TP^*$ can be written

$$w = w_i(dx^i)|_\rho. \tag{105}$$

### 3.3 Quotient spaces

Consider the general ODE:

$$f\left(x, \psi, \psi', \psi'', \psi^{[3]}, \dots\right) = 0, \tag{106}$$

where

$$\psi: X \to \gamma. \tag{107}$$

A set

$$S = \{x_0, x_1, x_2, \dots\} \subset X, \tag{108}$$

Such that

$$x_i = \rho\left(x_j\right) = x_j + 2\pi k_s \tag{109}$$

where $k_s$ is an integer, that is called an equivalence class. This leads to an Quotient space $\mathbb{R}/\backsim$, which is given by

$$\mathbb{R}/\backsim = \{[x_0], [x_1], [x_2], \dots\}. \tag{110}$$

It is a differentiable topological space. In our study, the image of this set, is also an equivalence class

$$\{[\psi(x_0)], [\psi(x_1)], [\psi(x_2)], \dots\}, \tag{111}$$

as such there is a homomorphism, and it extends to the derivative spaces.

$$\{\left[\psi^{(i)}(x_0)\right], \left[\psi^{(i)}(x_1)\right], \left[\psi^{(i)}(x_2)\right], \dots\} \tag{112}$$

for $i = 1, 2, 3, \dots$

## 4 Application of the differentiable manifolds method

To use the method of differentiable topological manifolds we first generate equations in the form of:

$$q'' = w^2 q. \tag{113}$$

We then look for

$$q^{(n+2)} = q^n = 0. \tag{114}$$

Differentiating (84) we get

$$(v')^2 + 2v'' + 2yv'v'' + yv''' = 0. \tag{115}$$

We again differentiate (116)

$$2\text{y}(v'')^2 + 3v''' + v'(4v'' + 2yv''') + yv^{(iv)} = 0. \tag{116}$$

When we use (115), we have

$$v'' = v^{(iv)} = 0. \tag{117}$$

$$3v''' + 2yv'v''' = 0. \tag{118}$$

This simplifies to

$$v'''(3 + 2yv') = 0. \tag{119}$$

This implies that

$$v''' = 0, \tag{120}$$

or

$$3 + 2yv' = 0, \tag{121}$$

$$v' = -\frac{3}{2y}. \tag{122}$$

Now, when we apply L'Hopital's Rule

$$\frac{v^{(iv)}}{v''} = \frac{v^{(v)}}{v'''}, \tag{123}$$

$$\frac{v^{(v)}}{v^{(iv)}} = \frac{v^{(''')}}{v^{('')}}. \tag{124}$$

It then becomes

$$\frac{\mathrm{d}}{\mathrm{dy}} \ln v^{(iv)} = \frac{d}{dy} \ln v''. \tag{125}$$

This gives

$$\ln v^{(iv)} = \ln v'' + \ln k. \tag{126}$$

We now end up getting

$$v^{(iv)} = kv''. \tag{127}$$

Applying (114), we then have

$$v^{(iv)} = w^2 v''. \tag{128}$$

From (129), we have

$$v''(y) = \frac{\mathrm{a}\sin(iwy+\alpha)}{iw}. \tag{129}$$

We then integrate (130)

$$v' = \frac{a}{w^2}[\sin(iwy)\cos(iwy) - \sin\alpha\cos\alpha] + b_1. \tag{130}$$

Integrating (131) again we obtain

$$v = \frac{a}{iw^3}[a\sin(iwy)\cos\alpha + \mathrm{a}\sin\alpha\cos(iwy)] + b_1 y + b_2\ . \tag{131}$$

Now substituting (131) in (122)

$$b_1 = \frac{1}{2(iw)^2}[-3(iw)^2 + 2ay\cos\alpha\cos(iwy) - 2ay\sin\alpha\sin(iwy)]\ . \tag{132}$$

Now use (121), (131) and (133) in (84) and integrate with respect to $y$.

$$\frac{3y}{4} + \frac{y^2 v}{2} - \frac{a(-2+\mathrm{I}0^2 y^2\omega^2)\cos[\phi+i\omega y]}{(i\omega)^4} + \frac{2ay\sin[\phi+i\omega y]}{(i\omega)^3} + C_1 = 0. \tag{133}$$

Now set $\sin[\phi + i\omega y] = 0$ and $\cos[\phi + i\omega y] = 1$, so that

$$\frac{3(iw)^4y + 2(iw)^4y^2v + 4(iw)^4C_1}{4(-2 + (iwy)^2)} - a[-2 + (iwy)^2] = 0. \quad (134)$$

We then find $a$ from (135)

$$a = \frac{3(iw)^4y+2(iw)^4y^2v+4(iw)^4C_1}{4(-2+(iwy)^2)}. \quad (135)$$

Integrating (134):

$$\frac{3y^2}{8} + \frac{y^3v}{6} - \frac{ay\cos(iwy+\alpha)}{(iw)^4} + \frac{6a\sin(iwy+\alpha)}{(iw)^5} - \frac{ay^2\sin(iwy+\alpha)}{(iw)^3} + yC_1 + C_2 = 0. \quad (136)$$

Now setting $\sin(iwy + \alpha) = 0$ and $\cos(iwy + \alpha) = 1$ gives

$f_1 = -45y^2 - 28y^3v - 72yC_1 - 24C_2$

and

$f_2 = 9y^4 + 4y^5v + 24y^3C_1 + 24y^2C_2$ .

We then solve for $w$:

$$w = \frac{\sqrt{2}\sqrt{f_1}}{\sqrt{f_2}}. \quad (137)$$

Substituting (138) in (136) gives

$$a = \frac{12(\mathrm{i})^4y(f_1)^2}{(f_2)^2} + \frac{8\mathrm{I}0^4y^2v(f_1)^2}{(f_2)^2} + \frac{16\mathrm{I}0^4C_1(f_1)^2}{(f_2)^2} \div 4(-2 + \frac{2\mathrm{I}0^2y^2(f_1)}{f_2}) . \quad (138)$$

Again setting $\cos(iwy + \alpha) = 1$ and $\sin[\phi + i\omega y] = 0$ in (133), thereafter substituting (138) and (139) in (133) to calculate $b_1$.

Also set $S_1 = \frac{12y(-45y^2-28y^3v-72yC_1-24C_2)^2}{(9y^4+4y^5v+24y^3C_1+24y^2C_2)^2}$,

$S_2 = \frac{8y^2v(-45y^2-28y^3v-72yC_1-24C_2)^2}{(9y^4+4y^5v+24y^3C_1+24y^2C_2)^2}$,

$S_3 = \frac{16C_1(-45y^2-28y^3v-72yC_1-24C_2)^2}{(9y^4+4y^5v+24y^3C_1+24y^2C_2)^2}$,

$S_1 + S_2 + S_3 = S_4$

and $S_5 = -2 - \frac{2y^2f_1}{f_2}$, eventually leads to

$$\mathrm{b}_1 = \frac{1}{y^2}(-y + \frac{{f_1}^2 S_4}{8{f_1}^2(S_5)} - \frac{\sqrt{(f_2{}^2 S_4}}{\left(8{f_1}^2 + S_5\right)^2} - \frac{2y^2\left(yv + y^2{f_2}^2(S_4)^2\right)}{256{f_1}^2(S_5)^2} - ({f_2}^3(S_4)/ (i24{f_1}^3(S_5)^2) + yf_2(S_4)/(8f_1(S_5) + C_3\ . \tag{139}$$

## 4.1 The solutions

From (132) and setting $b_2 = 0$, gives

$$\mathrm{v} = (\sinh[f_3]{f_2}^{3/2}(\frac{12i^4 y{f_1}^2}{{f_2}^2} + \frac{8i^4 y^2 v{f_1}^2}{{f_2}^2} + \frac{16i^4 C_1{f_1}^2}{{f_2}^2}))/(8\sqrt{2}{f_1}^{3/2}(-2 + \frac{2i^4 y^2 f_1}{f_2})) + \frac{1}{y}(-y + \frac{{f_2}^2(S_4)}{8{f_2}^2(S_5)} - \frac{\sqrt{\left(y - {f_2}^2(S_4)\right)}}{\left(8{f_1}^2(S_5)\right)^2} - 2y^2(yv + \frac{\frac{y^2{f_2}^2(S_4)^2}{256{f_1}^2(S_5)^2 - ({f_2}^3(S_4)^2}}{i24{f_1}^3(S_5)^2} + \frac{yf_1(S_4)}{8f_1(S_5)} + C_3\ , \tag{140}$$

where $f_3 = \frac{\sqrt{2}y\sqrt{f_1}}{\sqrt{9}}$.

Now setting $Z_3 = 24 - 45y^2 - 28y^3$,

$$Z_4 = -24y^2 + 9y^4 + 4y^5\ ,$$

$$Z_1 = (Z_4)^2(\frac{12y(Z_3)^2}{(Z_4)^2} + \frac{8y^2(Z_3)^2}{(Z_4)^2}),$$

$$Z_5 = -2 - \frac{2y^2(Z_3)}{Z_4}\ ,$$

$$Z_2 = (Z_3)^2(Z_5)\ ,$$

$$Z_6 = \frac{Z_1}{Z_2},$$

and $Z_7 = \frac{y^2}{256} - \frac{1}{i24} + \frac{y}{8}$, leads to

$$\mathrm{v} = \frac{1}{y}(-y + \frac{Z_6}{8} - \sqrt{(-2y^2(1 + y + (Z_7)Z_6)} + \left(y - \frac{Z_6}{8}\right)^2 + \frac{Z_1 \sinh[\frac{\sqrt{2}y\sqrt{Z_3}}{Z_4}]}{8\sqrt{2}(Z_3)^{3/2}Z_5}\ . \tag{141}$$

Now setting $h_1 = (-24t^2x^2 + 9t^4x^4 + 4t^5x^5)^2$,

$h_4 = (24 - 45t^2x^2 - 28t^3x^3)^2$,

$h_2 = \frac{12txh_4}{h_1} + \frac{8t^2x^2h_4}{h_1} - 2$,

$h_3 = \mathrm{h}_4(-2 - \frac{2t^2x^2\sqrt{h_4}}{\sqrt{h_1}}),$

$h_4 = -2t^2x^2(1 + tx + \frac{t^2x^2h_1h_2}{256h_3},$

$h_5 = \frac{\sqrt{2}tx\sqrt{\sqrt{h_4}}}{\sqrt{\sqrt{h_1}}}.$

then substituting (68) and (141) into (75) to find the value of $u$, gives the solution

$$\mathrm{u} = -\mathrm{t} - \mathrm{x} + \frac{1}{tx}(-tx + \frac{h_1h_2}{8h_3} - \tag{142}$$

$$\sqrt{h_4 - \frac{h_1h_2}{i24h_3} + \frac{txh_1h_2}{8h_3} - \left(tx - \frac{h_1h_2}{8h_3})^2\right)} + \frac{h_1h_2\sinh[h_5]}{8h_3}$$

It then follows that if we let $\nu = 1; \alpha = 1; \beta = 1; \gamma = 1; C_1 = 0; C_2 = -1; C_3 = 1$, the plot of (143) assumes the form in Figure 1.

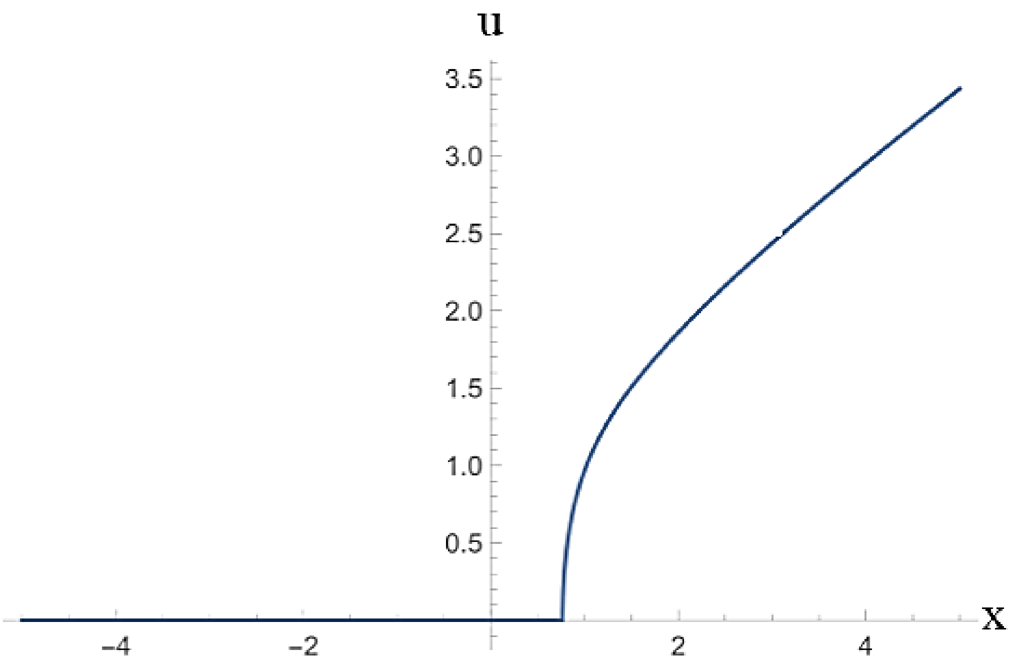


**Fig. 1:** The solution of $u$ in (143) with $t = 1; x \in [-5,5]$. It compares favourably with the one in Figure 2, obtained for the same assumed conditions, for the same equation (1), by Al-Ghafri.

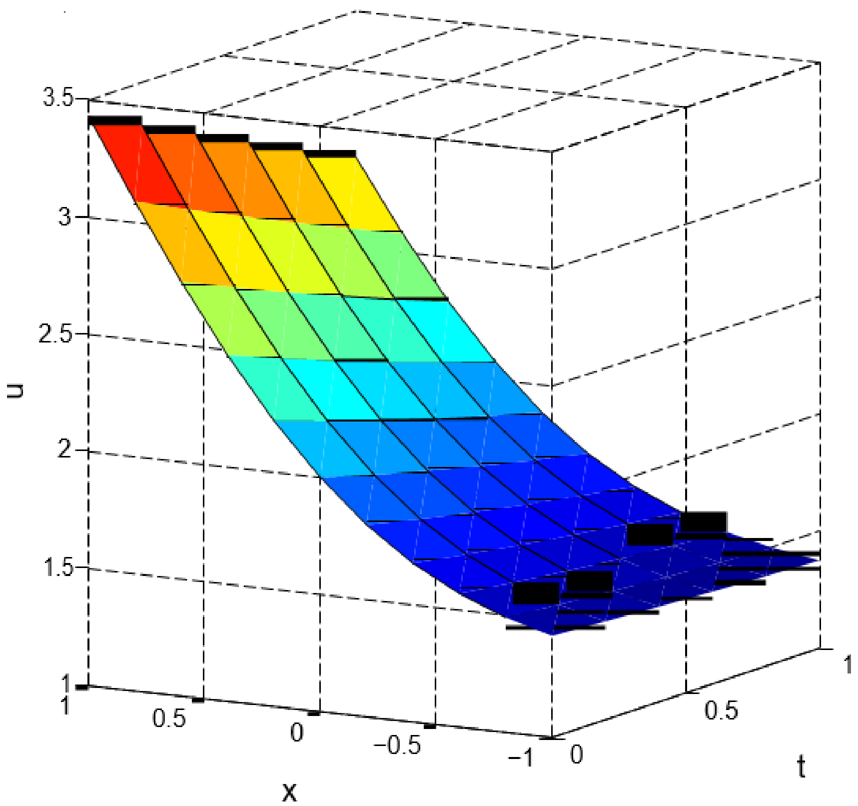


**Fig. 2:** The solution of u according to Al-Ghafri

## 5 Conclusion

When henry Thomas determined the partial differential equation (1), his objective was to understand the physical law that governs how zeolites release their ions in chemical solutions. It is unlikely that he could have succeeded, because that understanding should hinged on being able to obtain the analytical solutions of the equation, given arbitrary parameters, so far, that has not been possible, before it attracted our attention. Al-Ghafri' solution, plotted in Figure 2, is a special case solution. The next step after this, is to develop a method, aligned with the law, and protect it.

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