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Hankel Determinant of Second and Third Order for Functions with Derivative as Positive Real Part Associated with Multivalent Analytic Functions

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Abstract

The paper contains a derivation of an upper bound for second and third-order Hankel determinants for functions with derivative as positive real part, which is $p-$valent in nature.

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1. ORIGINATION

Let $\mathcal{A}_p$ with $p \in \mathbb{N} = \{1, 2, 3, \dots\}$ represent group of mappings f of the

$$f(z) = z^p + \sum_{n=0}^{\infty} a_p z^n \tag{1.1}$$

in $\mathcal{U}_d = \{z \in \mathcal{C} : |z| < 1\}$, denotes the open unit disc. Pommerenke [1] characterized the r^{th} - Hankel determinant of order n, for f (when $p = 1$) with $r, n \in \mathbb{N}$ namely

$$H_r(n) = \begin{vmatrix} a_n & a_{n+1} & \dots & a_{n+r-1} \\ a_{n+1} & a_{n+2} & \dots & a_{n+r} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+r-1} & a_{n+r} & \dots & a_{n+2r-2} \end{vmatrix} \tag{1.2}$$

The Fekete-Szegö functional is obtained for $r = 2$ and $n = 1$ in Eq. (1.2), denoted by $H_2(1)$). Further, sharp bounds to the functional $|a_2a_4 - a_3^2|$, obtained for $r = 2$ and $n = 2$ in Eq. (1.2), called as Hankel determinant of order two, given by

$$H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2a_4 - a_3^2.$$

In the recent years, the research on the estimation of an upper bound (UB) to $|H_2(2)|$ has been focused by many authors. The exact estimates of $|H_2(2)|$ for the functions namely, bounded turning, starlike and convex functions, symbolized as $\mathcal{R}, \mathcal{S}^*$ and $\mathcal{K}$, respectively, fulfilling the conditions $\mathrm{Re} f'(z) > 0$, $\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0$ and $\mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0$ in the unit disc $\mathcal{U}_d$, were proved by Janteng et al. [2, 3] and derived the bounds as $\frac{4}{9}, 1$, and $\frac{1}{8}$, respectively.

Choosing $r = 2$ and $n = p + 1$ in Eq. (1.2), we obtain Hankel determinant of second order for the $p-$valent function (see [4]), given by

$$H_2(p+1) = \begin{vmatrix} a_{p+1} & a_{p+2} \\ a_{p+2} & a_{p+3} \end{vmatrix} = a_{p+2}\, a_{p+3} - a_{p+2}^2.$$

The case $r = 3$ seems to be more tough than $r = 2$. A small number of papers have been dedicated to $H_3(1)$, named as the 3rd–order Hankel determinant obtained for $r = 3$ and $n = 1$ in Eq (1.2). Babalola [5] is the first one, who tried to estimate an UB for $|H_3(1)|$ for the classes $\mathcal{R}, \mathcal{S}^*$ and $\mathcal{K}$. As a consequence of [5], many papers containing results associated with the Hankel determinant of order 3 for specific subsets of holomorphic functions were obtained (see [6, 7, 8, 9, 5]). For our study, in this paper, we chose $H_2(p + 2)$ and $H_3(p)$, called the Hankel determinant of second and third order for the $p-$valent function, obtained for $r = 24$ and $n = p + 2$, and $r = 3$ and $n = p$ in Eq (1.2), given by

$$H_2(p+2) = \begin{vmatrix} a_{p+2} & a_{p+3} \\ a_{p+3} & a_{p+4} \end{vmatrix} = a_{p+2}\, a_{p+4} - a_{p+3}^2.$$

and

$$H_3(p) = \begin{vmatrix} a_p & a_{p+1} & a_{p+2} \\ a_{p+1} & a_{p+2} & a_{p+3} \\ a_{p+2} & a_{p+3} & a_{p+4} \end{vmatrix} \quad (a_p = 1).$$

Expanding the determinant in $H_3(P)$, we have

$$H_3(P) = a_p(a_{p+2}a_{p+4} - a_{p+3}^2) + a_{p+1}(a_{p+2}a_{p+3} - a_{p+1}a_{p+4}) \\ + a_{p+2}(a_{p+1}a_{p+3} - a_{p+2}^2). \quad (1.3)$$

Motivated with the results obtained by authors specified above, in the present paper, we estimate an UB to $H_3(p+2)$ and $|H_3(p)|$ for the bounded turning functions class, associated with $p-$valent functions, denoted by $\mathcal{R}_p$, given below.

Definition 1.1. For f Eq (1.1) to be in $\mathcal{R}_p$, if

$$\text{Re}\left\{\frac{zf'(z)}{pz^{p-1}}\right\} > 0, \quad z \in \mathcal{U}_d. \quad (1.4)$$

For the choice of $p = 1$, we obtain $\mathcal{R}_1 = \mathcal{R}$, introduced by Alexander [10] and MacGregor [11] carried a systematic study about the properties of these functions.

In proving our results, the following sharp estimates are needed, which are in the form of lemmas holds good for functions possessing positive real part. The collection $\mathcal{P}$, of all functions g, each is called as Caratheodory function [12] of the form,

$$g(z) = \sum_{t=1}^{\infty} c_t z^t, \quad (1.5)$$

holomorphic in $\mathcal{U}_d$ and Re$g(z) > 0$ for $z \in \mathcal{U}_d$.

Lemma 1.2. *([13]) If $g \in \mathcal{P}$, then the estimate $|c_i - \mu c_j c_{i-j}| \leq 2$, holds for $i, j \in \mathbb{N}$, with $i > j$ and $\mu \in [0, 1]$.*

Lemma 1.3. *([14]) If $g \in \mathcal{P}$, then the estimate $|c_i - c_j c_{i-j}| \leq 2$ holds for $i, j \in \mathbb{N}$, with $i > j$.*

Lemma 1.4. *([15]) If $g \in \mathcal{P}$ then the estimate $|c_t| \leq 2$, for each $t \in \mathbb{N}$ occurs for the function $h(z) = \dfrac{1+z}{1-z}$, $z \in \mathcal{U}_d$.*

Lemma 1.5. *([16]) If $g \in \mathcal{P}$, then $\left|c_2 c_4 - c_3^2\right| \leq 4 - \dfrac{1}{2}|c_2|^2 + \dfrac{1}{4}|c_2|^3$.*

In order to procure our results, we adopt the procedure framed through Libera and Zlotkiewicz [17].

2. IMPORTANT OUTCOMES

Theorem 2.1. *If $f \in \mathcal{R}_p$ then* $|H_3(p)| \leq \left[\dfrac{4p^2\left(6p^3+30p^2+29p+17\right)}{(p+1)\,(p+2)\,(p+3)^2\,(p+2)}\right]$.

Proof. For $f \in \mathcal{R}_p$, as per Definition 1.1

$$f'(z) = pz^{p-1}g(z),\ \ z \in \mathcal{U}_d. \tag{2.1}$$

Substitute the values for f and g in Eq (2.1), it simplifies to

$$a_{p+n} = \frac{pc_n}{p+n},\ \ n, p \in \mathbb{N}. \tag{2.2}$$

Putting the values of a_{p+n}, for $n = 1, 2, 3, 4$ from Eq (2.2) in Eq (1.3), after simplifying, we get

$$H_3(p) = p^2\left[\frac{c_2c_4}{(p+2)(p+4)} - \frac{pc_2^3}{(p+2)^2} - \frac{c_3^2}{(p+3)^2} - \frac{pc_1^2c_4}{(p+1)^2(p+4)} + \frac{2pc_1c_2c_3}{(p+1)(p+2)(p+3)}\right]. \tag{2.3}$$

On grouping the terms in Eq (2.3), in order to apply lemmas, we have

$$\begin{aligned}|H_3(p)| = p^2\Bigg[&\frac{pc_4(c_2-c_1^2)}{(p+1)^2(p+4)} - \frac{1}{(p+3)^2}c_3\left\{c_3 - \frac{6p}{(p+1)(p+2)}c_1c_2\right\} \\ &+\frac{p}{(p+2)^3}c_2(c_4-c_2^2) - \frac{2p^2}{(p+1)(p+2)(p+3)^2}c_2\left\{c_4 - c_1c_3\right\} \\ &+\frac{(p^6+6p^5+3p^4-30p^3-36p^2+24p+36)}{v}c_2c_4\Bigg],\end{aligned} \tag{2.4}$$

where $v = (p+1)^2(p+2)^3(p+3)^2(p+4)$.

By an appeal to the triangle inequality in Eq (2.4), we obtain

$$\begin{aligned}|H_3(p)| \leq p^2\Bigg[&\frac{p}{(p+1)^2(p+4)}\left|c_4\right|\left|c_2-c_1^2\right| + \frac{p}{(p+2)^3}\left|c_2\right|\left|c_4-c_2^2\right|\Bigg] \\ &+\frac{p}{(p+3^2)}\left|c_3\right|\left|c_3 - \frac{6p}{(p+1)(p+2)}c_1c_2\right| \\ &+\frac{2p^2}{(p+1)(p+2)(p+3)^2}\left|c_2\right|\left|c_4-c_1c_3\right| \\ &+\frac{(p^6+6p^5+3p^4-30p^3-36p^2+24p+36)}{v}\left|c_2\right|\left|c_4\right|.\end{aligned} \tag{2.5}$$

Upon using the lemmas given in 1.2, 1.3 and 1.4 in the above inequality, it simplifies to give the result of Theorem 2.1. Hence Theorem. □

Remark 2.2. For $p = 1$, the inequality in (2.6) coincides with the result obtained by Zaprawa [16].

Theorem 2.3. *If $f \in \mathcal{R}_p$, then* $H_3(p+2) \leq \left[\dfrac{4p^2}{(p+2)(p+4)}\right]$.

Proof. Substitute the values of a_{p+2}, a_{p+3} and a_{p+4} from Eq (2.2) in
$H_2(p+2)$, it simplifies to

$$\begin{aligned} H_2(P+2) = a_{p+2}a_{p+4} - a_{p+3}^2 &= p^2\left[\frac{c_2c_4}{(p+2)(p+4)} - \frac{c_3^2}{(p+3)^2}\right] \\ = p^2\left[\frac{c_2c_4}{(p+3)^3} - \frac{c_2c_4}{(p+3)^2} + \frac{c_2c_4}{(p+2)(p+4)} - \frac{c_3^2}{(p+3)^2}\right] \\ = p^2\left[\frac{c_2c_4 - c_3^2}{(p+3)^2} - \frac{c_2c_4}{(p+2)(p+4)(p+3)^2}\right]. \end{aligned}$$

Applying the same method as we carried in Theorem 2.1 and then using the lemmas 1.4 and 1.5, we obtain the result of Theorem 2.3. □

Remark 2.4. For $p = 1$, the inequality under Theorem 2.3, coincides with that of Zaprawa [6].

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