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φ-Superharmonic Functions on Infinite Random Walks

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Abstract

In a random walk $\{X, P : p(x, y)\}$ with a countable infinite state space X and P the matrix of transition probabilities, a basic problem is to determine whether the walk is recurrent or transient. Among different characterisations to solve this problem, one method uses the Laplace operator Δ. Now the Laplacian Δ (in the sense of distributions) plays an important role in classical potential theory starting with the study of subharmonic functions. In this paper we develop a parallel theory in the aspect of the random walk $\{X, P\}$, using an operator Δ_φ on X, which can be considered as a generalized version of the discrete Schrödinger operator. In this framework, for a function $\varphi(x) \geq 0$, we develop on X a theory of φ-superharmonic functions leading to φ-Dirichlet problem, φ-recurrence and φ-transience.

Keywords : recurrent and transient walk, φ-superharmonic functions, A_ϕ operator.

2020 AMS subject classification: 60J45, 31C20.

1. INTRODUCTION

In a random walk $\{X, P\}$ with a countable infinite state space X and $P = \{p(x, y)\}$ the matrix of transition probabilities $p(x, y)$, let $\varphi(x)$ be a density function on X. For a real-valued function $u(x)$ on X, $\varphi(x)u(x)$ is the weighted value at any state $x \in X$. The average function of $u(x)$ is defined as $Au(x) = \sum_{y \sim x} p(x, y)u(y)$, where $y \sim x$ denotes that y is a neighbour of x; the value $Au(x)$ is well defined, since we assume that any state x has only a finite number of neighbours y. We shall be more interested in the operator $A_\varphi u(x) = Au(x) - \varphi(x)u(x)$.

Remark that, when $\varphi \equiv 1$, A_φ is the laplace operator Δ on X; when $\varphi \not\equiv 0$ and $\varphi(x) > 1$, A_φ is a generalised version of the discrete Schrödinger operator on X; when $\varphi \not\equiv 0$, $\varphi(x) < 1$, A_φ represents a generalised version of the discrete Helmholtz operator on X. In this paper, we consider the case when $\varphi \geq 0$ only.
To study the effect of the operator A_φ on the real-valued functions on X, we adopt potential-theoretic methods on infinite graphs. Using the positive density function $\varphi(x)$, we define φ-harmonic,φ-superharmonic and φ-subharmonic functions and we try to determine the relationship between weighted value and average value of a real valued function on a random walk X. Some basic properties of φ-superharmonic functions are derived which also includes poisson modification of φ-superharmonic function. Greatest φ-harmonic minorant and Riesz-representation of positive φ-superharmonic functions are determined. In section 4, solution of Dirichlet problem is obtained by considering a connected finite subset of X. Potential theoretic concepts like Harnack property and domination principle are discussed. In section 5, relation between laplace operator-Δ and A_φ-operator is established.

2. PRELIMINARIES

Let $\{X, P\}$ be a random walk with a countable infinite number of states X and $P = \{p(x, y)\}$ is the probability transition matrix, where $p(x, y)$ denotes the transition probability from state x to state y. We assume $\{X, P\}$ is connected (i.e, for any two distinct states there exists a path connecting them), locally finite (every state in X has finite neighbours) and without self loops [1]. As usual, we shall take X as an infinite graph by defining $[x, y]$ as an edge iff $p(x, y) > 0$. We say two states x and y are neighbours if there exists an edge between them and it is denoted by $x \sim y$ and $\sum_{y\sim x} p(x, y) = 1$ for every $x \in X$ with $p(x, y) \geq 0$ such that $p(x, y) > 0$ if and only if $x \sim y$; $p(x, y) = 0$ if x and y are not neighbours [2].

Suppose F is a subset of an infinite random walk X, we say x is an interior vertex of F if and only if x and all its neighbours are in F. The set of all interior points of F is denoted by $\mathring{F}$ and $\partial F = F\backslash\mathring{F}$, where ∂F is referred to as the boundary of F [3] and [4]. For a positive density function $\varphi(x) > 0$ on X, we say that $\varphi(x)u(x)$ is the weighted value of $u(x)$ at x and $Au(x) = \sum_{y\sim x} p(x, y)u(y)$ is the average value of $u(x)$ at x. write $A_\varphi u(x) = Au(x) - \varphi(x)u(x)$

Definition: Let u be a real valued function defined on a subset F of X. Then u is said to be **φ-harmonic** on F if $A_\varphi u(x) = 0$ at every state $x \in \mathring{F}$ and u is said to be **φ-superharmonic** on F and **φ-subharmonic** on F if and only if $A_\varphi u(x) \leq 0$ and

$A_\varphi u(x) \geq 0$ at every state $x \in \overset{\circ}{F}$ respectively.

If s is φ-superharmonic and v is φ-subharmonic functions on F such that $s(x) \geq v(x)$, then h is a φ-harmonic function on F such that $s(x) \geq h(x) \geq v(x)$ and suppose there is another such φ-harmonic function h^1 between $s(x)$ and $v(x)$, then $h(x) \geq h^1(x)$ on F. Here h is called the **greatest φ-harmonic minorant (g.h.m.)** of s on F. If the greatest φ-harmonic minorant (g.h.m.) of a non-negative φ-superharmonic function p on F is 0 then p is called φ-potential.

A random walk is considered to be recurrent if the walk starting at a state z returns to z infinitely often; where as, the walk starting at a state z returning to state z only finitely often with probability one is said to be a transient walk [5].

3. SOME BASIC PROPERTIES OF φ- SUPERHARMONIC FUNCTION

Property 3.1. *If u_1, u_2 are φ-superharmonic on a set F, then $inf(u_1, u_2)$ is also φ-superharmonic on F.*

Proof. Let $u = inf(u_1, u_2)$
At a state z, suppose $u(z) = u_1(z)$
Then,

$$Au(z) = \sum_{y \sim z} p(z, y)u(y)$$

$$\leq \sum_{y \sim z} p(z, y)u_1(y) \leq \varphi(z)u_1(z)$$

$$= \varphi(z)u(z)$$

This implies $inf(u_1, u_2)$ is φ-superharmonic on F.

□

Property 3.2. *Let $u(x) > -\infty$ be a function on X such that $Au(x) \leq \varphi(x)u(x)$ for any state x in X. If $u(x)$ is real valued at some state z, then $u(x)$ is real valued on X, hence φ-superharmonic on X.*

Proof. Since $\varphi(z)u(z) \geq \sum_{y \sim z} p(z, y)u(y)$, then $u(y)$ is real valued for every $y \sim z$. Since X is connected, this implies that $u(x)$ is a real valued function on X, hence u is φ-superharmonic on X. □

Property 3.3. *Let $\{u_n\}$ be a sequence of φ-superharmonic functions on F and if $\lim_{n\to\infty} u_n(x) = u(x)$ is finite at every vertex in F, then u is φ-superharmonic on F [6].*

Proof. For $x \in \mathring{F}$,

$$A_\varphi u_n(x) \leq 0$$

$$\sum_{y\sim x} p(x,y)u_n(y) - \varphi(x)u_n(x) \leq 0$$

$$\sum_{y\sim x} p(x,y)u_n(y) \leq \varphi(x)u_n(x)$$

Taking limit $n \to \infty$ on both the sides,

$$\sum_{y\sim x} p(x,y) \lim_{n\to\infty} u_n(y) \leq \varphi(x) \lim_{n\to\infty} u_n(x), \; since\, X\, is\, locally\, finite,\, the\, sum\, is\, finite.$$

$$\sum_{y\sim x} p(x,y)u(y) - \varphi(x)u(x) \leq 0$$

Implies u is φ-superharmonic function on F. □

Property 3.4. *For a real-valued function $f(x)$ on X. Let $\mathscr{F}$ be the family of all φ-superharmonic functions $s(x)$ on X such that $s(x) \geq f(x)$. If $\mathscr{F}$ is non-empty then $u(x) = \inf_{s\in\mathscr{F}} s(x)$ is φ superharmonic on X.*

Proof. If s_1, s_2 are in $\mathscr{F}$, then $inf\{s_1, s_2\}$ also is in $\mathscr{F}$. Thus $\mathscr{F}$ is a lower-directed family; moreover X contains only a countable number of states. Hence there exist a decreasing sequence $\{s_n\}$ in $\mathscr{F}$, such that $inf_{s\in\mathscr{F}} s(x) = lim s_n(x)$ which is a φ superharmonic function on X.

□

Property 3.5. *Poisson Modification: Let $u(x)$ be a real-valued function on $\{X, P\}$, that is φ-superharmonic at a state z. Then there exists a function $u_z(x)$ on X such that $u_z(x) \leq u(x)$ on X; $u_z(x) = u(x)$ if $z \neq x$; and $u_z(x)$ is φ-harmonic at z.*

Proof. At state z, $u(x)$ is a φ-superharmonic function.
$\Rightarrow A_\varphi u(z) \leq 0$ at $z \in F$

$$\sum_{x\sim z} p(z,x)u(x) - \varphi(z)u(z) \leq 0$$

$$Au(z) - \varphi(z)u(z) \leq 0$$

$Au(z) \leq \varphi(z)u(z)$............(1)
Define,

$$u_z(x) = \begin{cases} \frac{Au(z)}{\varphi(z)} & if \quad x = z \\ u(x) & if \quad x \neq z \end{cases} \qquad on \quad X.$$

Then, (i) $u_z(x) \leq u(x)$ on X
(ii) $A_\varphi u_z(z) = 0$
For,

$$A_\varphi u_z(z) \Rightarrow \sum_{y \sim z} p(z,y)u_z(y) - \varphi(z)u_z(z)$$

If $x = z$, then

$$\Rightarrow \sum_{y \sim z} p(z,y)u(z) - \varphi(z)\frac{Au(z)}{\varphi(z)}$$

$$\Rightarrow Au(z) - Au(z) = 0$$

$$\Rightarrow A_\varphi u_z(z) = 0$$

$$\Rightarrow u_z(z) \quad is \quad \varphi - harmonic.$$

If $x \neq z$, $u_z(x) = u(x)$
If $x = z$,

$$u_z(z) = \frac{Au(z)}{\varphi(z)} \leq u(z)$$

$$u_z(z) \leq u(z)$$

Hence, $u_z(x) \leq u(x)$.

□

Property 3.6. *Greatest φ-harmonic minorant: Suppose $u(x) \geq v(x)$ on X where $u(x)$ is φ-superharmonic and $v(x)$ is φ-subharmonic on a subset F. Then there exists a φ-harmonic function $h(x)$ on F, $u(x) \geq h(x) \geq v(x)$ and if h_1 is any other φ-harmonic function between $u(x)$ and $v(x)$, then $h(x) \geq h_1(x)$ on F [7].*

Proof. Consider $\mathcal{F}$ to be the family of all φ-subharmonic functions $s(x)$ on F, such that $s(x) \leq u(x)$. We know that X is countable and $\mathcal{F}$ is an upper- directed family of φ-subharmonic functions. Consequently, there exists an increasing sequence $\{s_n(x)\}$ of functions in $\mathcal{F}$ such that $sup_{\mathcal{F}} s(x) = sup s_n(x) = h(x)$ which is a φ-subharmonic function on F and $h(x) \leq u(x)$. Actually, $h(x)$ is a φ-harmonic function. For, if $z \in \mathring{F}$, then the Poisson modification $h_z(x)$ is a φ-subharmonic function on F which

also belong to $\mathcal{F}$ so that $h_z(x) \geq h(x)$; but by construction $h(x)$ is the supremum. Hence $h_z(x) = h(x)$ which leads to the conclusion that $h(x)$ is φ-harmonic on F.
For the maximality of the function $h(x)$, note that if $h_1(x)$ is another such φ-harmonic minorant of $u(x)$, then $h_1(x) \in \mathcal{F}$ so that $h_1(x) \leq h(x)$. □

Property 3.7. *Riesz representation: Suppose $u(x)$ is a positive φ-superharmonic function on F. Then $u(x) = p(x) + h(x)$, where $p(x)$ is a non-negative φ-potential on F and $h(x)$ is a non-negative φ-harmonic function on F. This decomposition is unique.*

Proof. Let $h(x)$ be the greatest φ-harmonic minorant of $u(x)$ on F. Then $p(x) = u(x) - h(x)$ is a φ-potential on F, hence the decomposition. For the uniqueness, suppose $u(x) = p^*(x) + h^*(x)$ is another such decomposition, then $p(x) \geq h^*(x) - h(x)$ should imply that $h^*(x) - h(x) \leq 0$; similarly we prove that $h(x) \leq h^*(x)$. Then follows the uniqueness of decomposition. □

4. DIRICHLET PROBLEM

Theorem 4.1. *Dirichlet Problem: Let F be a connected finite subset of X on which a positive φ-superharmonic function exists. If $f(a)$ is a real-valued function on ∂F, then there exists a φ-harmonic function $h(x)$ on F such that $h(a) = f(a)$ for every $a \in \partial F$.*

Proof. Let $\xi(x) > 0$ be a φ-superharmonic function on X. Since F is a finite set, we can assume that $\xi(x) \geq 1$ on F. For a state z in ∂F, let $\delta_z(x)$ be the Dirac function on F. Let us consider a function $V(x)$ on F such that $V(x) = \xi(x)$ if $x \in \mathring{F}$ and $V(a) = \delta_z(a)$ if $a \in \partial F$. Note that $V(x)$ is a φ-superharmonic function on F. Let $\mathscr{F}$ be a family of all superharmonic functions on $s(x)$ on F such that $s(x) \geq V(x)$ on F.

Denote by $P(z, x) = inf_{\mathscr{F}} s(x)$; Then $P(z, x)$ is φ-harmonic on F, $P(z, x) = 1$ if $x = z$ and $P(z, x) = 0$ on $\partial F/\{z\}$.

Define now $h(x) = \sum_{a \in F} P(a, x) f(a)$ for $x \in F$.

By minimum principle for φ-harmonic functions on finite subsets, the uniqueness of the solution is proved. □

Remark 4.1. *In random walks, when $\varphi \equiv 1$, $P(z, x)$ represents the probability of a walker starting at the state $x \in F$ reaches the state $z \in \partial F$ before reaching any other state in ∂F.*

Proof. Let $\rho(a)$ be the probability of the walker starting at the state $a \in F$ and reaching $z \in \partial F$ before reaching any other state in ∂F. Then $\rho(z) = 1$ and $\rho(a) = 0$ for any $a \in \partial F/\{z\}$; if $x \in \mathring{F}$ then $P(x) = \sum_{y\sim x} p(x,y)\rho(y)$. Thus $\rho(x)$ is a φ-harmonic function on F when $\varphi \equiv 1$. By the uniqueness of the Dirichlet solution, $\rho(x) = p(z,x)$ □

Theorem 4.2. *Harnack property: Let x and y be two states on a subset F of X, there exists a constant $\alpha > 0$, such that $u(y) \leq \alpha u(x)$ for any non negative φ- superharmonic function u on F.*

Proof. Given a φ-superharmonic function u, $A_\varphi\, u(x) \leq 0$.

$$\Rightarrow \sum_{y\sim x} p(x,y)u(y) - \varphi(x)u(x) \leq 0$$

$$\sum_{y\sim x} p(x,y)u(y) \leq \varphi(x)u(x)$$

Let x, y be two states on X. Then a path $\{x, x_1, x_2, x_3, ..., x_n, y\}$ between x and y exists.

$$p(x,x_1)u(x_1) \leq \sum_{y\sim x} p(x,y)u(y) \leq \varphi(x)u(x)$$

$$p(x,x_1)u(x_1)_{\leq}\varphi(x)u(x)$$

$u(x_1) \leq \frac{\varphi(x)}{p(x,x_1)}u(x)$..............(1)

$$A_\varphi u(x_1) \leq 0$$

$$\sum_{y\sim x_1} p(x_1,y)u(y) \leq \varphi(x_1)u(x_1)$$

$$p(x_1,x_2)u(x_2) \leq \sum_{y\sim x_1} p(x_1,y)u(y) \leq \varphi(x_1)u(x_1)$$

$$u(x_2) \leq \frac{\varphi(x_1)\varphi(x)}{p(x,x_1)p(x_1,x_2)}u(x)$$

Similarly, we get

$$p(x_2,x_3)u(x_3) \leq \varphi(x_2)u(x_2)$$

$$u(x_3) \leq \frac{\varphi(x_2)\varphi(x_1)\varphi(x)}{p(x,x_1)p(x_1,x_2)p(x_2,x_3)}u(x)$$

Proceeding this way,

$$u(y) \leq \frac{\varphi(x_n)\varphi(x_{n-1})........\varphi(x)}{p(x_n,y)p(x_{n-1},x_n).........p(x,x_1)}u(x)$$

$$\Rightarrow u(y) \leq \alpha u(x)$$

□

Note: The same can be deduced for two disjoint finite subsets of X.

Theorem 4.3. *If there exists a positive φ-superharmonic function on X, then there exists a positive φ-harmonic function on X.*

Proof. For a positive φ-superharmonic function s on X and e is a fixed vertex on X.
Let $\{k_n\}_{n\geq 1}$ be an finite increasing sequence then,
$e \in \mathring{k}_n \subset k_n \subset \mathring{k}_{n+1} \subset k_{n+1}$ and $X = \cup k_n$.
Consider a function s_n on X.
such that,

$$s_n(x) = \begin{cases} u_n(x) & on \quad k_n \\ s(x) & on \quad X \backslash k_n \end{cases}$$

$u_n(x)$ is the dirichlet solution of k_n with boundary $s(x)$.

$$h_n(x) = \frac{s_n(x)}{s_n(e)}$$

h_n is φ-superharmonic on X.

Consequently, $A_\varphi h_n(x) = 0$ for $x \in \mathring{k}_n$ and $h_n(e) = 1$

By Harnack property, For any $y \in X$, $\alpha(y) > 0$ is a constant. Hence $u(y) \leq \alpha(y)u(e)$ for a positive φ-superharmonic function.
Certainly, for any $x \in X$, $h_n(x) \leq \alpha(x)h_n(e)$ and $h_n(x) \leq \alpha(x)$, that is $\{h_n(x)\}$ is a sequence of real numbers which is bounded. For X being a countable set, let us extract a subsequence $\{h_n'\}$ from $\{h_n\}$ so that for each $x \in X$,

$h(x) = \lim_{n\to\infty} h_n^{'}(x)$ exists .
For a finite set F in X, an integer m can be obtained such that $h_n^{'}$ is φ-harmonic at each vertex of F if $n \geq m$. Hence, h is φ-harmonic at each vertex of F. For an arbitrary finite set F, $h(x)$ is a non-negative φ-harmonic function on X. Since $h(z) = 1$, by the Minimum Principle, $h > 0$ on X. Hence proving the existence of a positive φ-harmonic function on X.

□

Theorem 4.4. *Domination principle: Let p be a φ-potential with φ-harmonic support U. If s is a non-negative φ-superharmonic function on X such that $s \geq p$ on U. Then $s \geq p$ on X.*

Proof. Suppose p is a φ-potential with φ-harmonic support U and s be a φ-superharmonic function on X such that $s \geq p$ on U.
Let $u = inf(s, p)$, then $A_\varphi u(x) \leq 0$.
$u \leq p$ on X (since p is φ-potential)
$u = p$ on U (since U harmonic support of S in X)
Suppose $v = p - u$ on X. Then for $a \in U$,

$$A_\varphi v(a) = \sum_{y\sim a} p(a, y)v(y) - \varphi(a)u(a)$$

$$= \sum_{y\neq z\sim a} p(a, y)v(y) + p(a, z)v(z) - \varphi(a)v(a)$$

$$\geq \sum_{y\neq z\sim a} p(a, y)v(y) + p(a, z)v(z) - \varphi(a)v(a)$$

$$\geq \sum_{y\sim a} p(a, y)v(y) - \varphi(a)v(a)$$

$$\geq 0$$

$$A_\varphi v(a) \geq 0$$

For $x \in X\backslash U$,

$$A_\varphi v(x) = A_\varphi p(x) - A_\varphi u(x) = 0 - A_\varphi u(x) \geq 0$$

$$\Rightarrow A_\varphi v(x) \geq 0$$

$\Rightarrow v$ is φ-subharmonic on X and $v \leq p$ on X.
Thus $v \leq 0$ on X, such that $p \leq u$ but, $u \leq p \Rightarrow p = u$.

Hence, $s \geq p$ on X [1]. □

5. WHEN $\varphi(X) \geq A\xi(X)/\xi(X)$ FOR SOME REAL VALUED FUNCTION $\xi(X) > 0$

When $\varphi(x) \geq A\xi(x)/\xi(x)$ for some real valued function $\xi(x) > 0$. In the following, assume that a function $\xi(x) > 0$ on X exists, such that $\varphi(x) \geq \frac{A\xi(x)}{\xi(x)}$.
Remark that if $\xi(x) \geq 1$ for all x in X, then this condition is satisfied with $\xi(x) = 1$; also, since $\xi(x)$ is a φ-superharmonic function on X by the assumption, from property (7) it follows that there is a function $\mu(x) > 0$ such that $\varphi(x) = A\mu(x)\backslash\mu(x)$.
Let $t(x, y) = p(x, y)\mu(y)$ for any pair of states x, y. Then $\{X, t(x, y)\}$ becomes an infinite network in the sense of Lecture Notes [3]. The Laplace operator Δ for this network is given by $\Delta u(x) = \sum_{y\sim x} t(x, y)[u(y) - u(x)]$.

Lemma 5.1. *For any real-valued function $u(x)$ on X, $A_\varphi u(x) = \Delta[\frac{u(x)}{\mu(x)}]$.*

Proof. $A_\varphi u(x) = Au(x) - \varphi(x)u(x)$

$$= Au(x) - \frac{A\mu(x)}{\mu(x)}u(x)$$

$$= \sum_{y\sim x} p(x, y)u(y) - \sum_{y\sim x} p(x, y)\mu(y)\frac{u(x)}{\mu(x)}$$

$$= \sum_{y\sim x} t(x, y)\frac{u(y)}{\mu(y)} - \sum_{y\sim x} t(x, y)\frac{u(x)}{\mu(x)}$$

$$= \sum_{y\sim x} t(x, y)(\frac{u(y)}{\mu(y)} - \frac{u(x)}{\mu(x)})$$

$$= \Delta[\frac{u(x)}{\mu(x)}]$$

. □

Consequence: From the above lemma, a real valued function $u(x)$ on X is φ-harmonic (respectively φ-superharmonic) at a state x iff $\frac{u(x)}{\mu(x)}$ is Δ-harmonic (respectively Δ-superharmonic) at x in the network $\{X, t(x, y)\}$.

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