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Preconditioned Fractional MEG Iterative Method for Solving the Time Fractional Advection-Diffusion Equation

Abdulkafi Mohammed Saeed[1*]

[1] *Department of Mathematics, College of Science,*
Qassim University, Buraydah, Kingdom of Saudi Arabia

Abstract

In recent decades, grouping strategies have been proven to possess characteristics that are able to reduce the spectral radius of the generated matrix resulting from the finite difference discretization of the partial differential equations, and therefore increase the convergence rates of the iterative algorithms. In this paper the development and formulation of new accelerated fractional explicit group iterative method which is called Preconditioned Fractional modified explicit group (PFMEG) for solving the 2D- time fractional advection-diffusion equation will be presented. Some new Fundamental theorems are established which are then analyze the convergence properties of the proposed method. Numerical experiments will also be conducted to confirm the agreement between the theoretical and the experimental results.

Keywords: Time fractional advection-diffusion equation, Crank–Nicolson scheme, Preconditioning method.

1. Introduction

Improved techniques using explicit group methods derived from the standard and skewed (rotated) finite difference operators have been developed over the last few years in solving the linear systems that arise from the discretization of the fractional partial differential equation [1-7]. Several preconditioned iterative methods reported in the literature have been used for improving the convergence rate of explicit group methods derived from the standard and rotated finite difference operators [8-23].

The time fractional advection-diffusion equation plays an important role in describing transport dynamics in complex systems which are governed by anomalous diffusion and non-exponential relaxation patterns [1]. Therefore, in this study, the formulation of new group iterative methods is presented in solving the following 2D time fractional advection diffusion equation,

$$\frac{\partial^\alpha u}{\partial t^\alpha} = a_x \frac{\partial^2 u}{\partial x^2} + a_y \frac{\partial^2 u}{\partial y^2} - b_x \frac{\partial u}{\partial x} - b_y \frac{\partial u}{\partial y} + f(x,y,t), \tag{1}$$

where $0 < \alpha < 1$, a_x, a_y, b_x, b_y are positive constants and $f(x,y,t)$ is nonhomogeneous term subject to the following initial and dirichlet boundary conditions,

$$\begin{aligned} &u(x,y,0) = g(x,y), \\ &u(0,y,t) = g_1(y,t), \quad u(1,y,t) = g_2(y,t) \\ &u(x,0,t) = g_3(x,t), \quad u(x,1,t) = g_4(x,t) \end{aligned} \tag{2}$$

The Caputo fractional derivative, D^α are of the order-α is expressed as follows [12]:

$$D^\alpha = \frac{1}{\Gamma(m-\alpha)} \int_0^x \frac{f^m(t)}{(x-t)^{\alpha-m+1}}\, dt\ ,\ \alpha > 0,\ x > 0,\ m \in \mathbb{N}, \tag{3}$$

where $\Gamma(.)$ is the Euler Gamma function and $m-1<\alpha<m$.

The domain is assumed to be uniform in both *x* and y directions with $h > 0$ be the space step and $k > 0$. The grid points in the space are denoted $x_i = i\,h,\ y_j = j\,h,\ \{i,j = 0,1,\ldots,n\}$ and the grid points for time are designated $t_k = k\tau,\ k = 0,1,\ldots,l$ to space and time for the positive integers n and l are respectively represented by $h = \frac{1}{n}$ and $\tau = \frac{T}{l}$. Discretization with regard to time fractional with utilization of second-order Crank-Nicolson finite difference approximations could obtained for equation (1) at the point of (x_i, y_j, t_k). By taking the average of the central difference approximations to the left side of equation (1) at the points (i,j,k) and $(i,j,k+1)$, the caputo time fractional approximation (3) can be transformed to the following form [6]

$$\frac{\partial^\alpha u(x_i,y_j,t_{k+1})}{\partial t^\alpha} = w_1 u^k + \sum_{s=1}^{k-1}[w_{k-s+1} - w_{k-s}]u_{i,j}^s - w_k u_{i,j}^0 + \sigma \frac{(u_{i,j}^{k+1} - u_{i,j}^k)}{2^{1-\alpha}} + 0(\tau^{2-\alpha}), \tag{4}$$

where $\sigma = \frac{1}{\tau^\alpha \Gamma(2-\alpha)}$, $w_s = \sigma\{(s+\frac{1}{2})^{1-\alpha} - (s-\frac{1}{2})^{1-\alpha}\}$

Using (4) in combination with the second-order Crank–Nicolson difference scheme for the right side of (1) will result in the following fractional standard point (FSP) formula

$$w_1 u^k + \sum_{s=1}^{k-1}[(w_{k-s+1} - w_{k-s})u^s_{i,j} - w_k u^0_{i,j} + \sigma\frac{(u^{k+1}_{i,j} - u^k_{i,j})}{2^{1-\alpha}} = \frac{a_x}{2}(\frac{u^{k+1}_{i-1,j} - 2u^{k+1}_{i,j} + u^{k+1}_{i+1,j}}{h^2} + \frac{u^k_{i-1,j} - 2u^k_{i,j} + u^k_{i+1,j}}{h^2}) \tag{5}$$
$$+\frac{a_y}{2}(\frac{u^{k+1}_{i,j-1} - 2u^{k+1}_{i,j} + u^{k+1}_{i,j+1}}{h^2} + \frac{u^k_{i,j-1} - 2u^k_{i,j} + u^k_{i,j+1}}{h^2}) - \frac{b_x}{2}(\frac{u^{k+1}_{i+1,j} - u^{k+1}_{i-1,j}}{2h} + \frac{u^k_{i+1,j} - u^k_{i-1,j}}{2h}) - \frac{b_y}{2}(\frac{u^{k+1}_{i,j+1} - u^{k+1}_{i,j-1}}{2h}$$
$$+\frac{u^k_{i,j+1} - u^k_{i,j-1}}{2h}) + f^{k+\frac{1}{2}}_{i,j} + O[\tau^{2-\alpha} + (\Delta x)^2 + (\Delta y)^2]$$

Equation (5) can be simplified to become as follows

$$(1 + s_x + s_y)u^{k+1}_{i,j} = (\frac{s_x}{2} + \frac{c_x}{4})u^{k+1}_{i-1,j} + (\frac{s_x}{2} - \frac{c_x}{4})u^{k+1}_{i+1,j} + (\frac{s_y}{2} + \frac{c_y}{4})u^{k+1}_{i,j-1} + (\frac{s_y}{2} - \frac{c_y}{4})u^{k+1}_{i,j+1} \tag{6}$$
$$+(1 - 2^{1-\alpha}w^*_1 - s_x - s_y)u^k_{i,j} + (\frac{s_x}{2} + \frac{c_x}{4})u^k_{i-1,j}(\frac{s_x}{2} - \frac{c_x}{4})u^k_{i+1,j} + (\frac{s_y}{2} + \frac{c_y}{4})u^k_{i,j-1}$$
$$+(\frac{s_y}{2} - \frac{c_y}{4})u^k_{i,j+1} + 2^{1-\alpha}w^*_k u^0_{i,j} + 2^{1-\alpha}\sum_{s=1}^{k-1}[(w^*_{k-s} - w^*_{k-s+1})u^s_{i,j} + m_0 f^{k+\frac{1}{2}}_{i,j}$$

where $m_0 = 2^{1-\alpha}t^\alpha\Gamma(2-\alpha)$, $s_x = \frac{a_x m_0}{h^2}, s_y = \frac{a_y m_0}{h^2}, c_x = \frac{b_x m_0}{h}, c_y = \frac{b_y m_0}{h}, w^*_s = [(s+\frac{1}{2})^{1-\alpha} - (s-\frac{1}{2})^{1-\alpha}].$

The main aim of this paper is to formulate a new fractional group iterative method known as Preconditioned Fractional Modified Explicit Decoupled Group (PFMEG) method in solving equations (1) and (2). The paper is organized in five sections: Section 2 describes the formulation of the fractional Explicit Group Method. In Section 3, the proposed accelerated version PFMEG approximation method will be given. Section 4 presented the stability and convergence analysis. In Section 5, the numerical results are presented in order to show the efficiency of the proposed method. Finally, the conclusion is given in Section 6.

2. Formulation of Fractional Modified Explicit Group (FMEG) Method

First, we have to formulate the fractional explicit group (FEG) method which can be obtained by applying equation (6) to any group of four points in the solution domain to generate a 4 × 4 system of equation as follows

$$\begin{pmatrix} d & -a_1 & 0 & -b_1 \\ -a_2 & d & -b_1 & 0 \\ 0 & -b_2 & d & -a_2 \\ -b_2 & 0 & -a_1 & d \end{pmatrix}\begin{pmatrix} u_{i,j} \\ u_{i+1,j} \\ u_{i+1,j+1} \\ u_{i,j+1} \end{pmatrix} = \begin{pmatrix} rhs_{i,j} \\ rhs_{i+1,j} \\ rhs_{i+1,j+1} \\ rhs_{i,j+1} \end{pmatrix}, \tag{7}$$

where $d = 1 + s_x + s_y$, $a_1 = (\frac{s_x}{2} - \frac{c_x}{4})$, $a_2 = (\frac{s_x}{2} + \frac{c_x}{4}), b_1 = (\frac{s_y}{2} - \frac{c_y}{4}), b_2 = (\frac{s_y}{2} + \frac{c_y}{4}),$

$$rhs_{i,j} = a_2[u^{k+1}_{i-1,j} + u^k_{i-1,j}] + b_2[u^{k+1}_{i,j-1} + u^k_{i,j-1}] + a_1 u^k_{i+1,j} + b_1 u^k_{i,j+1} + 2^{1-\alpha}w^*_k u^0_{i,j}$$
$$+ Ru^k_{i,j} + \sum_{s=1}^{k-1}2^{1-\alpha}[w^*_{k-s} - w^*_{k-s+1}]u^s_{i,j} + m_0 f^{k+\frac{1}{2}}_{i,j}$$

$$rhs_{i+1,j} = a_1[u_{i+2,j}^{k+1} + u_{i+2,j}^{k}] + a_2\, u_{i,j}^{k} + b_1 u_{i+1,j+1}^{k} + b_2[u_{i+1,j-1}^{k+1} + u_{i+1,j-1}^{k}]$$
$$+\sum_{s=1}^{k-1} 2^{1-\alpha}[w_{k-s}^{*} - w_{k-s+1}^{*}]u_{i+1,j}^{s} + 2^{1-\alpha}\, w_k^{*}\, u_{i+1,j}^{0} + Ru_{i+1,j}^{k} + m_0\, f_{i,j}^{k+\frac{1}{2}},$$

$$rhs_{i+1,j+1} = a_1[u_{i+2,j+1}^{k+1} + u_{i+2,j+1}^{k}] + b_1[u_{i+1,j+2}^{k+1} + u_{i+1,j+2}^{k}] + b_2 u_{i+1,j}^{k} + a_2 u_{i,j+1}^{k} + 2^{1-\alpha}\, w_k^{*}\, u_{i+1,j+1}^{0}$$
$$+ R u_{i+1,j+1}^{k} + \sum_{s=1}^{k-1} 2^{1-\alpha}[w_{k-s}^{*} - w_{k-s+1}^{*}]u_{i+1,j+1}^{s} + m_0\, f_{i+1,j+1}^{k+\frac{1}{2}},$$

$$rhs_{i,j+1} = a_2[u_{i-1,j+1}^{k+1} + u_{i-1,j+1}^{k}] + b_1[u_{i,j+2}^{k+1} + u_{i,j+2}^{k}] + a_1\, u_{i+1,j+1}^{k} + b_2\, u_{i,j}^{k} + 2^{1-\alpha}\, w_k^{*} u_{i,j+1}^{0}$$
$$+ 2^{1-\alpha}\sum_{s=1}^{k-1}[w_{k-s}^{*} - w_{k-s+1}^{*}]u_{i,j+1}^{s} + R\, u_{i,j+1}^{k} + m_0\, f_{i,j+1}^{k+\frac{1}{2}},$$

A four point FEG equation can be generated through a reversal of the matrix above

$$\begin{pmatrix} u_{i,j} \\ u_{i+1,j} \\ u_{i+1,j+1} \\ u_{i,j+1} \end{pmatrix} = \frac{1}{r}\begin{pmatrix} r_{11} & r_{12} & r_{13} & r_{14} \\ r_{21} & r_{11} & r_{14} & r_{24} \\ r_{31} & r_{32} & r_{11} & r_{21} \\ r_{32} & r_{42} & r_{12} & r_{11} \end{pmatrix}\begin{pmatrix} rhs_{i,j} \\ rhs_{i+1,j} \\ rhs_{i+1,j+1} \\ rhs_{i,j+1} \end{pmatrix}, \qquad (8)$$

where

$$r = \frac{1}{256}\{c_x^4 + c_y^4 + 8c_y^2(4 + 5s_x^2 + 8s_y + 3s_y^2 + 8s_x(1+s_y))] + 16[9s_x^4 + 48s_x^3(1+s_y) + (4 + 8s_y + 3s_y^2)^2$$
$$+ 2s_x^2(44 + 88\,s_y + 39s_y^2) + 16s_x(4 + 12s_y + 11s_y^2 + 3s_y^3)] + c_x^2[-2c_y^2 + 8(4 + 3s_x^2 + 8s_y + 5s_y^2$$
$$+ 8s_x(1+s_y))]\}$$

$$r_{11} = \frac{1}{16}(1 + s_x + s_y)[c_x^2 + c_y^2 + 4(4 + 3s_x^2 + 8s_y + 3s_y^2 + 8s_x(1+s_y))],$$

$$r_{12} = -\frac{1}{64}(c_x - 2s_x)[c_x^2 + c_y^2 + 4(4 + 3s_x^2 + 8s_y + 5s_y^2 + 8s_x(1+s_y))],$$

$$r_{13} = \frac{1}{8}(c_x - 2s_x)(c_y - 2s_y)(1 + s_x + s_y),$$

$$r_{14} = -\frac{1}{64}(c_y - 2s_y)[-c_x^2 + c_y^2 + 4(4 + 5s_x^2 + 8s_y + 3s_y^2 + 8s_x(1+s_y))],$$

$$r_{21} = \frac{1}{64}(c_x + 2s_x)[c_x^2 - c_y^2 + 4(4 + 3s_x^2 + 8s_y + 5s_y^2 + 8s_x(1+s_y))],$$

$$r_{24} = -\frac{1}{8}(c_x + 2s_x)(c_y - 2s_y)(1 + s_x + s_y),$$

$$r_{31} = \frac{1}{8}(c_x + 2s_x)(c_y + 2s_y)(1 + s_x + s_y),$$

$$r_{32} = \frac{1}{64}(c_y + 2s_y)[-c_x^2 + c_y^2 + 4(4 + 5s_x^2 + 8s_y + 3s_y^2 + 8s_x(1+s_y))],$$

$$r_{42} = -\frac{1}{8}(c_x - 2s_x)(c_y + 2s_y)(1 + s_x + s_y).$$

Now, we consider the nodal points with grid size $2h = \frac{2}{n}$. The standard fractional formula is generated through the application specific finite difference approximations with 2h-spaced points. Using the Caputo time fractional approximation (4) at the left-hand side of (1) and the second order Crank–Nicolson difference scheme with 2h-spaced points at the right hand side of (1), the following approximation formula is obtained:

$$\begin{aligned}
&w_1 u^k - w_k u_{i,j}^0 + \sum_{s=1}^{k-1}(w_{k-s+1} - w_{k-s})u_{i,j}^s + \sigma\frac{(u_{i,j}^{k+1} - u_{i,j}^k)}{2^{1-\alpha}} = \frac{a_x}{2}\left(\frac{u_{i-2,j}^{k+1} - 2u_{i,j}^{k+1} + u_{i+2,j}^{k+1}}{4h^2} + \frac{u_{i-2,j}^k - 2u_{i,j}^k + u_{i+2,j}^k}{4h^2}\right) \\
&+\frac{a_y}{2}\left(\frac{u_{i,j-2}^{k+1} - 2u_{i,j}^{k+1} + u_{i,j+2}^{k+1}}{4h^2} + \frac{u_{i,j-2}^k - 2u_{i,j}^k + u_{i,j+2}^k}{4h^2}\right) - \frac{b_x}{2}\left(\frac{u_{i+2,j}^{k+1} - u_{i-2,j}^{k+1}}{4h} + \frac{u_{i+2,j}^k - u_{i-2,j}^k}{4h}\right) \\
&-\frac{b_y}{2}\left(\frac{u_{i,j+2}^{k+1} - u_{i,j-2}^{k+1}}{4h} + \frac{u_{i,j+2}^k - u_{i,j-2}^k}{4h}\right) + f_{i,j}^{k+\frac{1}{2}} + O(\tau^{2-\alpha} + (\Delta x)^2 + (\Delta y)^2)
\end{aligned} \tag{9}$$

After simplification, it can be obtained the following

$$\begin{aligned}
&(1 + \frac{s_x + s_y}{4})u_{i,j}^{k+1} = (\frac{s_x}{8} + \frac{c_x}{8})u_{i-2,j}^{k+1} + (\frac{s_x}{8} - \frac{c_x}{8})u_{i+2,j}^{k+1} + (\frac{s_y}{8} + \frac{c_y}{8})u_{i,j-2}^{k+1} \\
&+(\frac{s_y}{8} - \frac{c_y}{8})u_{i,j+2}^{k+1} + (1 - 2^{1-\alpha}w_1^* - \frac{s_x + s_y}{4})u_{i,j}^k + (\frac{s_x}{2} + \frac{c_x}{8})u_{i-2,j}^k \\
&(\frac{s_x}{8} - \frac{c_x}{8})u_{i+2,j}^k + (\frac{s_y}{8} + \frac{c_y}{8})u_{i,j-2}^k + (\frac{s_y}{8} - \frac{c_y}{8})u_{i,j+2}^k + 2^{1-\alpha}w_k^* u_{i,j}^0 \\
&+2^{1-\alpha}\sum_{s=1}^{k-1}[(w_{k-s}^* - w_{k-s+1}^*)u_{i,j}^s + m_0 f_{i,j}^{k+\frac{1}{2}}
\end{aligned} \tag{10}$$

Applying (10) to any group of four points (i, j), (i + 2, j), (i + 2, j + 2) and (i, j + 2) in the solution domain will result in the following 4 × 4 system

$$\begin{pmatrix} d_1 & -a_1 & 0 & -b_1 \\ -a_2 & d_1 & -b_1 & 0 \\ 0 & -b_2 & d_1 & -a_2 \\ -b_2 & 0 & -a_1 & d_1 \end{pmatrix}\begin{pmatrix} u_{i,j} \\ u_{i+2,j} \\ u_{i+2,j+2} \\ u_{i,j+2} \end{pmatrix} = \begin{pmatrix} rhs_{i,j} \\ rhs_{i+2,j} \\ rhs_{i+2,j+2} \\ rhs_{i,j+2} \end{pmatrix}, \tag{11}$$

where: $d_1 = 1 + \frac{s_x + s_y}{4}$, $a_1 = (\frac{s_x - c_x}{8})$, $a_2 = (\frac{s_x + c_x}{8})$, $b_1 = (\frac{s_y - c_y}{8})$, $b_2 = (\frac{s_y + c_y}{8})$,

$$\begin{aligned}
rhs_{i,j} &= a_2[u_{i-2,j}^{k+1} + u_{i-2,j}^k] + b_2[u_{i,j-2}^{k+1} + u_{i,j-2}^k] + a_1 u_{i+2,j}^k + b_1 u_{i,j+2}^k + \sum_{s=1}^{k-1} 2^{1-\alpha}[w_{k-s}^* - w_{k-s+1}^*]u_{i,j}^s \\
&+ 2^{1-\alpha}w_k^* u_{i,j}^0 + Qu_{i,j}^k + m_0 f_{i,j}^{k+\frac{1}{2}},
\end{aligned}$$

$$\begin{aligned}
rhs_{i+2,j} &= a_1[u_{i+4,j}^{k+1} + u_{i+4,j}^k] + a_2 u_{i,j}^k + b_1 u_{i+2,j+2}^k + b_2[u_{i+2,j-2}^{k+1} + u_{i+2,j-2}^k] + \sum_{s=1}^{k-1} 2^{1-\alpha}[w_{k-s}^* - w_{k-s+1}^*]u_{i+2,j}^s \\
&+ 2^{1-\alpha}w_k^* u_{i+2,j}^0 + Qu_{i+2,j}^k + m_0 f_{i+2,j}^{k+\frac{1}{2}},
\end{aligned}$$

$$rhs_{i+2,j+2} = a_1[u^{k+1}_{i+4,j+2} + u^{k}_{i+4,j+2}] + b_1[u^{k+1}_{i+2,j+4} + u^{k}_{i+2,j+4}] + b_2 u^{k}_{i+2,j} + a_2 u^{k}_{i,j+2} + 2^{1-\alpha} w^{*}_{k} u^{0}_{i+2,j+2} + Q u^{k}_{i+2,j+2}$$
$$+ \sum_{s=1}^{k-1} 2^{1-\alpha}[w^{*}_{k-s} - w^{*}_{k-s+1}] u^{s}_{i+2,j+2} + m_0 f^{k+\frac{1}{2}}_{i+2,j+2},$$
$$rhs_{i,j+2} = a_2[u^{k+1}_{i-2,j+2} + u^{k}_{i-2,j+2}] + b_1[u^{k+1}_{i,j+4} + u^{k}_{i,j+4}] + a_1 u^{k}_{i+2,j+2} + b_2 u^{k}_{i,j} + 2^{1-\alpha} w^{*}_{k} u^{0}_{i,j+2}$$
$$+ 2^{1-\alpha} \sum_{s=1}^{k-1} [w^{*}_{k-s} - w^{*}_{k-s+1}] u^{s}_{i,j+2} + Q u^{k}_{i,j+2} + m_0 f^{k+\frac{1}{2}}_{i,j+2},$$

with $Q = (1 - 2^{1-\alpha} w^{*}_{1} - \frac{s_x + s_y}{4})$.

Hence, the four-point FMEG equation below is obtained by inverting (11), as follows

$$\begin{pmatrix} u^{k+1}_{i,j} \\ u^{k+1}_{i+2,j} \\ u^{k+1}_{i+2,j+2} \\ u^{k+1}_{i,j+2} \end{pmatrix} = \frac{1}{\text{cons tan t}} \begin{pmatrix} r^{*}_{11} & r^{*}_{12} & r^{*}_{13} & r^{*}_{14} \\ r^{*}_{21} & r^{*}_{11} & r^{*}_{14} & r^{*}_{24} \\ r^{*}_{31} & r^{*}_{32} & r^{*}_{11} & r^{*}_{21} \\ r^{*}_{32} & r^{*}_{42} & r^{*}_{12} & r^{*}_{11} \end{pmatrix} \begin{pmatrix} rhs_{i,j} \\ rhs_{i+2,j} \\ rhs_{i+2,j+2} \\ rhs_{i,j+2} \end{pmatrix}, \tag{12}$$

where:

$$\text{cons tan t} = \frac{1}{4096}(4096 + c_x^4 + c_y^4 + 4096 s_x + 1408 s_x^2 + 192 s_x^3 + 9 s_x^4 + 4096 s_y + 3072 s_x s_y + 704 s_x^2 s_y$$
$$+ 48 s_x^3 s_y + 1408 s_y^2 + 704 s_x s_y^2 + 78 s_x^2 s_y^2 + 192 s_y^3 + 48 s_x s_y^3 + 9 s_y^4 + 2 c_y^2 (64 + 5 s_x^2 + 32 s_y + 3 s_y^2$$
$$+ 8 s_x (4 + s_y)) + 2 c_x^2 ((64 - c_y^2 + 3 s_x^2 + 32 s_y + 5 s_y^2 + 8 s_x (4 + s_y))$$

$$r^{*}_{11} = \frac{1}{256}(4 + s_x + s_y)[64 + c_x^2 + c_y^2 + 32 s_x + 3 s_x^2 + 32 s_y + 8 s_x s_y + 3 s_y^2],$$

$$r^{*}_{12} = -\frac{1}{512}(c_x - s_x)[64 + c_x^2 - c_y^2 + 3 s_x^2 + 32 s_x + 32 s_y + 5 s_y^2 + 8 s_x s_y],$$

$$r^{*}_{13} = -\frac{1}{512}(c_x - s_x)[64 + c_x^2 - c_y^2 + 3 s_x^2 + 32 s_x + 32 s_y + 5 s_y^2 + 8 s_x s_y],$$

$$r^{*}_{14} = -\frac{1}{512}(c_y - s_y)[64 - c_x^2 + c_y^2 + 5 s_x^2 + 32 s_x + 32 s_y + 3 s_y^2 + 8 s_x s_y],$$

$$r^{*}_{21} = \frac{1}{512}(c_x + s_x)[64 + c_x^2 - c_y^2 + 3 s_x^2 + 32 s_x + 32 s_y + 5 s_y^2 + 8 s_x s_y],$$

$$r^{*}_{24} = -\frac{1}{128}(c_x + s_x)(c_y - s_y)[4 + s_x + s_y],$$

$$r^{*}_{31} = \frac{1}{128}(c_x + s_x)(c_y + s_y)[4 + s_x + s_y],$$

$$r^{*}_{32} = \frac{1}{512}(c_y + s_y)[64 - c_x^2 + c_y^2 + 5 s_x^2 + 32 s_x + 32 s_y + 3 s_y^2 + 8 s_x s_y],$$

$$r^{*}_{42} = -\frac{1}{128}(c_x - s_x)(c_y + s_y)[4 + s_x + s_y].$$

The solution domain of the above must be labelled in three different types of points (i.e., ●, □, ○) as shown in Figure 1 .

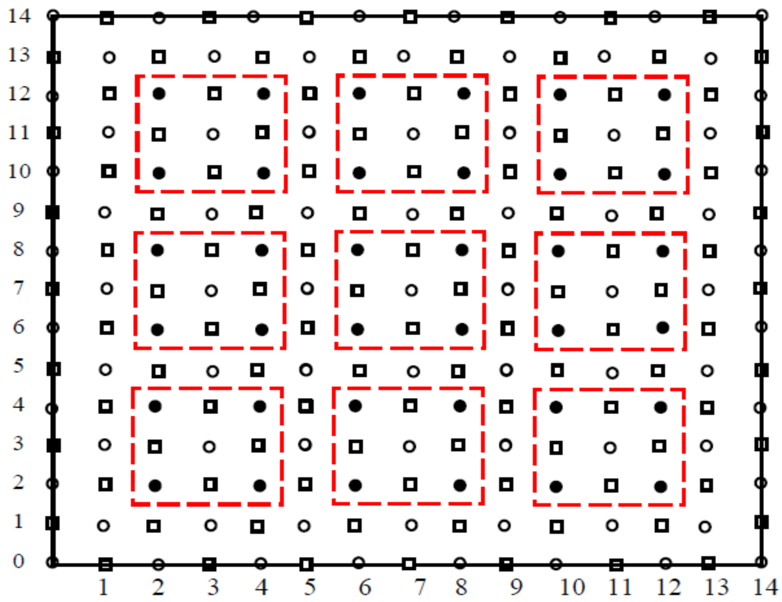

Figure 1 The solution domain of the four points-FMEG method

Figure 1 shows the construction of blocks of four points in the solution domain for the case n = 14. This method stood out amongst all the other tested methods that it is the most superior in terms of execution time, number of iterations and accuracy [23].

3. Preconditioned Fractional MEG Method

It is well known that preconditioners play a vital role in accelerating the convergence rates of iterative methods. A well-designed preconditioning of group iterative methods for solving partial differential equations (PDEs) and fractional partial differential equations (FPDEs) problems reduces the number of iterations to reach convergence [8-12]. Dramatic improvements are possible, but the difficulty is to construct the suitable preconditioner. In general, a good preconditioner should satisfy the following prosperities: the first one is that, the preconditioned system should be easy to solve and the second one is that the preconditioner should be cheap to construct and apply. Usually the system (11) is large and the matrix is sparse. Furthermore, matrix   can be write as

$$A = D - L - U \tag{13}$$

where D is diagonal matrix A, $-L$ is strictly lower triangular parts of A and $-U$ is strictly upper triangular parts of A. A preconditioner

$$P=\begin{pmatrix} 0 & 0 & -kb_2 & 0 \\ -ka_2 & 0 & 0 & 0 \\ 0 & -kb_2 & 0 & 0 \\ 0 & 0 & 0 & -ka_1 \end{pmatrix},$$

where $0\le k<2$ and a_1,a_2,b_2 same as in (11), is used to modify the original system (11) to the following system:

$$\begin{pmatrix} 0 & kb_2^2 & -kb_2d_1 & kb_2a_2 \\ -ka_2d_1 & ka_1a_2 & 0 & kb_1a_2 \\ kb_2a_2 & -kb_2d_1 & kb_1b_2 & 0 \\ ka_2b_2 & 0 & ka_1^2 & -ka_1d_1 \end{pmatrix}\begin{pmatrix} u_{i,j} \\ u_{i+2,j} \\ u_{i+2,j+2} \\ u_{i,j+2} \end{pmatrix}=\begin{pmatrix} -kb_2rhs_{i+2,j+2} \\ -ka_2rhs_{i,j} \\ -kb_2rhs_{i+2,j} \\ -ka_1rhs_{i,j+2} \end{pmatrix} \quad (14)$$

A preconditioner P is a matrix that transforms the original system (11) into new system (14) that is equivalent in the sense that it has the same solution, but that has more favourable spectral properties. Hence, the explicit four-point preconditioned FMEG equation is obtained by inverting (14) by the same manner as in (12).

4. Stability and Convergence Analysis

A scheme is considered to be stable if the errors cease to increase with the passing of time, and gradually become inconsequential as the computation progresses. Even though the spacing is different, Equation (6) gives rise to both the FEG and FMEG methods. Therefore, the stability of both methods can be analyzed in similar ways. Here, we will show the stability of the PFMEG scheme using eigenvalues of the generated matrices with mathematical induction.

Form (11) we obtain

$$AU^1=BU, \qquad k=0$$

$$AU^{k+1}=BU^k-c_1U^k+\sum_{s=1}^{k-1}c_{k-s}U^s+c_kU^0+b, \qquad k>0 \quad (15)$$

where

$$A=\begin{pmatrix} R_1 & R_2 & & 0 & 0 \\ R_3 & R_1 & R_2 & & 0 \\ & R_3 & \ddots & & \\ 0 & & & R_1 & R_2 \\ 0 & 0 & & R_3 & R_1 \end{pmatrix}, \quad B=\begin{pmatrix} S_1 & S_2 & & 0 & 0 \\ S_3 & S_1 & S_2 & & 0 \\ & & \ddots & & \\ 0 & & S_3 & S_1 & S_2 \\ 0 & 0 & & S_3 & S_1 \end{pmatrix}, \quad b=\begin{pmatrix} w_1 \\ w_1 \\ \vdots \\ w_1 \\ w_1 \end{pmatrix}$$

$$R_1 = \begin{pmatrix} G_1 & G_3 & & & \\ G_2 & G_1 & G_3 & & \\ & & \ddots & & \\ & & G_2 & G_1 & G_3 \\ & & & G_2 & G_1 \end{pmatrix},\ R_2 = \begin{pmatrix} G_4 & & & & \\ & G_4 & & & \\ & & \ddots & & \\ & & & G_4 & \\ & & & & G_4 \end{pmatrix},\ R_3 = \begin{pmatrix} G_5 & & & & \\ & G_5 & & & \\ & & \ddots & & \\ & & & G_5 & \\ & & & & G_5 \end{pmatrix}$$

$$S_1 = \begin{pmatrix} J_1 & J_2 & & & \\ J_3 & J_1 & J_2 & & \\ & & \ddots & & \\ & & J_3 & J_1 & J_2 \\ & & & J_3 & J_1 \end{pmatrix},\ S_2 = \begin{pmatrix} J_4 & & & & \\ & J_4 & & & \\ & & \ddots & & \\ & & & J_4 & \\ & & & & J_4 \end{pmatrix},\ S_3 = \begin{pmatrix} J_5 & & & & \\ & J_5 & & & \\ & & \ddots & & \\ & & & J_5 & \\ & & & & J_5 \end{pmatrix}$$

$$G_1 = \begin{pmatrix} d_1 & -a_{11} & 0 & -b_{11} \\ -a_{22} & d_1 & -b_{11} & 0 \\ 0 & -b_{22} & d_1 & -a_{22} \\ -b_{22} & 0 & -a_{11} & d_1 \end{pmatrix},\ G_2 = \begin{pmatrix} 0 & 0 & 0 & -b_{22} \\ 0 & 0 & -b_{22} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},\ G_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -b_{11} & 0 & 0 \\ -b_{11} & 0 & 0 & 0 \\ -b_{11} & 0 & 0 & 0 \end{pmatrix}$$

$$G_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -a_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & -a_{11} \\ 0 & 0 & 0 & 0 \end{pmatrix},\ G_5 = \begin{pmatrix} 0 & -a_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -a_{22} & 0 \end{pmatrix},\ J_1 = \begin{pmatrix} Q & a_{11} & 0 & b_{11} \\ a_{22} & Q & b_{11} & 0 \\ 0 & b_{22} & Q & a_{22} \\ b_{22} & 0 & a_{11} & Q \end{pmatrix}$$

$$J_2 = \begin{pmatrix} 0 & 0 & 0 & b_{22} \\ 0 & 0 & b_{22} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, J_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & b_{11} & 0 & 0 \\ b_{11} & 0 & 0 & 0 \end{pmatrix}, J_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ a_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{11} \\ 0 & 0 & 0 & 0 \end{pmatrix}, J_5 = \begin{pmatrix} 0 & a_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & a_{22} & 0 \end{pmatrix}$$

$$c_k = \begin{pmatrix} M_k & & & \\ & M_k & & \\ & & \ddots & \\ & & & M_k \end{pmatrix}, c_{k-s} = \begin{pmatrix} M_{k-s} & & & \\ & M_{k-s} & & \\ & & \ddots & \\ & & & M_{k-s} \end{pmatrix}, c_1 = \begin{pmatrix} M_1 & & & \\ & M_1 & & \\ & & \ddots & \\ & & & M_1 \end{pmatrix},\ s = 1,\dots,k-1$$

$$w_1 = \begin{pmatrix} l_1 \\ l_1 \\ \vdots \\ l_1 \\ l_1 \end{pmatrix},\ M_{k-s} = \frac{2^{1-\alpha}}{t^\alpha}\begin{pmatrix} (w^*_{k-s} - w^*_{k-s+1}) & 0 & 0 & 0 \\ 0 & (w^*_{k-s} - w^*_{k-s+1}) & 0 & 0 \\ 0 & 0 & (w^*_{k-s} - w^*_{k-s+1}) & 0 \\ 0 & 0 & 0 & (w^*_{k-s} - w^*_{k-s+1}) \end{pmatrix},$$

$$M_1 = \frac{2^{1-\alpha}}{t^\alpha}\begin{pmatrix} w^*_1 & 0 & 0 & 0 \\ 0 & w^*_1 & 0 & 0 \\ 0 & 0 & w^*_1 & 0 \\ 0 & 0 & 0 & w^*_1 \end{pmatrix}, M_k = \frac{2^{1-\alpha}}{t^\alpha}\begin{pmatrix} w^*_k & 0 & 0 & 0 \\ 0 & w^*_k & 0 & 0 \\ 0 & 0 & w^*_k & 0 \\ 0 & 0 & 0 & w^*_k \end{pmatrix}, l_1 = 2^{1-\alpha}\Gamma(2-\alpha)\begin{pmatrix} f_{i,j} \\ f_{i+2,j} \\ f_{i+2,j+2} \\ f_{i,j+2} \end{pmatrix}, i,j = 2,6,\dots,n-2.$$

The following lemma is important to prove the stability [6].

Lemma 1 In (15), the coefficients $w_s^*, s=1,2,...,k$ satisfy the following $1-w_{k-s}^* > w_{k-s+1}^*$, and $2-\sum_{s=1}^{k=1}[w_{k-s}^* - w_{k-s+1}^*]+w_k^* = w_1^*$. For simplicity, we assume:

$s_x = s_y = S = \frac{\Gamma(2-\alpha)2^{1-\alpha}}{h^2}$, $c_x = c_y = C = \frac{\Gamma(2-\alpha)2^{1-\alpha}}{h}$, $D_1 = \frac{1}{t\,a} + \frac{S}{2}$ and

$Q = \frac{1}{t\,a} - 2^{1-\alpha} w_1^* - \frac{S}{2}$.

Theorem 1 If $(\frac{1}{t^\alpha} - \frac{S}{2} - \frac{2^{1-\alpha}}{t^\alpha} w_1^* + \frac{\sqrt{-C^2+S^2}}{4}) > 0$ then the FMEG scheme (11) is stable.

Proof.

To prove the stability of (11), we assume that $u_{i,j}^k, i, j = 1,2,...,n$ *and* $k = 1,2,...,\ell$ are the approximate solution to the exact solution $U_{i,j}^k$ of (1), the error $\varepsilon_{i,j}^k = U_{i,j}^k - u_{i,j}^k$ be the error at time level k. From (15), the error satisfies

$$\begin{aligned} &AE^1 = BU, && k=0 \\ &AE^{k+1} = BE^k - C_1 E^k + \sum_{s=1}^{k-1} C_{k-s} U^s + C_k E^0 && k>0 \end{aligned} \tag{16}$$

where

$$E^{k+1} = \begin{pmatrix} E_1^{k+1} \\ E_1^{k+1} \\ \vdots \\ E_1^{k+1} \\ E_1^{k+1} \end{pmatrix}, \; E_1^{k+1} = \begin{pmatrix} \varepsilon_2^{k+1} \\ \varepsilon_4^{k+1} \\ \vdots \\ \varepsilon_{m-4}^{k+1} \\ \varepsilon_{m-2}^{k+1} \end{pmatrix}, \; \varepsilon_i^{k+1} = \begin{pmatrix} \varepsilon_{i,j}^{k+1} \\ \varepsilon_{i+2,j}^{k+1} \\ \varepsilon_{i+2,j+2}^{k+1} \\ \varepsilon_{i,j+2}^{k+1} \end{pmatrix}, \; i = 2,6,...,n-2, \;\; j = 2,6,...,n-2$$

From the above equations, the following are obtained:

$$A = G_1+G_2+G_3+G_4+G_5,\; B = H_1+H_2+H_3+H_4+H_5,\; C_1 = M_1, C_{k-s} = M_{k-s}, C_k = M_k \tag{17}$$

It can be seen that from (17), the eigenvalues of the matrices A, B, C_1, C_{k-s} and C_k are a_k, b_k, c_1, c_{k-s} and c_k respectively, where

$$a_k = \{\frac{1}{t^\alpha} + \frac{s}{2}, \frac{1}{t^\alpha} + \frac{s}{2}, \frac{1}{4}(\frac{4}{t^\alpha} + 2s - \sqrt{-c^2+s^2}), \frac{1}{4}(\frac{4}{t^\alpha} + 2s + \sqrt{-c^2+s^2})\},$$

$$b_k = \{\frac{1}{t^\alpha} - \frac{s}{2}, \frac{1}{t^\alpha} - \frac{s}{2}, \frac{1}{4}(\frac{4}{t^\alpha} - 2s - \sqrt{-c^2+s^2}), \frac{1}{4}(\frac{4}{t^\alpha} - 2s + \sqrt{-c^2+s^2})\}$$

$$c_1 = \frac{2^{1-\alpha}}{t^\alpha} w_1^*, \; c_{k-s} = \frac{2^{1-\alpha}}{t^\alpha}(w_{k-s}^* - w_{k-s+1}^*), \; c_k = \frac{2^{1-\alpha}}{t^\alpha} w_k^*$$

For $k=0,$

$$E^1 = A^{-1}BE^0 \;\Rightarrow \left\|E^1\right\| \le \rho(A^{-1}B)\left\|E^0\right\| = \left|\frac{(\frac{1}{t^\alpha}-\frac{s}{2}+\frac{\sqrt{-c^2+s^2}}{4})}{(\frac{1}{t^\alpha}+\frac{s}{2}+\frac{\sqrt{-c^2+s^2}}{4})}\right|\left\|E^0\right\| \Rightarrow \left\|E^1\right\| \le \left\|E^0\right\|$$

Supposing that: $\left\|E^s\right\| \le \left\|E^0\right\|, s=1,\ldots,k$

Need to prove this inequality holds for $s=k+1,$

$$E^{k+1} = A^{-1}(B-c_1)E^k + \sum_{s=1}^{k-1} A^{-1}C_{k-s}E^s + A^{-1}C_kE^0 \le \rho(A^{-1}(B-c_1))\left\|E^0\right\| + \sum_{s=1}^{k-1}\rho(A^{-1}C_{k-s})\left\|E^0\right\| + \rho(A^{-1}C_k)\left\|E^0\right\|$$

Using Lemma 1, we get

$$\left\|E^{k+1}\right\| \le \frac{\left|(\frac{1}{t^\alpha}-\frac{s}{2}-\frac{2^{1-\alpha}}{t^\alpha}w_1^*+\frac{\sqrt{-c^2+s^2}}{4})\right|}{(\frac{1}{t^\alpha}+\frac{s}{2}+\frac{\sqrt{-c^2+s^2}}{4})}\left\|E^0\right\| + \frac{\frac{2^{1-\alpha}}{t^\alpha}w_1^*}{(\frac{1}{t^\alpha}+\frac{s}{2}+\frac{\sqrt{-c^2+s^2}}{4})}\left\|E^0\right\| \Rightarrow \left\|E^{k+1}\right\| \le \left\|E^0\right\|$$

Therefore, under the conditions $(\frac{1}{t^\alpha}-\frac{s}{2}-\frac{2^{1-\alpha}}{t^\alpha}w_1^*+\frac{\sqrt{-c^2+s^2}}{4})>0$, the FMEG iterative scheme (11) is stable. Hence the proof is completed.

Remark 1 By using the same manner of theorem 1, we can easily prove that the PFMEG iterative scheme (14) is also conditionally stable.

Theorem 2 The FMEG iterative scheme (11) is convergent and therefore the PFMEG iterative scheme (14) is also convergent. The following estimate hold

$$\left\|e^{k+1}\right\| \le C_k^{-1}\{\tau^{2-\alpha}+(\vartriangle x)^2+(\vartriangle y)^2\} \text{ if } (\frac{1}{t^\alpha}-\frac{s}{2}-\frac{2^{1-\alpha}}{t^\alpha}w_1^*+\frac{\sqrt{-c^2+s^2}}{4})>0 \;.$$

Proof.

If we denote the truncation error at (x_i, y_j, t_{k+1}) by R^{k+1}, then from (5) we have

$$\left\|R^{k+\frac{1}{2}}\right\| \le C\{\tau^{2-\alpha}+(\vartriangle x)^2+(\vartriangle y)^2\}$$

Define $\xi^{k+1} = u(x_i, y_j, t_{k+1}) - U_{i,j}^{k+1},\ i, j = 1,2,\ldots,n,\ k = 1,2,\ldots,\ell$ and $e^{k+1} = (e_1^{k+1}, e_1^{k+1}, \ldots, e_1^{k+1})^T$, where

$e_1^{k+1}=(\xi_2^{k+1}\ \xi_4^{k+1}\ldots\xi_{m-4}^{k+1}\ \ \xi_{m-2}^{k+1})^T$, $\varepsilon_i^{k+1}=(\xi_{i,j}^{k+1}\ \ \xi_{i+2,j}^{k+1}\ \ \xi_{i+2,j+2}^{k+1}\ \ \xi_{i,j+2}^{k+1})^T$, $\{i,j\}=2,6,\ldots,n-2$

Substitution into (16) produces:

$$Ae^1=R^{\frac{1}{2}},\qquad Ae^{k+1}=Be^k-C_1e^k+\sum_{s=1}^{k-1}C_{k-s}e^{s-1}+R^{k+\frac{1}{2}}$$

By using mathematical induction to prove the above theorem, set $\left\|C_0^{-1}\right\|\le 1$.

For $k=0$, $Ae^1=R^{\frac{1}{2}}$, $\quad \left\|e^1\right\|\le\rho(A^{-1})\|R\|\le\left\|C_0^{-1}\right\|\{\tau^{2-\alpha}+(\triangle x)^2+(\triangle y)^2\}$

Assume that $\left\|e^k\right\|\le\left\|C_{k-1}^{-1}\right\|\left\|R^{k+\frac{1}{2}}\right\|$. Need to prove that the last inequality holds for $s=k+1$.

$$\because\ Ae^{k+1}=(B-C_1)e^k+\sum_{s=1}^{k-1}C_{k-s}e^{s-1}+R^{k+\frac{1}{2}}$$

$$\begin{aligned}\left\|e^{k+1}\right\|&\le\rho\{A^{-1}(B-C_1)\}\left\|e^k\right\|+\sum_{s=1}^{k-1}\rho(A^{-1}C_{k-s})\left\|e^{s-1}\right\|+\rho(A^{-1})\left\|R^{k+\frac{1}{2}}\right\|\\&=(\frac{2^{1-\alpha}}{t^\alpha}w_k^*)^{-1}\{\tau^{2-\alpha}+(\Delta x)^2+(\Delta y)^2\}\\&=\frac{k^\alpha k^{-\alpha}\tau^\alpha}{2^{1-\alpha}\{(k+\frac{1}{2})^{1-\alpha}-(k-\frac{1}{2})^{1-\alpha}\}}\{\tau^{2-\alpha}+(\Delta x)^2+(\Delta y)^2\}\end{aligned}$$

Hence, $\left\|e^{k+1}\right\|\le\dfrac{1}{2^{1-\alpha}(1-\alpha)}\{\tau^{2-\alpha}+(\Delta x)^2+(\Delta y)^2\}$.

Since $\lim\limits_{k\to\infty}\dfrac{k^{-\alpha}}{\{(k+\frac{1}{2})^{1-\alpha}-(k-\frac{1}{2})^{1-\alpha}\}}=\dfrac{1}{1-\alpha}$.

5. Numerical Results

In this section, we present numerical results for the proposed method applied to the time fractional initial boundary problem as the following [24]

$$\frac{\partial^\alpha u}{\partial t^\alpha}=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}-\frac{\partial u}{\partial x}-\frac{\partial u}{\partial y}+0.5\Gamma(3+\alpha)t^2e^{x+y}\tag{18}$$

where $\Omega=\{(x,y):0\le x\le 1,0\le y\le 1\}$ is the solution domain with the exact solution being $t^{2+\alpha} e^{x+y}$.

Different mesh sizes of 6, 10, 14 and 18 and various time steps, which satisfy the stability conditions, with a fixed relaxation factor (Gauss-Seidel relaxation scheme) of 1.0 were used to run the experiments. Preconditioned method was deemed efficient through investigations which revealed their superiority in the context of execution time (measured in seconds), number of iterations and maximum error with tolerance $\varepsilon = 10^{-5}$.

Table 1 The Comparison of the number of iterations, Execution times and maximum error for different time step and mesh size at $\alpha = 0.75$

Model Problem{ The time fractional initial boundary problem (18)}					
h	$\triangle t$	Method	Time	iterations	Max Error
$\frac{1}{6}$	$\frac{1}{100}$	FMEG	1.12	2	2.99E-3
		PFMEG	1.09	1	2.56E-3
$\frac{1}{10}$	$\frac{1}{350}$	FMEG	54.55	4	1.26E-3
		PFMEG	49.87	3	1.25E-3
$\frac{1}{14}$	$\frac{1}{850}$	FMEG	590.22	4	5.08E-4
		PFMEG	270.06	2	4.03E-4
$\frac{1}{18}$	$\frac{1}{1620}$	FMEG	3690.66	4	4.71E-4
		PFMEG	2130.42	2	4.65E-4

Table 2 The Comparison of the number of iterations, Execution times and maximum error for different time step and mesh size at $\alpha = 0.95$

Model Problem{ The time fractional initial boundary problem (18)}					
h	Δt	Method	Time	iterations	Max Error
$\frac{1}{6}$	$\frac{1}{100}$	FMEG	1.11	2	2.71E-3
		PFMEG	1.03	1	2.44E-3
$\frac{1}{10}$	$\frac{1}{350}$	FMEG	52.44	4	1.11E-3
		PFMEG	49.35	3	1.08E-3
$\frac{1}{14}$	$\frac{1}{850}$	FMEG	361.72	2	4.26E-4
		PFMEG	194.88	1	3.98E-4
$\frac{1}{18}$	$\frac{1}{1620}$	FMEG	1678.53	2	2.61E-4
		PFMEG	1275.91	1	2.05E-4

From tables (1) and (2), we can observe that the proposed preconditioned system (PFMEG) is superior than original system FMEG methods in term of the number of iterations and execution times which yield very encouraging results.

6. Conclusion

In this work, we have formulated new preconditioned iterative method based on FMEG method for solving the time fractional initial boundary problem. From observation of all experimental results, it can be concluded that the proposed scheme may be a good alternative to solve this type of equations and many other numerical problems. The stability and convergence of the proposed method were analyzed using the matrix form with mathematical induction. Furthermore, the idea of this introduced method can be extended to solve other types of fractional initial boundary problems which will be reported separately in the future.

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