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An Accelerated Numerical Solution of Elliptic Equations by using Nine Explicit Group Iterative Method

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Abstract

It has been shown in recent decades that grouping strategies can help reduce the spectral radius of the generated matrix resulting from the discretization of partial differential equations (PDEs) using finite difference techniques. As a result of this reduction of the spectral radius, the convergence rates of the iterative algorithms were increased. This paper will present the development and formulation of nine explicit groups SOR/AOR methods for solving Elliptic PDEs. The convergence analysis of the proposed method will be introduced. In addition, numerical experiments will be conducted to show the most superior Explicit Group method for solving PDEs of Elliptic Type.

Keywords: Poisson's and Laplace's equations; SOR and AOR Methods; Nine-Point Group Method.

2021 MSC: 34A08; 34B15; 34A12; 47H10.

1. INTRODUCTION:

The partial differential equations arise in many applications, such as elasticity, fluid mechanics, and many other areas [1]-[5]. In the last few years, improved techniques have been developed to resolve the linear systems resulting from the discernment of the partial differential equations by using explicit group methods derived from the standard

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and rotated finite differential operators [6]-[12]. For improving the convergence rate of these group methods, many strategies and iterative methods mentioned in the literature have been used [13]-[15]. The investigation of the effectiveness of 4-point Explicit Group Successive Over-Relaxation (EGSOR), 4-point Explicit Group Accelerated Over- Relaxation (EGAOR) have been done in recent years [16]. This study introduced the combination of 4 explicit groups with SOR and AOR methods, giving us encouraging results. It found that AOR is superior to SOR despite computational effort more than SOR due to the look completion and iteration and execution time reduction. This research will focus on the development formulation of a nine-point explicit group with the AOR iterative method. Furthermore, we will compare our results with a precise study that uses an explicit group with SOR to solve the two-dimensional elliptic (Laplace and Poisson) equations.

Consider 2D-Poisson equation as:

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y), (x,y) \in \Omega. \tag{1.1}$$

with specific Dirichlet boundary conditions $U(x,y) = g(x,y), (x,y) \in \partial\Omega$. At the point (x_i, y_j), equation (1.1) can be roughly approximated. Suppose a rectangular grid at this point (x_i, y_j) plane with equal grid spacing h is used in both directions $x = ih, y = jh(i, j = 0, 1, \ldots N)$ by neglecting the terms of $O(h^2)$, we get the simplest approximation of (1.1), known as the five-point standard formulation:

$$u_{i,j+1} + u_{i,j-1} + u_{i+1,j} + u_{i-1,j} - 4u_{ij} = h^2 f_{ij}. \tag{1.2}$$

An accurate analysis of convergence properties of the SOR and AOR methods is possible if the matrix A is consistently ordered in the following sense

Definition 1.1. *A matrix A is a generalized (q,r)-consistently ordered matrix (a GCO(q,r)-matrix) if* $\Delta = \det(\alpha^q E + \alpha^{-r} F - kD)$ *is independent of* α *for all* $\alpha \neq 0$ *and for all k. Here, D = diag A and E and F are strictly lower and strictly upper triangular matrices, respectively, such that:* $A = D - E - F$.

Definition 1.2. *[21] A matrix A of the form eq(3.1) is said to be generally consistently ordered (π,q,r) or simply GCO (π,q,r), where q and r are positive integers, if for the partitioning π of A the diagonal submatrices* $A_{(ii)}, i = 1, 2, .., p(\geq 2)$*, are non-singular, and the eigenvalues of* $B_J(\alpha) = \alpha^r L + \alpha^{-q} U$ *are indpendent of* α*, for all* $\alpha \neq 0$*, where L and U are strict blocks lower and upper triangular parts of A, respectively.*

This work aims to find the most efficient group method for solving elliptic partial differential equations. For this purpose, the formulation of nine explicit groups (NEG) with the AOR iterative method will be presented. Furthermore, we will compare the four-point group and nine-point group SOR iterative methods with the four-point group and nine-point group AOR iterative methods.

The following is the contour of this paper: Section 2 provides an overview of the Improvement of the accuracy of the differential equation. Then, convergence analysis related to the studied group methods will be introduced in section 3. In section 4, the formulation of nine explicit groups SOR methods will be given to solve Poisson's Equation. The formulation of nine explicit groups AOR method will be introduced in section 5. In section 6, illustrative examples to justify the results will be presented. The final remarks, discussion, and conclusions will be given in sections 7 and 8, respectively.

2. IMPROVEMENT OF THE ACCURACY:

The accuracy of the differential equation can also be enhanced by using a higher-order finite-difference approximation that minimizes truncation error[17]. Laplace's and Poisson's equations were introduced and listed in[18]. Its derivative mode is indicated below by establishing a nine-point finite difference approximation to Poisson's equation, which has a truncation error of order h^4.
Denote the Laplacian

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, (x,y) \in \Omega \tag{2.1}$$

and let

$$\xi = h\frac{\partial}{\partial x}, \eta = h\frac{\partial}{\partial y}, \vartheta^2 = h\frac{\partial^2}{\partial x \partial y},$$

So that

$$\begin{aligned} &\xi^2 + \eta^2 = h^2\nabla^2, \xi\eta = h^2\vartheta^2, \\ &\xi^4 + \eta^4 = \left(\xi^2 + \eta^2\right)^2 - 2\xi^2\eta^2 = h^4\left(\nabla^4 - 2\vartheta^4\right). \end{aligned}$$

As Taylor's series can be written as

$$U(x+h) = (1 + h\frac{d}{dx} + \ldots + \frac{h^n}{n!}\frac{d^n}{dx^n} + \ldots)U(x) = \left(e^{h(d/dx)}\right)U(x),$$

It follows that

$$U_1 = e^{\xi}U_0, U_2 = e^{\eta}U_0, U_3 = e^{-\xi}U_0, U_1 = e^{\xi}U_0, U_5 = e^{\xi+\eta}U_0, \ldots.$$

Since in their derivatives the equation of Poisson is symmetric, the following symmetrical sums are determined;

$$S_1 = U_1 + U_2 + U_3 + U_4, S_2 = U_5 + U_6 + U_7 + U_8,$$

and

$S_3 = U_9 + U_{10} + U_{11} + U_{12}.$

It can be shown that $U_1, U_2 \ldots$ is substituted by U_0 as the following

$$\begin{aligned}
S_1 &= 4U_0 + h^2\nabla^2 U_0 + \frac{1}{12}h^4\left(\nabla^4 - 2\vartheta^4\right)U_0 + \frac{1}{360}h^6(\nabla^6 - 3\vartheta^2\nabla^2)U_0 + \ldots, \\
S_2 &= 4U_0 + h^2\nabla^2 U_0 + \frac{1}{6}h^4\left(\nabla^4 + 4\vartheta^4\right)U_0 + \frac{1}{180}h^6(\nabla^6 + 12\vartheta^4\nabla^2)U_0 + \ldots, \\
S_3 &= 4U_0 + 4h^2\nabla^2 U_0 + \frac{4}{3}h^4\left(\nabla^4 - 2\vartheta^4\right)U_0 + \frac{8}{45}h^6(\nabla^6 - 3\vartheta^4\nabla^2)U_0 + \ldots,
\end{aligned} \tag{2.2}$$

Between S_1 and S_2 is eliminated of $\vartheta^4 U_0$, resulting in

$$\nabla^2 U_0 = \frac{4S_1 + S_2 - 20U_0}{6h^2} - \frac{1}{12}h^2\nabla^4 U_0 + O\left(h^4\right).$$

The second term at the right for Poisson's equation is known as $\nabla^2 U = f, so\, \nabla^4 U = \nabla^2 f$. The coefficients of h^2 and h^4 disappear from Laplace's equation. Thus, a finitely different representation of Laplace's equation of nine points is more precise since the order of the error truncates it in $S_1 - 4u_0 = 0$ is of order h^2. The most convenient way of exhibiting this nine-point formula approximating Poisson's equation $\nabla^2 u = f$, is by the 'molecular' display

$$\begin{pmatrix} 1 & 4 & 1 \\ 4 & -20 & 4 \\ 1 & 4 & 1 \end{pmatrix} u = 6h^2 f + \frac{1}{2}h^4\nabla^2 f.$$

3. CONVERGENCE ANALYSIS

In this section, we will introduce several preliminary relevant theorems and lemmas, which are needed to prove the convergence properties of the solution resulting from the mentioned iterative methods. The spectral radius of a matrix is denoted by $\rho(.)$, which is defined as the largest of the moduli of the eigenvalues of the iteration matrix. This spectral radius of a matrix plays an important role in studying these convergence properties [19]-[20].

It can be seen that the resulted coefficient block matrix A of AU=B obtained from the discretization of 4- and 9- point group iterative methods can be partitioned in the following block form :

$$A_m = \begin{bmatrix} A_{m_{11}} & A_{m_{12}} & & & \\ A_{m_{21}} & A_{m_{22}} & A_{m_{23}} & & \\ & A_{m_{32}} & A_{m_{33}} & \ddots & \\ & & \ddots & \ddots & A_{m_{(p-1)p}} \\ & & & A_{m_{p(p-1)}} & A_{m_{pp}} \end{bmatrix} \quad (3.1)$$

with p = (N-2), where $A_{m_{ii}} \in \not\subset^{n_i,n_i}_{\pi,p}$, i = 1,2,. . . ,p, and $\sum_{i=1}^{p} n_i = n$. Let $A_m = D_m - E_m - F_m$, where $D_m = diag(A_{m_{11}}, A_{m_{22}}, ..., A_{m_{pp}})$ and

$$E_m = E_{m_{ij}} = \begin{cases} -A_{m_{ij}} & \text{for} \quad j < i \\ 0 \quad \text{for} & j \geq i, \end{cases}, F_m = F_{m_{ij}} = \begin{cases} -A_{m_{ij}} & \text{for} \quad j > i \\ 0 \quad \text{for} & j \leq i, \end{cases} \quad (3.2)$$

Are block matrices consisting of the block diagonal, strict block lower triangular, and strict block upper triangular parts of A. Here, the diagonal entries $A_{m_{ii}}$ are nonsingular. The block Jacobi iteration matrix is $B_J(A_m) = D_m{}^{-1}(E_m + F_m) = L_m + U_m$, where $L_m = D_m^{-1}E_m$, $U_m = D_m^{-1}F_m$, while the block Gauss-seidel iteration matrix is $B_{GS}(A_m) = (I_m - L_m)^{-1}U_m$. The Block Successive Over-Relaxation method (BSOR) iteration matrix is therefore

$$T_{\ell_w} = (I_m - wL_m)^{-1}\{(1-w)I_m + wU_m\} \quad (3.3)$$

The general form of BSOR is called the Block Accelerated Over-Relaxation (BAOR) method, and it can be written as:

$$T_{\ell_{r,\omega}} = (I_m - rD_m{}^{-1}L_m)^{-1}[(1-\omega)I_m + (\omega - r)D_m{}^{-1}L_m + \omega D_m{}^{-1}U_m] \quad (3.4)$$

Since the matrix A of Eq.(3.1) is π- consistently ordered and possesses property $A^{(\pi)}$, the theory of block SOR is valid for this iterative method and, therefore, convergent [6].

Lemma 3.1. *[22] Suppose $A_m = I_m – L_m – U_m$ is a GCO(π,q,r), where $–L_m$ and $–U_m$ are strictly lower and upper triangular matrices, respectively. Let T_{ℓ_w} be the block iteration matrix of the SOR method given by Eq (3.3). If $1 < w < 2$, then the block SOR method converges, that is. $\rho(T_{\ell_w}) < 1$.*

Proof. Let the matrix A with partitioning π be given as in (3.1) and let the block SOR iteration matrix T_{ℓ_w} be given as in (3.3),

Set $B'_{\ell_w} = (I - |wL_m|)^{-1}\{|1-w|\,I_m + |w|\,|U_m|\}$
Clearly, we can see that $|T_{\ell_w}| < B'_{\ell_w}$ and hence we can conclude that $\rho(T_{\ell_w}) \leqslant \rho(B'_{\ell_w})$. By assumption that $B'_{\ell_w} = \bar{M}_m^{-1}\bar{N}_m$, it can be proved that $\rho(B'_{\ell_w}) < 1$. Hence, $\rho(T_{\ell_w}) < 1$, which completes the proof. □

Theorem 3.1. *[23] Let A be the coefficient matrix of the following system of equations*

$$Ax = b \tag{3.5}$$

, and be an element of the matrix set G of G=$\{A \in C^{n,n} / |B^A| = |L| + |U|$ *is a* $GCO(s,q) -$ *matrix*$\}$*, Then, for any r and* ω *satisfying:*

$$\begin{cases} 0 < \omega < 2 \\ (|1-\omega+r| + |1-r|)^q (1+|1-\omega|)^s \bar{\mu}^p < \pi(2-\omega)^p, \end{cases} \tag{3.6}$$

where $p := s + q$ *and* $\bar{\mu} := \rho(B^A)$*, the block AOR method, applied to the matrix of equation Eq (3.5) ,converges* $(\rho(T_{\ell_{r,\omega}}) < 1)$.

Proof. The block AOR method, applied to the matrix equation $Ax = b$,is, as usual, defined by

$$\begin{cases} x^{(m+1)} = \ell_{r,\omega} x^{(m)} + c_{r,\omega},\ m = 0,1,2,\ldots \\ \ell_{r,\omega} \equiv \ell^A_{r,\omega} := (I - rL)^{-1}[(I-\omega)I + (\omega - r)L + \omega U] \\ c_{r,\omega} := \omega(I - rL)^{-1}D^{-1}b \end{cases} \tag{3.7}$$

when $(r,\omega) = (0,1),(1,1)$ and (ω,ω) the AOR method is reduced to Jacobi, Gauss-Seidel, and SOR methods, in that order.

$$\ell_{r,\omega} = I - \omega(I - rL)^{-1}D^{-1}A and c_{r,\omega} = (I - \ell_{r,\omega})A^{-1}b \tag{3.8}$$

The convergence domains for the following preconditioned iterative methods,

$$\Im_r = (I - rL)^{-1}D^{-1}A = (I - rL)^{-1}(I - L - U) \tag{3.9}$$

can be determined if and only if $\lambda and \tau$ are eigenvalues of $\ell_{r,\omega} and\ \Im_r$ respectively with,

$$\lambda = 1 - \omega\tau \tag{3.10}$$

Moreover, it can be seen that

$$\rho(\ell_{r,\omega}) \prec 1 iff \tau \neq 0 and |1 - \omega\tau| \prec 1. \tag{3.11}$$

Let the matrices $M, N, \tilde{M} and \tilde{N}$ be defined by

$$\begin{cases} M := I - zL - \hat{z}U \\ N := (1-z)L + (1-\hat{z})U \\ \tilde{M} := I - |z|\,|L| - |\hat{z}|\,|U| \\ \tilde{N} := |1-z|\,|L| + |1-\hat{z}|\,|U| \end{cases} \tag{3.12}$$

where z and $\hat{z}$ are any complex-valued parameters. Satisfying

$$(|z| + |1-z|)^q (|\hat{z}| + |1-\hat{z}|)^s \bar{\mu}^p \prec 1, \tag{3.13}$$

there holds

$$\rho(M^{-1}N) \leq \rho(\tilde{M}^{-1}\tilde{N}) \prec 1, \tag{3.14}$$

assuming now that z satisfies

$$z = 1 - (1-\hat{z})(1-r), \hat{z} \neq 1 \tag{3.15}$$

it is thus apparent that, for all $z and \hat{z}$ satisfying (3.12) and (3.14), indicates that

$$\left| \frac{1-\tau}{1 + \frac{\hat{z}}{1-\hat{z}}\tau} \right| \prec 1, \tag{3.16}$$

for any eigenvalue $\tau of \Im_r$ of (3.9), this relationship implies that $\tau \neq 0$ and that, for $\hat{z} \prec 1$,

$$\frac{2\,\mathrm{Re}(\tau)}{|\tau|^2} \succ 1 - \frac{\hat{z}}{1-\hat{z}}. \tag{3.17}$$

A combination of (3.11) and (3.17) yields that, for all satisfying

$$\begin{cases} (|z| + |1-z|)^q (|\hat{z}| + |1-\hat{z}|)^s \bar{\mu}^p < 1 \\ z = 1 - (1-\hat{z})(1-r) \\ \hat{z} \leq (1-\omega)/(2-\omega),\ 0 < \omega < 2, \end{cases} \tag{3.18}$$

when $z and \hat{z}$ is set, $z = \frac{1-\omega+r}{2-\omega}$ and $\hat{z} = \frac{1-\omega}{2-\omega}$ observe that (3.18) holds for all $r and \omega$ satisfying (3.6), and hence the proof follows. □

4. FORMULATION OF NINE EXPLICIT GROUP SOR METHOD

In the group 9-point method, the mesh points are grouped in blocks of nine. The points involved in updating are also using the standard five-point formula. The solution domain is divided into groups of nine points. In matrix notation, the system of nine equations can be written as

$$\begin{bmatrix} 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 4 & 0 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 4 & -1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 4 & -1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 4 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 4 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 \end{bmatrix} \begin{bmatrix} u_{ij} \\ u_{i+1,j} \\ u_{i+2,j} \\ u_{i,j+1} \\ u_{i+1,j+1} \\ u_{i+2,j+1} \\ u_{i,j+2} \\ u_{i+1,j+2} \\ u_{i+2,j+2} \end{bmatrix} = \begin{bmatrix} u_{i-1,j} + u_{i,j-1} - h^2 f_{ij} \\ u_{i+1,j-1} - h^2 f_{i+1,j} \\ u_{i+2,j-1} + u_{i+3,j} - h^2 f_{i+2,j} \\ u_{i-1,j+1} - h^2 f_{i,j+1} \\ -h^2 f_{i+1,j+1} \\ u_{i+3,j+1} - h^2 f_{i+2,j+1} \\ u_{i-1,j+2} + u_{i,j+3} - h^2 f_{i,j+2} \\ u_{i+1,j+3} - h^2 f_{i+1,j+2} \\ u_{i+3,j+2} + u_{i+2,j+3} - h^2 f_{i+2,j+2} \end{bmatrix} \tag{4.1}$$

The inverse form of the above system is:

$$\begin{bmatrix} u_{ij} \\ u_{i+1,j} \\ u_{i+2,j} \\ u_{i,j+1} \\ u_{i+1,j+1} \\ u_{i+2,j+1} \\ u_{i,j+2} \\ u_{i+1,j+2} \\ u_{i+2,j+2} \end{bmatrix} = \begin{bmatrix} \frac{67}{224} & \frac{22}{224} & \frac{7}{224} & \frac{22}{224} & \frac{14}{224} & \frac{22}{224} & \frac{22}{224} & \frac{22}{224} & \frac{22}{224} \\ \frac{11}{112} & \frac{37}{112} & \frac{11}{112} & \frac{7}{112} & \frac{14}{112} & \frac{7}{112} & \frac{3}{112} & \frac{5}{112} & \frac{3}{112} \\ \frac{7}{224} & \frac{22}{224} & \frac{67}{224} & \frac{6}{224} & \frac{14}{224} & \frac{22}{224} & \frac{3}{224} & \frac{6}{224} & \frac{7}{224} \\ \frac{11}{112} & \frac{7}{112} & \frac{3}{112} & \frac{37}{112} & \frac{14}{112} & \frac{5}{112} & \frac{11}{112} & \frac{7}{112} & \frac{3}{112} \\ \frac{1}{16} & \frac{2}{16} & \frac{1}{16} & \frac{2}{16} & \frac{6}{16} & \frac{2}{16} & \frac{1}{16} & \frac{2}{16} & \frac{1}{16} \\ \frac{3}{112} & \frac{7}{112} & \frac{11}{112} & \frac{5}{112} & \frac{14}{112} & \frac{37}{112} & \frac{3}{112} & \frac{7}{112} & \frac{11}{112} \\ \frac{7}{224} & \frac{6}{224} & \frac{3}{224} & \frac{22}{224} & \frac{14}{224} & \frac{6}{224} & \frac{67}{224} & \frac{22}{224} & \frac{7}{224} \\ \frac{3}{112} & \frac{5}{112} & \frac{3}{112} & \frac{7}{112} & \frac{14}{112} & \frac{7}{112} & \frac{11}{112} & \frac{37}{112} & \frac{11}{112} \\ \frac{3}{224} & \frac{6}{224} & \frac{7}{224} & \frac{6}{224} & \frac{14}{224} & \frac{22}{224} & \frac{7}{224} & \frac{22}{224} & \frac{67}{224} \end{bmatrix} \begin{bmatrix} u_{i-1,j} + u_{i,j-1} - h^2 f_{ij} \\ u_{i+1,j-1} - h^2 f_{i+1,j} \\ u_{i+2,j-1} + u_{i+3,j} - h^2 f_{i+2,j} \\ u_{i-1,j+1} - h^2 f_{i,j+1} \\ -h^2 f_{i+1,j+1} \\ u_{i+3,j+1} - h^2 f_{i+2,j+1} \\ u_{i-1,j+2} + u_{i,j+3} - h^2 f_{i,j+2} \\ u_{i+1,j+3} - h^2 f_{i+1,j+2} \\ u_{i+3,j+2} + u_{i+2,j+3} - h^2 f_{i+2,j+2} \end{bmatrix}$$

Hence, the explicit 9-point group iterative equations are given by:

$$\begin{aligned} u_{ij} &= \frac{1}{224}[67t_1 + 22t_2 + 7t_7 - 14t_0 + 6t_5 + 3t_6], \\ u_{i+1,j} &= \frac{1}{112}[37t_{19} + 11t_8 + 7t_9 - 14t_0 + 5t_{20} + 3t_{10}], \\ u_{i+2,j} &= \frac{1}{224}[67t_3 + 22t_{13} + 7t_{18} - 14t_0 + 6t_{14} + 3t_4], \\ u_{i,j+1} &= \frac{1}{112}[37t_{21} + 11t_{15} + 7t_{16} - 14t_0 + 5t_{22} + 3t_{17}], \\ u_{i+1,j+1} &= \frac{1}{16}[2t_{11} - 6t_0 + t_{12}], \\ u_{i+2,j+1} &= \frac{1}{112}[37t_{22} + 11t_{17} + 7t_{16} - 14t_0 + 5t_{21} + 3t_{15}], \\ u_{i,j+2} &= \frac{1}{224}[67t_4 + 22t_{14} + 7t_{18} - 14t_0 + 6t_{13} + 3t_3], \\ u_{i+1,j+2} &= \frac{1}{112}[37t_{20} + 11t_{10} + 7t_9 - 14t_0 + 5t_{19} + 3t_8], \\ u_{i+2,j+2} &= \frac{1}{224}[67t_6 + 22t_5 + 7t_7 - 14t_0 + 6t_2 + 3t_1] \end{aligned} \tag{4.2}$$

where:

$t_0 = h^2 f_{i+1,j+1}$, $t_1 = u_{i-1,j} + u_{i,j-1} - h^2 f_{i+1,j+1}$,

$t_2 = u_{i+1,j-1} + u_{i-1,j+1} - h^2 f_{i+1,j} - h^2 f_{i,j+1},$ $t_3 = u_{i+2,j-1} + u_{i+3,j} - h^2 f_{i+2,j},$
$t_4 = u_{i-1,j+2} + u_{i,j+3} - h^2 f_{i,j+2},$ $t_5 = u_{i+3,j+1} + u_{i+1,j+3} - h^2 f_{i+2,j+1} - h^2 f_{i+1,j+2},$
$t_6 = u_{i+3,j+2} + u_{i+2,j+3} - h^2 f_{i+2,j+2},$ $t_7 = t_3 + t_4,$
$t_8 = t_1 + t_3,$ $t_9 = u_{i+3,j+1} + u_{i-1,j+1} - h^2 f_{i+2,j+1} - h^2 f_{i,j+1},$
$t_{10} = t_4 + t_6,$ $t_{11} = t_2 + t_5,$
$t_{12} = t_8 + t_{10},$ $t_{13} = u_{i+1,j-1} + u_{i+3,j+1} - h^2 f_{i+1,j} - h^2 f_{i+2,j+1},$
$t_{14} = u_{i-1,j+1} + u_{i+1,j+3} - h^2 f_{i,j+1} - h^2 f_{i+1,j+2},$ $t_{15} = t_1 + t_4,$
$t_{16} = u_{i+1,j-1} + u_{i+1,j+3} - h^2 f_{i+1,j} - h^2 f_{i+1,j+2},$ $t_{17} = t_3 + t_6,$
$t_{18} = t_1 + t_6,$ $t_{19} = u_{i+1,j-1} - h^2 f_{i+1,j},$
$t_{20} = u_{i+1,j+3} - h^2 f_{i+1,j+2},$ $t_{21} = u_{i-1,j+1} - h^2 f_{i,j+1},$
$t_{22} = u_{i+3,j+1} - h^2 f_{i+2,j+1}.$

The nine-point group was also performed by Evans & Yousif [10]. This method proceeds with an iterative evaluation of the solution in nine points throughout the entire solution domain using all nine equations (4.2). The process is continued until convergence is achieved. It is well known that the AOR method is a general form of SOR method due to the two accelerating parameters which including in its scheme. In the following section, the combination of nine-point group scheme with AOR method will be formulated to obtain more superior method than the original one.

5. FORMULATION OF NINE EXPLICIT GROUP AOR METHOD

In this section, we present the construction of a nine-point group AOR iterative scheme. We know that AOR iterative scheme can easily be generated if we know the corresponding SOR iterative scheme. For example, from equation (4.2), we can build the nine-point SOR iterative scheme as follows:

$$
\begin{aligned}
u_{ij}^{(k+1)} &= \frac{1}{224}[\omega(67t_1 + 22t_2 + 7t_7 - 14t_0 + 6t_5 + 3t_6)] + (1-\omega)u_{ij}^{(k)},\\
u_{i+1,j}^{(k+1)} &= \frac{1}{112}[\omega(37t_{19} + 11t_8 + 7t_9 - 14t_0 + 5t_{20} + 3t_{10})] + (1-\omega)u_{i+1,j}^{(k)},\\
u_{i+2,j}^{(k+1)} &= \frac{1}{224}[\omega(67t_3 + 22t_{13} + 7t_{18} - 14t_0 + 6t_{14} + 3t_4)] + (1-\omega)u_{i+2,j}^{(k)},\\
u_{i,j+1}^{(k+1)} &= \frac{1}{112}[\omega(37t_{21} + 11t_{15} + 7t_{16} - 14t_0 + 5t_{22} + 3t_{17})] + (1-\omega)u_{i,j+1}^{(k)},\\
u_{i+1,j+1}^{(k+1)} &= \frac{1}{16}[\omega(2t_{11} - 6t_0 + t_{12})] + (1-\omega)u_{i+1,j+1}^{(k)},\\
u_{i+2,j+1}^{(k+1)} &= \frac{1}{112}[\omega(37t_{22} + 11t_{17} + 7t_{16} - 14t_0 + 5t_{21} + 3t_{15})] + (1-\omega)u_{i+2,j+1}^{(k)},\\
u_{i,j+2}^{(k+1)} &= \frac{1}{224}[\omega(67t_4 + 22t_{14} + 7t_{18} - 14t_0 + 6t_{13} + 3t_3)] + (1-\omega)u_{i,j+2}^{(k)},
\end{aligned}
$$

$$u^{(k+1)}_{i+1,j+2} = \frac{1}{112}[\omega(37t_{20}+11t_{10}+7t_9-14t_0+5t_{19}+3t_8)]+(1-\omega)u^{(k)}_{i+1,j+2},$$
$$u^{(k+1)}_{i+2,j+2} = \frac{1}{224}[\omega(67t_6+22t_5+7t_7-14t_0+6t_2+3t_1)]+(1-\omega)u^{(k)}_{i+2,j+2}, \tag{5.1}$$

where

$$t_0 = h^2 f_{i+1,j+1}, \qquad t_1 = u^{(k+1)}_{i-1,j} + u^{(k+1)}_{i,j-1} - h^2 f_{i,j},$$
$$t_2 = u^{(k+1)}_{i+1,j-1} + u^{(k+1)}_{i-1,j+1} - h^2 f_{i+1,j} - h^2 f_{i,j+1}, \qquad t_3 = u^{(k+1)}_{i+2,j-1} + u^{(k)}_{i+3,j} - h^2 f_{i+3,j} - h^2 f_{i+2,j},$$
$$t_4 = u^{(k+1)}_{i-1,j+2} + u^{(k)}_{i,j+3} - h^2 f_{i,j+2}, \qquad t_5 = u^{(k)}_{i+3,j+1} + u^{(k)}_{i+1,j+3} - h^2 f_{i+2,j+1} - h^2 f_{i+1,j+2},$$
$$t_6 = u^{(k)}_{i+2,j+2} + u^{(k)}_{i+2,j+3} - h^2 f_{i+2,j+2}, \qquad t_7 = t_3 + t_4,$$
$$t_8 = t_1 + t_3, \qquad t_9 = u^{(k)}_{i+3,j+1} + u^{(k+1)}_{i-1,j+1} - h^2 f_{i+2,j+1} - h^2 f_{i,j+1},$$
$$t_{10} = t_4 + t_6, \qquad t_{11} = t_2 + t_5,$$
$$t_{12} = t_8 + t_{10}, \qquad t_{13} = u^{(k+1)}_{i+1,j-1} + u^{(k)}_{i+3,j+1} - h^2 f_{i+1,j} - h^2 f_{i+2,j+1},$$
$$t_{14} = u^{(k+1)}_{i-1,j+1} + u^{(k)}_{i+1,j+3} - h^2 f_{i,j+1} - h^2 f_{i+1,j+2}, \qquad t_{15} = t_1 + t_4,$$
$$t_{16} = u^{(k+1)}_{i+1,j-1} + u^{(k)}_{i+1,j+3} - h^2 f_{i+1,j} - h^2 f_{i+1,j+2}, \qquad t_{17} = t_3 + t_6,$$
$$t_{18} = t_1 + t_6, \qquad t_{19} = u^{(k+1)}_{i+1,j-1} - h^2 f_{i+1,j},$$
$$t_{20} = u^{(k)}_{i+1,j+3} - h^2 f_{i+1,j+2}, \qquad t_{21} = u^{(k+1)}_{i-1,j+1} - h^2 f_{i,j+1},$$
$$t_{22} = u^{(k)}_{i+3,j+1} - h^2 f_{i+2,j+1},$$

The coefficient for expressions $u^{(k+1)}_{i-1,j}, u^{(k+1)}_{i+1,j-1}, u^{(k+1)}_{i+1,j-1}, u^{(k+1)}_{i+2,j-1}, u^{(k+1)}_{i-1,j+2}$ and $u^{(k+1)}_{i-1,j+1}$ contained in L. To construct the AOR scheme, we have to change these expressions to $u^{(k)}_{i-1,j}, u^{(k)}_{i+1,j-1}, u^{(k)}_{i+1,j-1}, u^{(k)}_{i+2,j-1}, u^{(k)}_{i-1,j+2}$ and $u^{(k)}_{i-1,j+1}$. After that, add expressions $\alpha r(u^{(k+1)}_{i-1,j} - u^{(k)}_{i-1,j}), \alpha r(u^{(k+1)}_{i,j-1} - u^{(k)}_{i,j-1}), \alpha r(u^{(k+1)}_{i+1,j-1} - u^{(k)}_{i+1,j-1}),$
$\alpha r(u^{(k+1)}_{i+2,j-1} - u^{(k)}_{i+2,j-1}), \alpha r(u^{(k+1)}_{i-1,j+2} - u^{(k)}_{i-1,j+2})$ and $\alpha r(u^{(k+1)}_{i-1,j+1} - u^{(k)}_{i-1,j+1})$ correspond the SOR iterative scheme, where the coefficient is for those expressions. Hence, nine-point group AOR iterative scheme can be written as:

$$u^{(k+1)}_{ij} = \frac{1}{224}[\omega(67t_1+22t_2+7t_7-14t_0+6t_5+3t_6)+r(67c_7+22c_8+7c_9]+(1-\omega)u^{(k)}_{ij},$$
$$u^{(k+1)}_{i+1,j} = \frac{1}{112}[\omega(37t_{19}+11t_8+7t_9-14t_0+5t_{20}+3t_{10})+r(37c_3+11c_{10}+7c_4+3c_6]+(1-\omega)u^{(k)}_{i+1,j},$$
$$u^{(k+1)}_{i+2,j} = \frac{1}{224}[\omega(67t_3+22t_{13}+7t_{18}-14t_0+6t_{14}+3t_4)+r(67c_5+22c_3+7c_7+6c_4+3c_6]+(1-\omega)u^{(k)}_{i+2,j},$$
$$u^{(k+1)}_{i,j+1} = \frac{1}{112}[\omega(37t_{21}+11t_{15}+7t_{16}-14t_0+5t_{22}+3t_{17})+r(37c_4+11c_{12}+7c_3+3c_5]+(1-\omega)u^{(k)}_{i,j+1},$$
$$u^{(k+1)}_{i+1,j+1} = \frac{1}{16}[\omega(2t_{11}-6t_0+t_{12})+r(2c_8+c_{11}]+(1-\omega)u^{(k)}_{i+1,j+1},$$
$$u^{(k+1)}_{i+2,j+1} = \frac{1}{112}[\omega(37t_{22}+11t_{17}+7t_{16}-14t_0+5t_{21}+3t_{15})+r(11c_5+7c_3+5c_4+3c_{12}]+(1-\omega)u^{(k)}_{i+2,j+1},$$

$$u^{(k+1)}_{i,j+2} = \frac{1}{224}[\omega(67t_4+22t_{14}+7t_{18}-14t_0+6t_{13}+3t_3)+r(67c_6+22c_4+7c_7+6c_3+3c_5]+(1-\omega)u^{(k)}_{i,j+2},$$
$$u^{(k+1)}_{i+1,j+2} = \frac{1}{112}[\omega(37t_{20}+11t_{10}+7t_9-14t_0+5t_{19}+3t_8)+r(11c_6+7c_4+5c_3+3c_{10}]+(1-\omega)u^{(k)}_{i+1,j+2},$$
$$u^{(k+1)}_{i+2,j+2} = \frac{1}{224}[\omega(67t_6+22t_5+7t_7-14t_0+6t_2+3t_1)+r(7c_9+6c_8+3c_7]+(1-\omega)u^{(k)}_{i+2,j+2} \tag{5.2}$$

where

$t_0 = h^2 f_{i+1,j+1}$, $t_1 = u^{(k)}_{i-1,j} + u^{(k)}_{i,j-1} - h^2 f_{i,j}$,

$t_2 = u^{(k)}_{i+1,j-1} + u^{(k)}_{i-1,j+1} - h^2 f_{i+1,j} - h^2 f_{i,j+1}$, $t_3 = u^{(k)}_{i+2,j-1} + u^{(k)}_{i+3,j} - h^2 f_{i+3,j} - h^2 f_{i+2,j}$,

$t_4 = u^{(k)}_{i-1,j+2} + u^{(k)}_{i,j+3} - h^2 f_{i,j+2}$, $t_5 = u^{(k)}_{i+3,j+1} + u^{(k)}_{i+1,j+3} - h^2 f_{i+2,j+1} - h^2 f_{i+1,j+2}$,

$t_6 = u^{(k)}_{i+2,j+2} + u^{(k)}_{i+2,j+3} - h^2 f_{i+2,j+2}$, $t_7 = t_3 + t_4$,

$t_8 = t_1 + t_3$ $t_9 = u^{(k)}_{i+3,j+1} + u^{(k)}_{i-1,j+1} - h^2 f_{i+2,j+1} - h^2 f_{i,j+1}$,

$t_{10} = t_4 + t_6$, $t_{11} = t_2 + t_5$,

$t_{12} = t_8 + t_{10}$, $t_{13} = u^{(k)}_{i+1,j-1} + u^{(k)}_{i+3,j+1} - h^2 f_{i+1,j} - h^2 f_{i+2,j+1}$,

$t_{14} = u^{(k)}_{i-1,j+1} + u^{(k)}_{i+1,j+3} - h^2 f_{i,j+1} - h^2 f_{i+1,j+2}$, $t_{15} = t_1 + t_4$,

$t_{16} = u^{(k)}_{i+1,j-1} + u^{(k)}_{i+1,j+3} - h^2 f_{i+1,j} - h^2 f_{i+1,j+2}$, $t_{17} = t_3 + t_6$,

$t_{18} = t_1 + t_6$, $t_{19} = u^{(k)}_{i+1,j-1} - h^2 f_{i+1,j}$,

$t_{20} = u^{(k)}_{i+1,j+3} - h^2 f_{i+1,j+2}$, $t_{21} = u^{(k)}_{i-1,j+1} - h^2 f_{i,j+1}$,

$t_{22} = u^{(k)}_{i+3,j+1} - h^2 f_{i+2,j+1}$, $c_1 = u^{(k+1)}_{i-1,j} - u^{(k)}_{i-1,j}$,

$c_2 = u^{(k+1)}_{i,j-1} - u^{(k)}_{i,j-1}$, $c_3 = u^{(k+1)}_{i+1,j-1} - u^{(k)}_{i+1,j-1}$,

$c_4 = u^{(k+1)}_{i-1,j+1} - u^{(k)}_{i-1,j+1}$, $c_5 = u^{(k+1)}_{i+2,j-1} - u^{(k)}_{i+2,j-1}$,

$c_6 = u^{(k+1)}_{i-1,j+2} - u^{(k)}_{i-1,j+2}$, $c_7 = c_1 + c_2$,

$c_8 = c_3 + c_4$, $c_9 = c_5 + c_6$,

$c_{10} = c_5 + c_7$, $c_{11} = c_{10} + c_6$,

$c_{12} = c_6 + c_7$.

After constructing the AOR scheme for each method, we see that the nine-point group iterative scheme is the most complicated scheme. This is because it involves many variables and expressions. However, as mentioned in section 2, this proposed scheme has more accuracy than the other four-point group scheme. Furthermore, the superiority of the 9-point EGAOR scheme for solving the 2D Poisson and Laplace equations in terms of the number of iterations and execution time will be justified from the following numerical experiments.

6. ILLUSTRATIVE EXAMPLES

To compare the four-point group and nine-point group iterative methods, some numerical experiments have been performed. These methods were implemented to two model examples like the following,

Example 6.1. *Consider the Poisson equation,*

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = (x^2 + y^2)e^{xy}, \tag{6.1}$$

with $u(x,0) = u(0,y) = 1,\ u(x,1) = e^x, u(1,y) = e^y, 0 \leq x, y \leq 1.$
The exact solution for this problem is $u(x,y) = e^{xy}$.

Example 6.2. *Consider the Laplace equation,*

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, (x,y) \in \Omega, \tag{6.2}$$

with the Dirichlet boundary conditions

$$\begin{aligned} &u(x,0) = \sin \pi x && 0 \leq x \leq 1, \\ &u(0,y) = u(1,y) = u(x,1) = 0 && 0 \leq x, y \leq 1, \end{aligned}$$

The exact solution for this problem is $u(x,y) = \sin \pi x \{\cosh \pi y - \left(\frac{\cosh \pi}{\sinh \pi}\right) \sinh \pi y\}$.

The tolerance used was $\varepsilon = 10^{-5}$, and the parameter acceleration ω was chosen to give the smallest number of iterations between 1 and 2.The computer processing unit was Intel(R) Core(TM) i7- 7500U CPU with a memory of 8 Gb, and the software used to implement and generate the results was MATLAB. We have computed the average absolute errors and record the number of iterations and elapsed time for convergence with different sizes of grids. Furthermore, the spectral radius of the resulted matrices will be calculated for all tested group iterative methods. Due to the Matlab program's use, a larger number of processors were used in this experiment 12, 46,86, 106, 146, 186, 226, 350, and 426.

Table 1: Comparison of four-point EGSOR and nine-point EGSOR iterative methods of the Poisson equation (Example 1)

	Four-Point Group (SOR)					Nine-Point Group (SOR)				
N	ω	k	t	$\rho(J)$	E	ω	k	t	$\rho(J)$	E
12	1.3520	22	0.021156	0.9184	$6.2312e-06$	1.3520	13	0.0135	0.8776	$3.5391e-06$
46	1.7846	76	0.040845	0.9951	$9.6553e-06$	1.7846	51	0.0197	0.9927	$6.9877e-06$
86	1.8797	138	0.106233	0.9986	$9.3103e-06$	1.8797	111	0.0454	0.9980	$8.7251e-06$
106	1.9015	169	0.174041	0.9991	$8.1532e-06$	1.9015	117	0.0700	0.9987	$8.2433e-06$
146	1.9277	227	0.181077	0.9995	$9.1567e-06$	1.9277	182	0.2253	0.9993	$8.0770e-06$
186	1.9429	287	0.227092	0.9997	$9.7462e-06$	1.9429	206	0.3259	0.9996	$9.4477e-06$
226	1.9528	344	0.309206	0.9998	$9.6445e-06$	1.9528	252	0.3628	0.9997	$9.6069e-06$
350	1.9693	528	0.796697	0.9999	$9.5333e-06$	1.9693	418	0.7957	0.9999	$9.7627e-06$
426	1.9747	644	2.142877	0.9999	$9.3794e-06$	1.9747	476	1.5589	0.9999	$9.5685e-06$

Table 2: Comparison of four-point EGAOR and nine-point EGAOR iterative methods of the Poisson equation (Example 1)

Four-Point Group (AOR)						
N	ω	r	k	t	$\rho(J)$	E
12	1.4331	1.4537	17	$0.0085s$	0.9184	$9.7737e-06$
46	$1.792-1.763$	1.8270	69	$0.0261s$	0.9951	$9.8840e-06$
86	$1.847-1.853$	1.9050	124	$0.0920s$	0.9986	$9.9985e-06$
106	$1.880-1.867$	1.9220	152	$0.0990s$	0.9991	$9.8045e-06$
146	$1.903-1.843$	1.9430	208	$0.1782s$	0.9995	$9.8531e-06$
186	$1.919-1.917$	1.9550	263	$0.2521s$	0.9997	$9.9905e-06$
226	$1.931-1.927$	1.9620	318	$0.4317s$	0.9998	$9.9684e-06$
350	$1.954-1952$	1.9753	488	$0.9133s$	0.9999	$9.9444e-06$
426	$1.935-1.929$	1.9800	587	$1.9163s$	0.9999	$9.9413e-06$
Nine-Point Group (AOR)						
N	ω	r	k	t	$\rho(J)$	E
12	$1.365-1.305$	1.3780	13	$0.0016s$	0.8776	0
46	$1.779-1.760$	1.7780	48	$0.0346s$	0.9927	$9.9268e-06$
86	$1.800-1.846$	1.8840	100	$0.0473s$	0.9986	$9.9778e-06$
106	$1.871-1.895$	1.9000	111	$0.1890s$	0.9987	$9.2575e-06$
146	$1.901-1.879$	1.9300	166	$0.3660s$	0.9987	$9.9082e-06$
186	$1.939-1.900$	1.9410	194	$0.5760s$	0.9997	$9.1504e-06$
226	$1.898-1.895$	1.9510	231	$0.6791s$	0.9997	$9.9932e-06$
350	$1.915-1913$	1.9710	384	$0.9233s$	0.9999	$9.6516e-06$
426	$1.905-1.861$	1.9740	404	$2.7046s$	0.9999	$9.9758e-06$

Table 3: Comparison of four-point EGSOR and nine-point EGSOR iterative methods of the Laplace equation (Example 2)

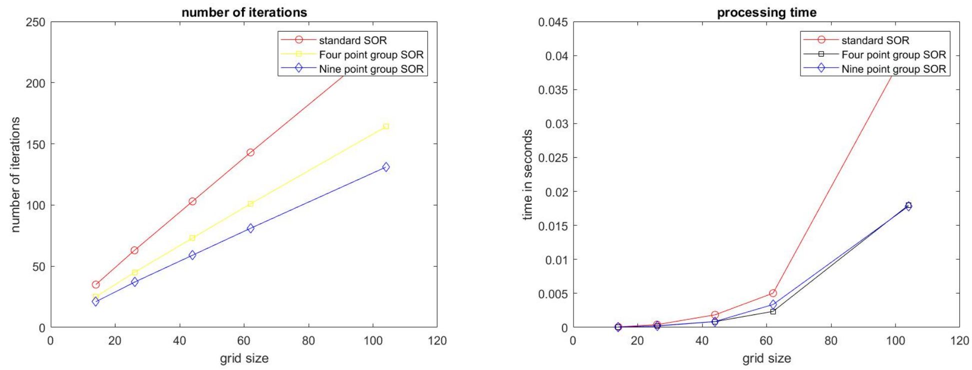
	Four-Point Group (SOR)					Nine-Point Group (SOR)				
N	ω	k	t	$\rho(J)$	E	ω	k	t	$\rho(J)$	E
12	1.3520	17	0.020608	0.9184	$8.9420e-06$	1.3520	12	0.0111	0.8776	$5.8474e-06$
46	1.7846	59	0.024852	0.9951	$9.3668e-06$	1.7846	48	0.0249	0.9927	$9.3343e-06$
86	1.8797	109	0.074012	0.9986	$8.8669e-06$	1.8797	91	0.0616	0.9980	$7.0504e-06$
106	1.9015	136	0.195734	0.9991	$9.6677e-06$	1.9015	113	0.1426	0.9987	$9.3321e-06$
146	1.9277	187	0.350675	0.9995	$8.9961e-06$	1.9277	152	0.2990	0.9993	$9.8924e-06$
186	1.9429	238	0.370068	0.9997	$8.7862e-06$	1.9429	195	0.5102	0.9996	$9.3307e-06$
226	1.9528	288	0.485994	0.9998	$9.4881e-06$	1.9528	240	0.6482	0.9997	$9.9608e-06$
350	1.9693	445	0.917831	0.9999	$9.7127e-06$	1.9693	371	1.1700	0.9999	$9.2169e-06$
426	1.9747	545	2.520613	0.9999	$9.6303e-06$	1.9747	448	2.6002	0.9999	$9.9481e-06$

Table 4: Comparison of four-point EGAOR and nine-point EGAOR iterative methods of the Laplace equation (Example 2)

Four-Point Group (AOR)						
N	ω	r	k	t	$\rho(J)$	E
12	1.4331	1.4537	16	0,0310	0.9184	$9.7737e-06$
46	$1.792-1.763$	1.8270	60	$0.0267s$	0.9951	$9.8840e-06$
86	$1.847-1.853$	1.9050	107	$0.0707s$	0.9986	$9.9985e-06$
106	$1.880-1.867$	1.9220	130	$0.1971s$	0.9991	$9.8045e-06$
146	$1.903-1.843$	1.9430	165	$0.3293s$	0.9995	$9.8531e-06$
186	$1.919-1.917$	1.9550	216	$0.3606s$	0.9997	$9.9905e-06$
226	$1.931-1.927$	1.9620	254	$0.7630s$	0.9998	$9.9684e-06$
350	$1.954-1952$	1.9753	384	$1.1427s$	0.9999	$9.9444e-06$
426	$1.935-1.929$	1.9800	456	$2.3024s$	0.9999	$9.9413e-06$
Nine-Point Group (AOR)						
N	ω	r	k	t	$\rho(J)$	E
12	$1.365-1.305$	1.3780	13	$0.0469s$	0.8776	$5.5395e-06$
46	$1.779-1.760$	1.7780	46	$0.0696s$	0.9927	$9.2977e-06$
86	$1.800-1.846$	1.8840	83	$0.0645s$	0.9986	$9.6766e-06$
106	$1.871-1.895$	1.9000	101	$0.1478s$	0.9987	$9.9325e-06$
146	$1.901-1.879$	1.9300	139	$0.3278s$	0.9987	$9.6865e-06$
186	$1.939-1.900$	1.9410	186	$0.6307s$	0.9997	$9.0283e-06$
226	$1.898-1.895$	1.9510	204	$0.6413s$	0.9997	$9.9028e-06$
350	$1.915-1913$	1.9710	327	$1.1698s$	0.9999	$9.9259e-06$
426	$1.905-1.861$	1.9740	366	$2.4964s$	0.9999	$9.8728e-06$

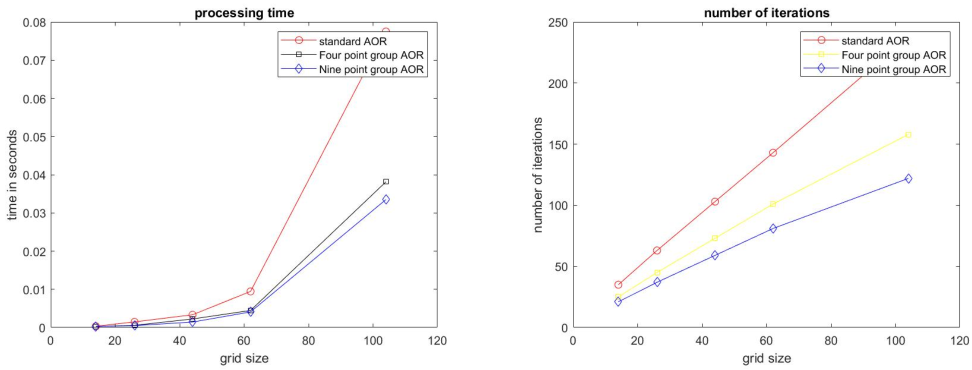
Tables 1 and 3 represent the Comparison of 4-point EGSOR and 9-point EGSOR, and

Table 2,4 represents the Comparison of 4-point EGAOR and 9-point EGAOR, where N: the number of squares, r: the second parameter of AOR, k: the number of iterations, e: the maximum errors and T represents CPU time. The convergence of the iteration methods relies on the spectral radius, which is the largest moduli of the iteration matrix's eigenvalues. Through figures 1,2,3, and 4, the progress of the nine-point EGAOR iterative method in reducing time and number of iterations among the other studied methods becomes clear.



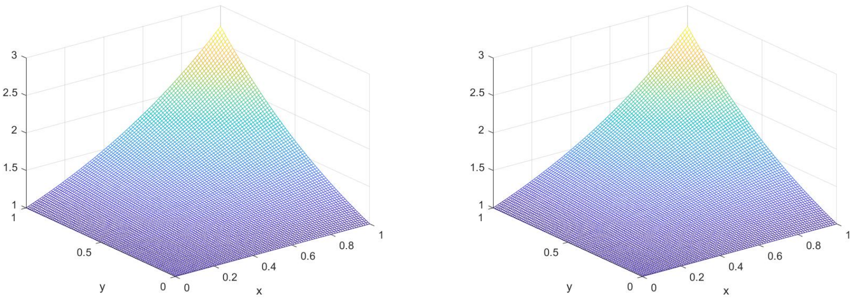
(a) **(b)**

Figure 1: Comparison of the CPU time (t) for Standard Five-Point SOR, Four-point EGSOR, and Nine points EGSOR iterative methods of the Poisson equation.



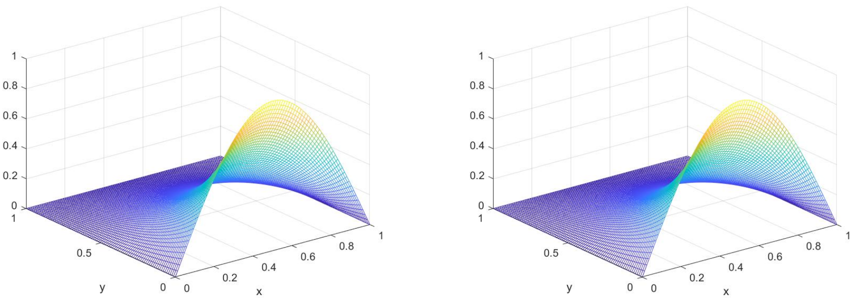
(a) **(b)**

Figure 2: Comparison of the CPU time (t) for Standard Five-Point AOR, Four-point EGAOR, and Nine points EGAOR iterative methods of the Poisson equation.



(a) **(b)**

Figure 3: (a) approximation solution of EGSOR for N=86,(b) approximation solution EGAOR for N=86 of the Poisson equation.



(a) **(b)**

Figure 4: (a) approximation solution of EGSOR for N=86,(b) approximation solution EGAOR for N=86 of the Laplace equation.

7. DISCUSSION OF RESULTS

The results in tables 1 and 3 reveal that the nine-point group SOR is superior to the four-point group SOR in solving both Laplace and Poisson equations. Furthermore, we included our experiments with the standard five-point to further illustrate the progress of group methods. Fig.1 compares the number of iterations between these methods. The figure explained the minimum number of iterations given in the prerequired nine-point group SOR method, and the difference became apparent when the value of N increased. Tables 2, 4, and Fig.2 show that the 4-pointEGAOR scheme is slightly higher than the nine-point EG in terms of the number of iterations and execution time, indicating that the 9-point EGAOR is more efficient than the corresponding 4-point EGAOR method. It can be observed that among the two group methods presented, the 9-pointEGAOR scheme requires lesser execution timings than the existing 4-pointEGAOR method.

The experiments above show the superiority of the 9-pointEGAOR scheme for solving the 2D Poisson and Laplace equations. Furthermore, the surveillance in Tables 1,2,3 and 4 shows that the 9-pointEGAOR method has the best convergent rate compared with the 4-point EGSOR, 9-pointEGSOR, and 4-pointEGAOR scheme when applied to the examples. It is noticeable through all experiments that the spectral radius is always less than one, which is consistent with the convergence theories of these methods in previous studies [20].

8. CONCLUSIONS

The results presented in Tables 1,2,3, and 4, along with the results in Fig1,2,3and 4, indicate a significant reduction of the total computing effort of the 9-point EGAOR method for solving elliptic PDEs. Therefore, we can conclude that the 9-point EGAOR approach provides significant economies for solving several PDEs. This paper provides an alternative group iterative method that can solve more complex problems useful to study in the future.

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