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Solving the Septic equation $x^7+\alpha x+\beta=0$

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Abstract

According to Galois' theory, and the Abel-Ruffini theorem, polynomials of degree five and above, cannot be solved in radicals. We demonstrate here that the solutions for the septic equation are nested in quaternions. Both Abel and Ruffini could not have conjured this, because quaternions came into existence after they had both passed.

Keybords: Septic Equations, Differential Equations, Quaternions, Topological Differentiable Manifolds.

1. INTRODUCTION

The theorem credited to Niels Henrik Abel (1802-1829) and Paolo Ruffini (1765-1822), the Abel–Ruffini theorem, suggests that the equation of the form

$$a_5x^5+a_4x^4+a_3x^3+a_2x^2+a_1x+a_0=0, \tag{1}$$

the quintic equation, cannot be solved in radicals. This implies that even those of higher degree are also unsolvable. We addressed it in [1]. Our attention here is on the septic equation

$$\begin{aligned}&a_7x^7+a_6x^6+a_5x^5+a_4x^4+a_3x^3+a_2x^2\\&+a_1x+a_0=0,\end{aligned} \tag{2}$$

with constant coefficients $a_7,a_6,a_5,a_4,a_3,a_2,a_1,a_0$. It being an odd-degreed polynomial like (1) means it can be approached the same way, without the need to attend to the sectic equation

$$\begin{aligned}&a_6x^6+a_5x^5+a_4x^4+a_3x^3+a_2x^2\\&+a_1x+a_0=0,\end{aligned} \tag{3}$$

which lies between them.

Despite the theorem declaration, some equations were found to be solvable. Amongst the pioneers of this idea, there is George Neville Watson (1886–1965). Spearman and Williams [2], also with Lavallee [3], followed in his footsteps. See also [4], [5], [6] and [4]. They proposed that quintic equations of the form

$$a_5x^5 + a_1x + a_0 = 0, \tag{4}$$

are solvable for rational values of a_0, a_1 and a_5. Dummit [7] also considered this equation. Kobayashi and Nakagawa [8] also did.

There are other properties that can be used. Nikos Bagis [9] proposed Gamma functions and nested radical functions. The property Kulkarni [10] introduced, required that the sum of its four roots be equal to the sum of its remaining three roots. A similar property was used by Faggal and Lazard [11]. They considered polynomials whose roots are the sum of two different roots of the input (for the quintics) or the sum of three different roots of the input (for the septics).

The property we used in [1] is that (1) has a real root. Equation (2) shares this property. What this means is that if the roots can be expressed in the form

$$x = \tilde{x} + i\tilde{y}, \tag{5}$$

for real values of $\tilde{x}$ and $\tilde{y}$, where i is the element for complex numbers, then there is a root for which

$$\tilde{y} = 0. \tag{6}$$

We used this result to generate useful conditions for resolving (1).

This is a completely new approach, nobody has ever tried it, except us. We now apply it to the less complicated form of (2), namely

$$x^7 + \alpha x + \beta = 0. \tag{7}$$

The outline of the paper, is that we first transform (2) into a seventh-order linear ordinary differential equation, with constant coefficients. This we do in the next section, Section 2.

Section 3 is an application of our approach to the quadratic equation, because it is much simpler than (7) intend. We assume that at this scale, the method will be much more comprehensible, and a preparation for Section 4. Section 5 is on numerical tests.

2. THE TRANSFORMATION OF EQUATION (2) INTO A DIFFERENTIAL EQUATION

To transform (2) into a differential equation, we first introduce the factor $\exp(x\xi)$ to the equation. That is,

$$\begin{aligned}&(a_7x^7+a_6x^6+a_5x^5+a_4x^4+a_3x^3+a_2x^2\\&+a_1x+a_0)e^{x\xi}=0,\end{aligned} \tag{8}$$

or

$$\begin{aligned}&a_7x^7e^{x\xi}+a_6x^6e^{x\xi}+a_5x^5e^{x\xi}+a_4x^4e^{x\xi}+a_3x^3e^{x\xi}\\&+a_2x^2e^{x\xi}+a_1xe^{x\xi}+a_0e^{x\xi}=0,\end{aligned} \tag{9}$$

so that

$$\begin{aligned}&a_7\frac{d^7}{d\xi^7}e^{x\xi}+a_6\frac{d^6}{d\xi^6}e^{x\xi}+a_5\frac{d^5}{d\xi^5}e^{x\xi}+a_4\frac{d^4}{d\xi^4}e^{x\xi}\\&+a_3\frac{d^3}{d\xi^3}e^{x\xi}+a_2\frac{d^2}{d\xi^2}e^{x\xi}+a_1\frac{d}{d\xi}e^{x\xi}\\&+a_0e^{x\xi}=0.\end{aligned} \tag{10}$$

Letting

$$y=e^{x\xi} \tag{11}$$

with $y=y(\xi)$, suggests

$$\begin{aligned}&y^{(7)}=\frac{d^7e^{x\xi}}{d\xi^7},\ y^{(6)}=\frac{d^6e^{x\xi}}{d\xi^6},\ y^{(5)}=\frac{d^5e^{x\xi}}{d\xi^5},\\&y^{(4)}=\frac{d^4e^{x\xi}}{d\xi^4},\ y^{(3)}=\frac{d^3e^{x\xi}}{d\xi^3},\ y''=\frac{d^2e^{x\xi}}{d\xi^2}\\&y'=\frac{de^{x\xi}}{d\xi},\end{aligned} \tag{12}$$

leading to the ordinary differential equation

$$\begin{aligned}&a_7y^{(7)}+a_6y^{(6)}+a_5y^{(5)}+a_4y^{(4)}+a_3y^{(3)}+a_2y''+\\&a_1y'+a_0y=0.\end{aligned} \tag{13}$$

3. AN ILLUSTRATIVE EXAMPLE: THE QUADRATIC EQUATION

The quadratic equation

$$ax^2 + bx + c = 0, \tag{14}$$

with the parameters a, b and c, has the solutions

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{15}$$

What we wish to achieve here is demonstrate that the method we are proposing, to be used on the septic equation, lead to the same result.

First we introduce the factor (11) as before, so that

$$ay'' + by' + cy = 0. \tag{16}$$

There is a popular belief that this differential equation is trivial, and that it can easily be solved. Centuries ago, Leonhard Euler (1707–1783), proposeded a formula that it is supposed to solve it. It is still use today, a taught to all students in the sciences, mathematics and the applications, even in very advanced studies and research. See [12], [13], [14], [15] and [16].

We argue in [17] and [18] that there is an error in Euler's formula, and that the correct solution is

$$y = Ae^{-b/(2a)} \frac{\sin(i\omega\xi + \phi)}{i\omega}, \tag{17}$$

where A and ϕ are constants of integration, and

$$\omega = \frac{\pm\sqrt{b^2 - 4ac}}{2a}. \tag{18}$$

Equating (11) to (17) by removing the common y gives

$$e^{x\xi} = Ae^{-\xi b/(2a)} \frac{\sin(i\omega\xi + \phi)}{i\omega}, \tag{19}$$

so that

$$x\xi = \ln\left[\frac{A}{i\omega}\right] - \frac{b}{2a}\xi + \ln\left[\sin(i\omega\xi + \phi)\right], \tag{20}$$

or

$$x = \frac{\ln\left[\frac{A}{i\omega}\right]}{\xi} - \frac{b}{2a} + \frac{\ln\left[\sin(i\omega\xi + \phi)\right]}{\xi}. \tag{21}$$

This can be written in the form

$$x = \frac{\ln\left[\frac{A}{i\omega}\right]}{\xi} - \frac{b}{2a} + \frac{\ln\left[-\frac{e^{\omega\xi - i\phi} - e^{-\omega\xi + i\phi}}{2i}\right]}{\xi}. \tag{22}$$

The limit as $\xi \to \pm\infty$ gives

$$x = -\frac{b}{2a} - \omega \frac{e^{\omega\xi - i\phi} + e^{-\omega\xi + i\phi}}{e^{\omega\xi - i\phi} - e^{-\omega\xi + i\phi}}. \tag{23}$$

That is,

$$x = -\frac{b}{2a} - \omega \frac{e^{\omega\xi - i\phi}}{e^{\omega\xi - i\phi}}, \tag{24}$$

and subsequently

$$x = -\frac{b}{2a} - \omega, \tag{25}$$

or

$$x = -\frac{b}{2a} - \frac{\pm\sqrt{b^2 - 4ac}}{2a}. \tag{26}$$

This result is the same as the one in (15).

4. SOLVING (??)

An equivalent set arises from (8) when

$$y = y'' = 0. \tag{27}$$

This can be expressed in the form

$$\frac{y''}{y} = \frac{y^{(3)}}{y'}, \tag{28}$$

which integrates into

$$y = a\frac{\sin(i\omega\xi + \phi)}{i\omega}, \tag{29}$$

where A, ϕ and ω are constants of integration. Substituting this result into (13) gives an equation that determines ω. That is

$$-A\,(i\omega)^7\,\frac{\cos(i\omega[\xi + \phi])}{i\omega} + A\beta\cos(i\omega[\xi + \phi]) = 0. \tag{30}$$

This gives

$$(i\omega)^7 - \beta = 0, \tag{31}$$

a very trivial septic equation.

Note that (13) can be expressed in the form

$$\begin{aligned} -a_7 y^{(7)} &= a_6 y^{(6)} + a_5 y^{(5)} + a_4 y^{(4)} + a_3 y^{(3)} \\ &\quad + a_2 y'' + a_1 y' + a_0 y. \end{aligned} \tag{32}$$

That is,

$$\begin{aligned} y^{(7)} &= \frac{a_6 y^{(6)} + a_5 y^{(5)} + a_4 y^{(4)} + a_3 y^{(3)}}{-a_7} \\ &\quad + \frac{a_2 y'' + a_1 y' + a_0 y}{-a_7}, \end{aligned} \tag{33}$$

or

$$\begin{aligned} \frac{d^7 e^{x\xi}}{d\xi^7} &= \frac{a_6 y^{(6)} + a_5 y^{(5)} + a_4 y^{(4)} + a_3 y^{(3)}}{-a_7} \\ &\quad + \frac{a_2 y'' + a_1 y' + a_0 y}{-a_7}, \end{aligned} \tag{34}$$

so that

$$\begin{aligned} x^7 e^{x\xi} = x^7 y = &\quad \frac{a_6 y^{(6)} + a_5 y^{(5)} + a_4 y^{(4)} + a_3 y^{(3)}}{-a_7} \\ &\quad + \frac{a_2 y'' + a_1 y' + a_0 y}{-a_7}. \end{aligned} \tag{35}$$

Hence,

$$\begin{aligned} x^7 = &\quad \frac{a_6 y^{(6)} + a_5 y^{(5)} + a_4 y^{(4)} + a_3 y^{(3)}}{-a_7 y} \\ &\quad + \frac{a_2 y'' + a_1 y' + a_0 y}{-a_7 y}. \end{aligned} \tag{36}$$

That is,

$$\begin{aligned} x^7 &= -\left(\frac{a_6}{a_7}\frac{y^{(6)}}{y} + \frac{a_4}{a_7}\frac{y^{(4)}}{y} + \frac{a_2}{a_7}\frac{y''}{y} + \frac{a_0}{a_7}\right) \\ &\quad - \left(\frac{a_5}{a_7}\frac{y^{(5)}}{y} + \frac{a_3}{a_7}\frac{y^{(3)}}{y} + \frac{a_1}{a_7}\frac{y'}{y}\right). \end{aligned} \tag{37}$$

From (29), that is

$$y = a\frac{\sin(i\omega\xi+\phi)}{i\omega}, \tag{38}$$

we get

$$\begin{aligned}
y' &= (i\omega)a\frac{\cos(i\omega\xi+\phi)}{i\omega}, &(39)\\
y'' &= -(i\omega)^2 a\frac{\sin(i\omega\xi+\phi)}{i\omega}, &(40)\\
y^{(3)} &= -(i\omega)^3 a\frac{\cos(i\omega\xi+\phi)}{i\omega}, &(41)\\
y^{(4)} &= (i\omega)^4 a\frac{\sin(i\omega\xi+\phi)}{i\omega}, &(42)\\
y^{(5)} &= (i\omega)^5 a\frac{\cos(i\omega\xi+\phi)}{i\omega}, &(43)\\
y^{(6)} &= -(i\omega)^6 a\frac{\sin(i\omega\xi+\phi)}{i\omega}. &(44)\\
& &(45)
\end{aligned}$$

This then suggests that

$$\begin{aligned}
\frac{y^{(6)}}{y} &= \frac{-(i\omega)^6 a\frac{\sin(i\omega\xi+\phi)}{i\omega}}{a\frac{\sin(i\omega\xi+\phi)}{i\omega}} = -(i\omega)^6 = -\omega^6, &(46)\\
\frac{y^{(4)}}{y} &= \frac{(i\omega)^4 a\frac{\sin(i\omega\xi+\phi)}{i\omega}}{a\frac{\sin(i\omega\xi+\phi)}{i\omega}} = (i\omega)^4 = \omega^4, &(47)\\
\frac{y''}{y} &= \frac{-(i\omega)^2 a\frac{\sin(i\omega\xi+\phi)}{i\omega}}{a\frac{\sin(i\omega\xi+\phi)}{i\omega}} = -(i\omega)^2 = \omega^2, &(48)
\end{aligned}$$

so that

$$\begin{aligned}
& \frac{a_6}{a_7}\frac{y^{(6)}}{y} + \frac{a_4}{a_7}\frac{y^{(4)}}{y} + \frac{a_2}{a_7}\frac{y''}{y} + \frac{a_0}{a_7}\\
= & -\frac{a_6}{a_7}\omega^6 + \frac{a_4}{a_7}\omega^4 + \frac{a_2}{a_7}\omega^2 + \frac{a_0}{a_7}. &(49)
\end{aligned}$$

Similarly,

$$\begin{aligned}
\frac{y^{(5)}}{y} &= \frac{(i\omega)^5 a\frac{\cos(i\omega\xi+\phi)}{i\omega}}{a\frac{\sin(i\omega\xi+\phi)}{i\omega}} = i\omega^5\cot(i\omega\xi+\phi), &(50)\\
\frac{y^{(3)}}{y} &= \frac{-(i\omega)^3 a\frac{\cos(i\omega\xi+\phi)}{i\omega}}{a\frac{\sin(i\omega\xi+\phi)}{i\omega}} = i\omega^2\cot(i\omega\xi+\phi), &(51)\\
\frac{y'}{y} &= \frac{(i\omega)a\frac{\cos(i\omega\xi+\phi)}{i\omega}}{a\frac{\sin(i\omega\xi+\phi)}{i\omega}} = i\omega\cot(i\omega\xi+\phi), &(52)
\end{aligned}$$

so that

$$\frac{a_5}{a_7}\frac{y^{(5)}}{y}+\frac{a_3}{a_7}\frac{y^{(3)}}{y}+\frac{a_1}{a_7}\frac{y'}{y}$$
$$i\left(\frac{a_5}{a_7}\omega^5+\frac{a_3}{a_7}\omega^2+\frac{a_1}{a_7}\omega\right)\cot(i\omega\xi+\phi) \quad (53)$$

Subsequently

$$\begin{aligned} x^7 \;=\; & i\left(\frac{a_5}{a_7}\omega^5+\frac{a_3}{a_7}\omega^2+\frac{a_1}{a_7}\omega\right)\cot(i\omega\xi+\phi) \\ & -\frac{a_6}{a_7}\omega^6+\frac{a_4}{a_7}\omega^4+\frac{a_2}{a_7}\omega^2+\frac{a_0}{a_7}. \end{aligned} \quad (54)$$

4.1. The radical root

4.1.1 Determining ω for (13)

The conditions (27) and (38) reduce (13) to

$$a_7y^{(7)}+a_5y^{(5)}+a_3y^{(3)}+a_1y'=0, \quad (55)$$

so that

$$i\left(-a_7\omega^7+a_5\omega^5-a_3\omega^3+a_1\omega\right)\cos(i\omega\xi+\phi)=0. \quad (56)$$

That is, $\omega=0$, or

$$a_7\omega^6+a_5\omega^4-a_3\omega^2+a_1=0. \quad (57)$$

One of the six roots is

$$\omega=-\sqrt{w_3} \quad (58)$$

with

$$\begin{aligned} w_3 = \; & \frac{\sqrt[3]{\sqrt{w_2}+w_1}}{3\sqrt[3]{2}a_7}+\frac{\sqrt[3]{2}a_5^2}{3a_7\sqrt[3]{\sqrt{w_2}+w_1}} \\ & -\frac{\sqrt[3]{2}a_3}{\sqrt[3]{\sqrt{w_2}+w_1}}-\frac{a_5}{3a_7}, \end{aligned} \quad (59)$$

$$w_2=4\left(3a_3a_7-a_5^2\right)^3+\left(-2a_5^3+9a_3a_7a_5-27a_1a_7^2\right)^2 \quad (60)$$

and

$$w_1=-2a_5^3+9a_3a_7a_5-27a_1a_7^2. \quad (61)$$

4.1.2 Determining ξ and ϕ

As suggested in (6), we have to set the imaginary part of the expression on the right of (54) to zero. Doing so determines ξ and ϕ. The resulting equations lead expressions containing the number $\sqrt{-1}$, but it cannot be assigned to i, because we are equating the latter to zero. A better choice is j, a quaternion element.

We get expressions for a and b by varying the radical signs. This is not different from the process followed when extracting quadratic roots, with $-\surd$ assigned to one root, and $+\surd$ assigned to another. They are

$$a = \mathrm{Re}\left(j\left(2a_7\frac{A_{13}A_{15}}{\left(a_7^2-i\right)A_{16}}\right)^{1/7}\right) \tag{62}$$

and

$$b = \mathrm{Re}\left(-a_7\frac{B_{14}B_{16}j}{\left(a_7+(-1)^{3/4}\right)B_{17}\left(B_8-B_{11}\right)}\right)^{1/7}. \tag{63}$$

The parameters are

$$A_1 = \coth^{-1}\left(\frac{1+\frac{2j\left(a_7+\sqrt[4]{-1}\right)}{2(-1)^{3/4}-2ja_7}}{-1+\frac{2j\left(a_7+\sqrt[4]{-1}\right)}{2(-1)^{3/4}-2ja_7}}\right), \tag{64}$$

$$A_2 = \log\left(\frac{(1+j)\sqrt{a_7+\sqrt[4]{-1}}}{\sqrt{2(-1)^{3/4}-2ja_7}}\right), \tag{65}$$

$$A_3 = -1+\frac{2j\left(a_7+\sqrt[4]{-1}\right)}{2(-1)^{3/4}-2ja_7}, \tag{66}$$

$$A_4 = 1+\frac{2j\left(a_7+\sqrt[4]{-1}\right)}{2(-1)^{3/4}-2ja_7}, \tag{67}$$

$$A_5 = \sinh\left(A_1-(1-j)A_2\right), \tag{68}$$

$$A_6 = \cosh\left(A_1-(1-j)A_2\right), \tag{69}$$

$$A_7 = \cos\left(4\left(-A_2 + jA_1\right) - 4jA_2\right), \tag{70}$$

$$\begin{aligned} A_8 &= -a_7A_5^2 + a_7A_6^2 + \frac{\sqrt[4]{-1}A_5}{\sqrt{1-\frac{A_3^2}{A_4^2}}} \\ &\quad + \frac{(-1)^{3/4}A_3A_5}{\sqrt{1-\frac{A_3^2}{A_4^2}}A_4} - \frac{(-1)^{3/4}A_6}{\sqrt{1-\frac{A_3^2}{A_4^2}}} \\ &\quad - \frac{\sqrt[4]{-1}A_3A_6}{\sqrt{1-\frac{A_3^2}{A_4^2}}A_4}, \end{aligned} \tag{71}$$

$$A_9 = a_7^2A_7 + (-1)^{3/4}a_7\sin\left(4\left(-A_2 + jA_1\right) + (1-3i)A_2\right), \tag{72}$$

$$\begin{aligned} A_{10} &= a_7^2A_7 - 2(-1)^{3/4}a_7\sin\left((1+j)A_2\right) \\ &\quad + 2(-1)^{3/4}a_7\sin\left(4\left(-A_2jA_1\right) + (1-3j)A_2\right) \\ &\quad + i\cos\left(4\left(-A_2jA_1\right) + (2-2j)A_2\right), \end{aligned} \tag{73}$$

$$\begin{aligned} A_{11} &= \sqrt{T_1 + T_2 + T_3}, \\ T_1 &= \frac{a_7A_5^2}{A_8} - \frac{\sqrt[4]{-1}A_5}{\sqrt{1-\frac{A_3^2}{A_4^2}}A_8}, \\ T_2 &= \frac{a_7A_6^2}{A_8} - \frac{\sqrt[4]{-1}A_3A_6}{\sqrt{1-\frac{A_3^2}{A_4^2}}A_4A_8}, \\ T_3 &= \frac{\sqrt{-a_7^2 + A_{10} - 2i\cos\left((2+2i)A_2\right) - i}}{\sqrt{2}A_8}, \end{aligned} \tag{74}$$

$$A_{12} = \cosh\left(\log\left(-A_{11}\right) - (1+j)A_2\right), \tag{75}$$

$$\begin{aligned} A_{13} &= jA_5\cosh\left(\log\left(-A_{11}\right) - (1-i)A_2\right) \\ &\quad - iA_6\sinh\left(\log\left(-A_{11}\right) - (1-j)A_2\right), \end{aligned} \tag{76}$$

$$\begin{aligned} A_{14} &= -A_5\cosh\left(\log\left(-A_{11}\right) - 2A_2\right) \\ &\quad + (1-j)a_7\sqrt{1+ja_7^2}\sinh\left(\log\left(-A_{11}\right) - (1+i)A_2\right), \end{aligned} \tag{77}$$

$$
\begin{aligned}
A_{15} \;=\;& \sqrt[4]{-1}(1-i)\sqrt{1+ja_7^2}A_{12} \\
&-a_7^3 A_6 \sinh\left(\log\left(-A_{11}\right)-2A_2\right) \\
&-ia_7^2 A_6 \sinh\left(\log\left(-A_{11}\right)-2A_2\right) \\
&+ja_7 A_6 \sinh\left(\log\left(-A_{11}\right)-2A_2\right) \\
&+a_7^3 A_5 \left(-\cosh\left(\log\left(-A_{11}\right)-2A_2\right)\right) \\
&-ia_7^2 A_5 \cosh\left(\log\left(-A_{11}\right)-2A_2\right) \\
&+ia_7 A_5 \cosh\left(\log\left(-A_{11}\right)-2A_2\right) \\
&+A_{14}-A_6 \sinh\left(\log\left(-A_{11}\right)-2A_2\right), \qquad (78)
\end{aligned}
$$

$$
\begin{aligned}
A_{16} \;=\;& \cos\left(2\left(iA_1-(1+j)A_2\right)\right) \\
&-\cosh\left(2\left(\log\left(-A_{11}\right)-A_2\right)+2jA_2\right), \qquad (79)
\end{aligned}
$$

$$
B_1 = \coth^{-1}\left(\frac{1-\frac{2j\left(a_7+(-1)^{3/4}\right)}{2\sqrt[4]{-1}+2ja_7}}{-1-\frac{2j\left(a_7+(-1)^{3/4}\right)}{2\sqrt[4]{-1}+2ja_7}}\right), \qquad (80)
$$

$$
B_2 = -(1-j)\log\left(\frac{(1-i)\sqrt{a_7+(-1)^{3/4}}}{\sqrt{2\sqrt[4]{-1}+2ja_7}}\right), \qquad (81)
$$

$$
B_3 = \log\left(\frac{(1-i)\sqrt{a_7+(-1)^{3/4}}}{\sqrt{2\sqrt[4]{-1}+2ja_7}}\right), \qquad (82)
$$

$$
\begin{aligned}
B_4 \;=\;& -2\sqrt[4]{-1}a_7 \sin\left(4\left(-B_3+jB_1\right)-4jB_3\right) \\
&+a_7^2 \cos\left(4\left(-B_3+jB_1\right)-4jB_3\right) \\
&-a_7^2-i\cos\left(4\left(-B_3+jB_1\right)-4jB_3\right)-i, \qquad (83)
\end{aligned}
$$

$$
\begin{aligned}
B_5 \;=\;& -a_7 \sinh^2\left(B_1+B_2\right)+a_7\cosh^2\left(B_1+B_2\right) \\
&-(-1)^{3/4}\sinh^2\left(B_1+B_2\right) \\
&+(-1)^{3/4}\cosh^2\left(B_1+B_2\right), \qquad (84)
\end{aligned}
$$

$$
\begin{aligned}
B_6 \;=\;& \frac{a_7\sinh^2\left(B_1+B_2\right)}{B_5}+\frac{a_7\cosh^2\left(B_1+B_2\right)}{B_5} \\
&-\frac{2(-1)^{3/4}\sinh\left(B_1+B_2\right)\cosh\left(B_1+B_2\right)}{B_5} \\
&+\frac{\sqrt{B_4}}{\sqrt{2}B_5}, \qquad (85)
\end{aligned}
$$

$$B_7 = \cosh(B_1 + B_2) - \sinh(B_1 + B_2), \tag{86}$$

$$B_8 = \cos(2(jB_1 - (1+j)B_3)), \tag{87}$$

$$B_9 = \cosh\left(-\log\left(-\sqrt{B_6}\right) + (1+j)B_3\right), \tag{88}$$

-

$$B_{10} = \cosh(B_1 + B_2), \tag{89}$$

$$B_{11} = Cosh[2B3 + 2(-B_3 + Log[-\sqrt[B_6]{}]))], \tag{90}$$

$$B_{12} = \sinh(B_1 + B_2), \tag{91}$$

$$B_{13} = \sinh\left(-\log\left(-\sqrt{B_6}\right) + (1+j)B_3\right), \tag{92}$$

$$B_{14} = \sqrt{B_4}B_7(B_{10}B_{13} - B_{12}B_9), \tag{93}$$

$$B_{15} = -\sqrt{2}B_{10}^2 - 4\sqrt{2}B_{12}B_{10} - \sqrt{2}B_{12}^2, \tag{94}$$

$$\begin{aligned}
B_{16} = & -2\sqrt{2}a_7^3B_{10}B_{12} \\
& -(-1)^{3/4}\sqrt{2}a_7^2B_{10}^2 \\
& -(-1)^{3/4}\sqrt{2}a_7^2B_{12}^2 - 6(-1)^{3/4}\sqrt{2}a_7^2B_{10}B_{12} \\
& -2j\sqrt{2}a_7^2B_{10}B_{12} + \sqrt[4]{-1}\sqrt{2}a_7B_{10}^2 \\
& +j\sqrt{2}a_7B_{10}^2 + \sqrt[4]{-1}\sqrt{2}a_7B_{12}^2 + i\sqrt{2}a_7B_{12}^2 \\
& +6\sqrt[4]{-1}\sqrt{2}a_7B_{10}B_{12} + 4j\sqrt{2}a_7B_{10}B_{12} \\
& +a_7^2\sqrt{B_4} + (-1)^{3/4}a_7\sqrt{B_4} + ja_7\sqrt{B_4} \\
& +B_{15} - \sqrt[4]{-1}\sqrt{B_4},
\end{aligned} \tag{95}$$

$$\begin{aligned}
B_{17} = & -2a_7B_{10}^2 - 2a_7B_{12}^2 + 4(-1)^{3/4}B_{12}B_{10} - \sqrt{2}\sqrt{B_4}, \\
B_{18} = & -2\sqrt[4]{-1}a_7\sin(4(-B_3 + jB_1) - 4jB_3) \\
& +a_7^2\cos(4(-B_3 + jB_1) - 4jB_3) \\
& -a_7^2 - j\cos(4(-B_3 + jB_1) - 4jB_3) - i.
\end{aligned} \tag{96}$$

4.2. Roots of the imaginary component

Setting

$$\psi_i = \psi_i(\xi, \phi) = 0, \tag{97}$$

with the condition that ϕ as $\xi = \xi_0 \pm 2N\pi$ as N approaches infinity, leads to two polynomials in ϕ and ξ_0, respectively.

Under normal circumstances, without the Abel-Ruffini theorem, this result should solve (2). Instead, we get two values, complying to the following axiom and theorems.

Axiom 1. *There exists two radical numbers a and b, such that a line that connects the points* $(a; f(a))$ *and* $(b; f(b))$ *passes through the interior point* $(x; 0)$*, a root of the septic function*

$$\begin{aligned} f(x) = {} & a_7x^7 + a_6x^6 + a_5x^5 + a_4x^4 + a_3x^3 \\ & + a_2x^2 + a_1x + a_0. \end{aligned} \tag{98}$$

Theorem 1. *If a line that joins the points* $(a; f(a))$ *and* $(b; f(b))$ *passes through the interior point* $(x_r; 0)$*, for the function (98) , then the line is*

$$\begin{aligned} & x = c + \\ & m\left(f(x) - f\left(\frac{a-x}{f(a)-f(x)}f(x) + a - \frac{f(a)}{f'(a)}\right)\right), \end{aligned} \tag{99}$$

$$\tag{100}$$

or

$$\begin{aligned} & x = c + \\ & m\left(f(x) - f\left(\frac{b-x}{f(b)-f(x)}f(x) + b - \frac{f(b)}{f'(b)}\right)\right), \end{aligned}$$

$$\tag{101}$$

where c is the x-intercept and m is given

$$\frac{\left[\frac{a-x}{f(a)-f(x)} - \frac{b-x}{f(b)-f(x)}\right] f(x) + a - b - \frac{f(a)}{f'(a)} + \frac{f(b)}{f'(b)}}{f\left(\frac{a-x}{f(a)-f(x)}f(x) + a - \frac{f(a)}{f'(a)}\right) - f\left(\frac{b-x}{f(b)-f(x)}f(x) + b - \frac{f(b)}{f'(b)}\right)}.$$

$$\tag{102}$$

Theorem 2. *If a line that joins the points $(a;f(a))$ and $(b;f(b))$ passes through the interior point $(x_r;0)$, for the function (98) , then x_r is given by*

$$x_r = \frac{a-b-\frac{f(a)}{f'(a)}+\frac{f(b)}{f'(b)}}{f\left(a-\frac{f(a)}{f'(a)}\right)-f\left(b-\frac{f(b)}{f'(b)}\right)}\left(-f\left(a-\frac{f(a)}{f'(a)}\right)\right) +a-\frac{f(a)}{f'(a)}, \tag{103}$$

or

$$x_r = \frac{a-b-\frac{f(a)}{f'(a)}+\frac{f(b)}{f'(b)}}{f\left(a-\frac{f(a)}{f'(a)}\right)-f\left(b-\frac{f(b)}{f'(b)}\right)}\left(-f\left(b-\frac{f(b)}{f'(b)}\right)\right) +b-\frac{f(b)}{f'(a)}. \tag{104}$$

5. NUMERICAL EXPERIMENTS

Here we conduct some tests based on the fifth-degree polynomial. The parameter in the first is the irrational number π, while the second has the most irrational number ϕ, called the golden number.

5.1. The case $a_0 = 1/\pi$

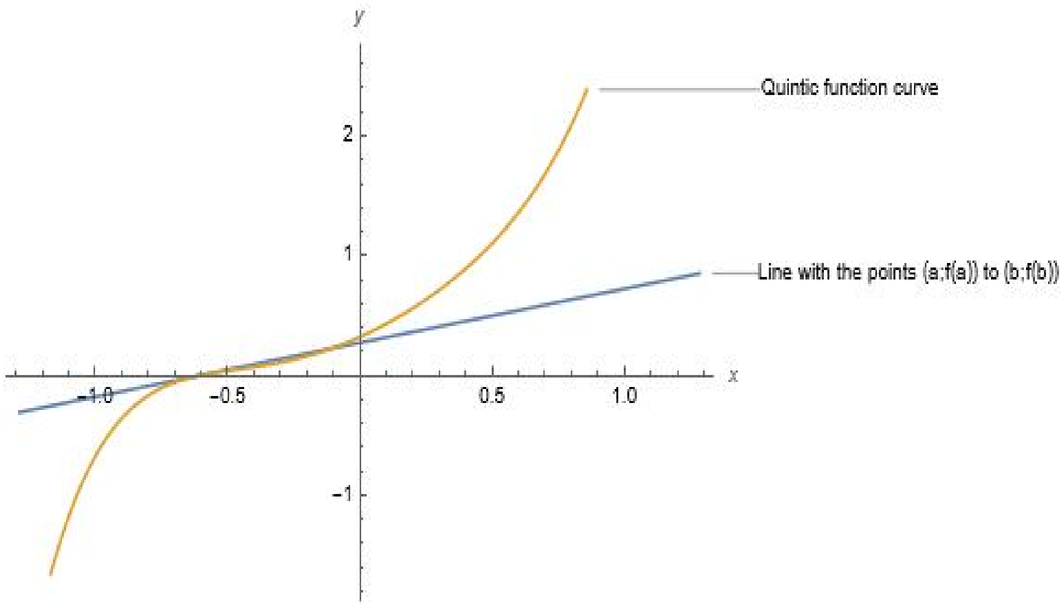

Figure 1: An llustration on π. An irrational number.

For the equation $x^5+x^2+x+1/\pi=0$, we have the values for a and b given by

$$a=-0.6012227544458956 \tag{105}$$

and

$$b=-0.6012313313588129. \tag{106}$$

Substituting these into (103) or (104) gives

$$x_r=-0.6012235335340362. \tag{107}$$

The program Mathematica's

$$N[x^5+x^2+x+\frac{1}{\pi}==0,\{x\}][[1]],$$

gives

$$x_n=-0.6012235335340362. \tag{108}$$

5.2. The most irrational number ϕ

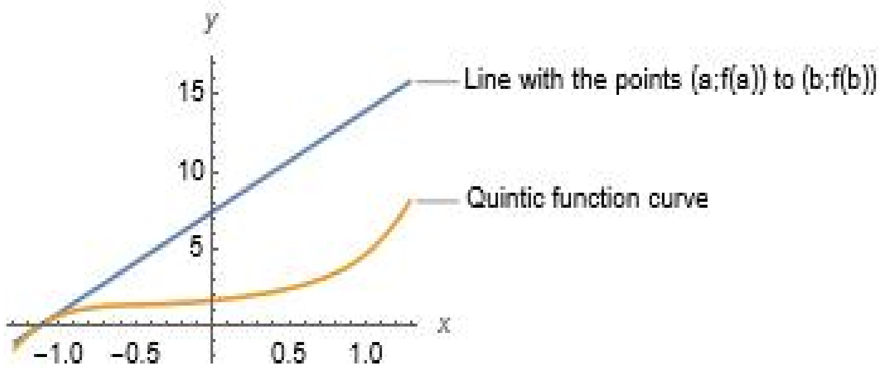

Figure 2: An illustration of on the golden number ϕ. The most irrational number.

For the equation $x^5+x^2+x+\phi=0$, we have the values for a and b given by

$$a=-1.1184269147201447 \tag{109}$$

and

$$b=-1.1185013933400367. \tag{110}$$

Substituting these into (103) or (104) gives

$$x_r = -1.1185006651209661. \tag{111}$$

The program Mathematica's

$$N[x^5 + x^2 + x + \phi == 0, \{x\}][[1]],$$

gives

$$x_n = -1.1185006651209661. \tag{112}$$

6. CONCLUSION

In this study we determined the roots of the septic equation by using developing a procedure based on the property that it has a real root. As a demonstration that it works, we conducted two numerical test. The first was on the quintic equation with the irrational number π. In the equation it was inverted to $1/\pi$, just to turn the test stiffer. The second has ϕ, the most irrational number.

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