Advances in Dynamical Systems and Applications (ADSA).
ISSN 0973-5321, Volume 16, Number 2, (2021) pp. 1757-1776

http://www.ripublication.com/adsa.htm

Homogenization of Stochastic Variational Inequality PDE with Nonlinear Neumann Boundary Condition

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Abstract

We establish homogenization results of a stochastic variational inequality semilinear PDE with a nonlinear Neumann boundary condition. Our approach is entirely probabilistic, and extends the result of Pardoux and Ouknine published in [8].

Keywords :Viscosity Solution - Backward Stochastic Differential Equations - Neumann Boundary Condition - Local time - Homogenization.

Mathematics Subjection Classification: 60H15 , 35K00 , 35B27

1. INTRODUCTION

The theory of homogenisation has developed considerably in recent years and is a discipline in its own right. In their book Bensoussan, Lions and Papanicolaou [2] have shown that this can be done by probabilistic methods (see also [5]). Since then, this approach has developed in parallel with the analytical approach.The objective of this paper is to study the homogenisation of Inequality Variations Semilinear second order parabolic semilinear (IVS) with periodical coefficients. Our purely probabilistic approach generalises the work of Pardoux and Ouknin [8] to the case considered in based on the interpretation of IVS solutions in terms of the solutions of the EDSRs considered (cf. Pardoux-Rascanu [7]). The paper is organised as follows: Section 2 contains some reminders and hypotheses for the rest. We deal with the convergence of the family of processes $(Y^\varepsilon, Z^\varepsilon, U^\varepsilon)$ solution of on BSDEs reflected in Section 3. In Section 4, we give an application for the homogenisation of a parabolic IVS with periodic coefficients. We consider the following semi-linear partial differential equation (PDE) in a domain $\boldsymbol{D} = \left\{(x_1, x_2, \ldots, x_d) \in \mathbb{R}^d \ : \ x_1 > 0\right\}$, with non-linear Neumann boundary condition on $\partial \boldsymbol{D}$. For each $\varepsilon > 0$, we consider,

$$\begin{cases} \dfrac{\partial u^\varepsilon}{\partial t}(t,x) - L_\varepsilon u^\varepsilon(t,x) \\ -f\left(\frac{x}{\varepsilon}, u^\varepsilon(t,x)\right) - \frac{1}{\varepsilon} e\left(\frac{x}{\varepsilon}, u^\varepsilon(t,x)\right) \in \partial\phi\left(u^\varepsilon(t,x)\right), \ x \in \boldsymbol{D}, \ 0 < t \\ \Gamma_\varepsilon u^\varepsilon(t,x) + h\left(\frac{x}{\varepsilon}, u^\varepsilon(t,x)\right) \in \partial\phi\left(u^\varepsilon(t,x)\right), \qquad x \in \partial\boldsymbol{D}, \ 0 \le t \\ u^\varepsilon(0,x) = g(x), \qquad x \in \partial\boldsymbol{D} \end{cases} \tag{1}$$

where

- $e : \mathbb{R}^d \times \mathbb{R} \longrightarrow \mathbb{R}$ is a measurable mapping, which is periodic, of period one in each direction in the first argument, continuous in the second argument uniformly with respect to the first, and satisfies :

$$\int_{\mathbb{T}^d} e(x,y) m(dx) = 0, \qquad \forall y \in \mathbb{R} \tag{2}$$

  when m is the unique invariant measure on the torus $\mathbb{T}^d$.
  Suppose e be twice continuously differentiable in y, uniformly with respect to x, and there exists a constant K such that :

$$|e(x,y)| + \left|\frac{\partial}{\partial y} e(x,y)\right| + \left|\frac{\partial^2}{\partial y^2} e(x,y)\right| \le K, \quad \forall x \in \mathbb{R}^d, \ y \in \mathbb{R} \tag{3}$$

- $f : \mathbb{R}^d \times \mathbb{R} \longrightarrow \mathbb{R}$, $g : \mathbb{R}^d \longrightarrow \mathbb{R}$ and $h : \mathbb{R}^d \times \mathbb{R} \longrightarrow \mathbb{R}$ are sufficiently smooth functions. Equivalently the coefficients can be seen as periodic functions with respect to the first variable with period one in each direction on $\mathbb{R}^d$ and are such that for some $c > 0$, $p > 0$, $\mu \in \mathbb{R}$, $\beta < 0$, and all $x \in \mathbb{R}^d$, $y, y' \in \mathbb{R}$:

$$|g(x)| \le c\,(1 + |x|^p) \tag{4}$$

$$|f(x,y)| \le c \tag{5}$$

$$(y - y')\,[f(x,y) - f(x,y')] \le \mu\,|y - y'| \tag{6}$$

$$(y - y')\,[h(x,y) - h(x,y')] \le \beta\,|y - y'| \tag{7}$$

$$|h(x,y)| \le c \tag{8}$$

Assumptions and definitions

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space on which a d-dimensional Brownian motion $\left(B^1, \ldots, B^d\right)$ is defined. Let $\mathbb{E}$ be the corresponding expectation operator.

The differential operator L_ε inside $\boldsymbol{D}$ is given by :

$$L_\varepsilon = \frac{1}{2}\sum_{i,j}^{d} a_{i,j}\left(\frac{x}{\varepsilon}\right)\frac{\partial^2}{\partial x_i x_j} + \frac{1}{\varepsilon}\sum_{i=1}^{d} b_i\left(\frac{x}{\varepsilon}\right)\frac{\partial}{\partial x_i} + \sum_{i=1}^{d} c_i\left(\frac{x}{\varepsilon}\right)\frac{\partial}{\partial x_i} \tag{9}$$

This operator is the generator of the reflected $(L_\varepsilon, \Gamma_\varepsilon)$-diffusion (see Tanaka [13]) :

$$\begin{cases} dX_t^\varepsilon = \sigma\left(\frac{X_t^\varepsilon}{\varepsilon}\right) dB_t + \frac{1}{\varepsilon} b\left(\frac{X_t^\varepsilon}{\varepsilon}\right) dt + c\left(\frac{X_t^\varepsilon}{\varepsilon}\right) dt + \gamma\left(\frac{X_t^\varepsilon}{\varepsilon}\right) d\varphi_t^\varepsilon, \quad 0 < t, \\ X_t^{1,\varepsilon} \geq 0,\ \varphi^\varepsilon \text{ is continuous and increasing, } \displaystyle\int_0^t X_s^{1,\varepsilon} d\varphi_s^\varepsilon = 0, \quad 0 < t, \\ X_0^\varepsilon = x \end{cases} \tag{10}$$

where $X^{1,\varepsilon}$ denotes the first component of the process X^ε. We recall that $\boldsymbol{D} = \mathbb{R}_+^* \times \mathbb{R}^{d-1}$, so that X^ε lives is $\boldsymbol{D}$, that is, $X^{1,\varepsilon}$ remains non-negative and φ^ε increases when and only when $X^{1,\varepsilon}$ is zero, just to keep it non-negative.

The associated differential operator on $\partial\boldsymbol{D}$ is defined as :

$$\Gamma_\varepsilon := \sum_{i=1}^{d} \gamma_i\left(\frac{x}{\varepsilon}\right)\frac{\partial}{\partial x_i} \tag{11}$$

The function $\gamma :\ \partial\boldsymbol{D}\left(\cong \mathbb{R}^{d-1}\right) \longrightarrow \mathbb{R}^d$ is smooth and periodic of period one in each direction and satisfies : $\gamma^1(x) = 1$.

We suppose that $\sigma :\ \mathbb{R}^d \longrightarrow \mathbb{R}^{d\times d}$, $b :\ \mathbb{R}^d \longrightarrow \mathbb{R}^d$ and $c :\ \mathbb{R}^d \longrightarrow \mathbb{R}^d$ are smooth and periodic of period one in each direction.
L is uniformly elliptic and the matrix $a(x) = (a_{ij}(x))$ can be factored as $\sigma^{\mathrm{t}}(x)\sigma(x)/2$.
Let us define $\tilde{X}_t^\varepsilon := \frac{1}{\varepsilon} X_{\varepsilon^2 t}^\varepsilon$ and $\tilde{\varphi}_t^\varepsilon := \frac{1}{\varepsilon}\varphi_{\varepsilon^2 t}^\varepsilon$, then we get with a new standard d-dimensional Brownian motion $\{B_t^\varepsilon :\ t \geq 0\}$, which in fact depends on ε:

$$\begin{cases} d\tilde{X}_t^\varepsilon = \sigma\left(\tilde{X}_t^\varepsilon\right) dB_t^\varepsilon + b\left(\tilde{X}_t^\varepsilon\right) dt + \varepsilon c\left(\tilde{X}_t^\varepsilon\right) dt + \gamma\left(\tilde{X}_t^\varepsilon\right) d\tilde{\varphi}_t^\varepsilon,\ 0 < t, \\ \tilde{X}_t^{1,\varepsilon} \geq 0,\ \tilde{\varphi} \text{ is continuous and increasing, } \displaystyle\int_0^t \tilde{X}_s^{1,\varepsilon} d\tilde{\varphi}_s^\varepsilon = 0,\ 0 < t, \\ \tilde{X}_0^\varepsilon = \frac{x}{\varepsilon} \end{cases} \tag{12}$$

From now, we consider the $\mathbb{T}^d$-values process $\tilde{X}$ which operator is given by :

$$\tilde{L}_\varepsilon := \frac{1}{2}\sum_{i,j}^{d} a_{i,j}(x)\frac{\partial^2}{\partial x_i x_j} + \sum_{i=1}^{d} b_i(x)\frac{\partial}{\partial x_i} + \varepsilon\sum_{i=1}^{d} c_i(x)\frac{\partial}{\partial x_i}, \quad x \in \mathbb{T}^d \tag{13}$$

This operator converges to L where

$$L := \frac{1}{2}\sum_{i,j}^{d} a_{i,j}(x)\frac{\partial^2}{\partial x_i x_j} + \sum_{i=1}^{d} b_i(x)\frac{\partial}{\partial x_i}, \quad x \in \mathbb{T}^d \tag{14}$$

2. WEAK CONVERGENCE (SDE AND BSDE)

By a simple adaptation of the results in Pardoux and Ouknine [8], one can observe that the $\mathbb{T}^d$-values $\tilde{X}^\varepsilon$, is an homogeneous Feller process with values in a compact set, and this process has a unique invariant measure whose density is strictly positive. Thereafter, we set m the invariant measure associated of L on $\mathbb{T}^d$ and we set m_0 the invariant measure associated of the differential operator on $\mathbb{T}^{d-1}$.

Throughout, we suppose that

$$\int_{\mathbb{T}^d} b(x)m(dx) = 0 \tag{15}$$

Let $\hat{b}$ be the solution of Poisson equation : $L\hat{b} = -b$, and let us introduce the process $\hat{X}_t^\varepsilon$ defined as:

$$\begin{aligned}\hat{X}_t^\varepsilon &= X_t^\varepsilon + \varepsilon\left[\hat{b}\left(\frac{X_t}{\varepsilon}\right) - \hat{b}\left(\frac{x}{\varepsilon}\right)\right]\\ &= x + \int_0^t \left(I + \nabla\hat{b}\right)\sigma\left(\frac{X_s^\varepsilon}{\varepsilon}\right)dB_s + \int_0^t \left(I + \nabla\hat{b}\right)c\left(\frac{X_s^\varepsilon}{\varepsilon}\right)ds\\ &\quad + \int_0^t \left(I + \nabla\hat{b}\right)\gamma_\varepsilon\left(\frac{X_s^\varepsilon}{\varepsilon}\right)d\varphi_s^\varepsilon\end{aligned} \tag{16}$$

Let us write (16) in coordinate form :

$$\begin{cases}\hat{X}_t^{1,\varepsilon} = x_1 + \int_0^t \left(1 + \nabla\hat{b}_1\right)\sigma_1\left(\frac{X_s^\varepsilon}{\varepsilon}\right)dB_s^1 + \int_0^t \left(1 + \nabla\hat{b}_1\right)c_1\left(\frac{X_s^\varepsilon}{\varepsilon}\right)ds\\ \qquad + \int_0^t \left(1 + \nabla\hat{b}_1\right)\left(\frac{X_s^\varepsilon}{\varepsilon}\right)d\varphi_s^\varepsilon\\ \text{and}\\ \hat{X}_t^{j,\varepsilon} = x_j + \int_0^t \left(1 + \nabla\hat{b}_j\right)\sigma_j\left(\frac{X_s^\varepsilon}{\varepsilon}\right)dB_s^j + \int_0^t \left(1 + \nabla\hat{b}_j\right)c_j\left(\frac{X_s^\varepsilon}{\varepsilon}\right)ds\\ \qquad \text{for all} \quad j = 2, \ldots, d\end{cases} \tag{17}$$

Then there exists a bounded and smooth solution η of the PDE with Neumann-type boundary condition :

$$\begin{cases} L\eta = 0 & \text{in } \boldsymbol{D}\\ \gamma\cdot\nabla\eta = \left(1 + \nabla\hat{b}_1\right) - \int_{\mathbb{T}^{d-1}}\left(1 + \nabla\hat{b}_1\right)(x)m_0(dx) & \text{on } \boldsymbol{\partial D}\end{cases} \tag{18}$$

Taking such a solution η, we have by Itô :

$$\begin{aligned}\varepsilon\left[\eta\left(\frac{X_t^\varepsilon}{\varepsilon}\right)-\eta\left(\frac{x}{\varepsilon}\right)\right]&=\int_0^t\nabla\eta\sigma\left(\frac{X_s^\varepsilon}{\varepsilon}\right)dB_s+\int_0^t\nabla\eta c\left(\frac{X_s^\varepsilon}{\varepsilon}\right)ds\\&+\int_0^t\left(1+\nabla\hat{b}_1\right)\left(\frac{X_s^\varepsilon}{\varepsilon}\right)d\varphi_s^\varepsilon-\varphi_t^\varepsilon\int_{\mathbb{T}^{d-1}}\left(1+\nabla\hat{b}_1\right)(x)m_0(dx)\end{aligned}\tag{19}$$

Putting (19) into the first component of (17) we have

$$\begin{aligned}\hat{X}_t^{1,\varepsilon}=&x_1+\int_0^t\left(1+\nabla\hat{b}_1\right)\sigma_1\left(\frac{X_s^\varepsilon}{\varepsilon}\right)dB_s^1+\int_0^t\left(1+\nabla\hat{b}_1\right)c_1\left(\frac{X_s^\varepsilon}{\varepsilon}\right)ds\\&+\varphi_t^\varepsilon\int_{\mathbb{T}^{d-1}}\left(1+\nabla\hat{b}_1\right)(x)m_0(dx)\\&\underbrace{-\int_0^t\nabla\eta\sigma\left(\frac{X_s^\varepsilon}{\varepsilon}\right)dB_s-\int_0^t\nabla\eta c\left(\frac{X_s^\varepsilon}{\varepsilon}\right)ds+\varepsilon\left[\eta\left(\frac{X_t}{\varepsilon}\right)-\eta\left(\frac{x}{\varepsilon}\right)\right]}_{A_\varepsilon(t)}\end{aligned}\tag{20}$$

From (19) and with a well knowing Skorohod problem (see Pilipenko [12] or Tanaka [13]) it is easy to show that the term

$$\lim_{\varepsilon\to0}\mathbb{E}\left\{\max_{0\le t\le T}|A_\varepsilon(t)|\right\}=0\tag{21}$$

Before proceeding, we introduce some definition :

$$\begin{aligned}a_0&=\int_{\mathbb{T}^d}\left[\left(I+\nabla\hat{b}\right)a\left(I+\nabla\hat{b}\right)^*\right](x)m(dx),\\c_0&=\int_{\mathbb{T}^d}\left[\left(I+\nabla\hat{b}\right)c\right](x)m(dx),\\L_0&=\tfrac{1}{2}\sum_{i,j=1}^d(a_0)_{ij}\frac{\partial^2}{\partial x_i\partial x_j}+\sum_{i=1}^dc_0^i\frac{\partial}{\partial x_i}\\\Gamma_0&=\gamma_0\cdot\nabla\quad\text{with}\quad\gamma_0=\int_{\mathbb{T}^{d-1}}\left[\left(I+\nabla\hat{b}\right)\gamma\right](x)m_0(dx).\end{aligned}$$

As in [8], we have the

Theorem 2.1. *Under the condition (15), the $(L_\varepsilon,\Gamma_\varepsilon)$-reflected diffusion process X^ε converges in law to the (L_0,Γ_0)-reflected diffusion process X as $\varepsilon\downarrow0$. Moreover, on the space $C\left([0,T],\mathbb{R}^{2d+1}\right)$ equipped with the sup-norm topology,*

$$\left(X^\varepsilon,M_t^{X^\varepsilon},\varphi^\varepsilon\right)\longrightarrow\left(X,M^X,\varphi\right)$$

where

- $M_t^{X^\varepsilon}=\int_0^t\left(I+\nabla\hat{b}\right)\sigma\left(\frac{X_s^\varepsilon}{\varepsilon}\right)dB_s$

- *M^X is the martingale part of X*
- *φ (resp. φ^ε) is the local time of X^1 (resp. $X^{1,\varepsilon}$)*

It follows moreover easily from the result in Tanaka ([13]) that

Remark 2.2. Under the conditions of the (2.1) ,for any $p \leq 1$

$$\sup_{\varepsilon} \mathbb{E}\left(|X_t^\varepsilon|^p + \varphi_t^\varepsilon\right) < +\infty.$$

Let $\overline{X}$ denote the unique diffusion process with values in the d-dimensional torus $\mathbb{T}^d$, whose generator is the operator L.

We now consider a BSDE, for each fixed $(t,x) \in [0,T] \times \overline{\boldsymbol{D}}$,
let $\left\{ (Y_s^\varepsilon, Z_s^\varepsilon, U_s^\varepsilon)\,; 0 \leq s \leq T \right\}$ be the solution of the BSDE

$$\begin{aligned} Y_s^\varepsilon =& g\left(X_t^\varepsilon\right) + \int_s^t f\left(\frac{X_r^\epsilon}{\varepsilon}, Y_r^\varepsilon\right) dr + \frac{1}{\varepsilon}\int_s^t e\left(\frac{X_r^\varepsilon}{\varepsilon}, Y_r^\epsilon\right) dr \\ &+ \int_s^t h\left(\frac{X_r^\varepsilon}{\varepsilon}, Y_r^\varepsilon\right) d\varphi_r^\varepsilon - \int_s^t Z_r^\varepsilon dB_r - \int_s^t U_r^\varepsilon dr \\ &(Y^\varepsilon, U^\varepsilon) \in \boldsymbol{Gr}\left(\partial\phi\right) d\mathbb{P} \times dt \quad \text{on} \quad \Omega \times [0,t]\,. \end{aligned} \tag{22}$$

Where g is continuous, with polynomial growth and values in a bounded and convex open $\boldsymbol{\Theta}$ in $\mathbb{R}^d$ and

$$\phi(x) = \begin{cases} 0 & \text{if} \quad x \in \overline{\boldsymbol{\Theta}} \\ +\infty & \quad \text{else.} \end{cases} \tag{23}$$

$$\begin{gathered} \partial\phi\left(u\right) = \left\{u^* \in \mathbb{R}^d : \langle u^*, v-u\rangle + \phi(u) \leq \phi(v)\ ,\ \forall v \in \mathbb{R}^d\right\} \\ \text{Dom}\left(\partial\phi\left(u\right)\right) = \left\{u \in \mathbb{R}^d : \partial\phi\left(u\right) \neq \emptyset\right\} \end{gathered}$$

$$\boldsymbol{Gr}\left(\partial\phi\left(u\right)\right) = \left\{(u,u^*) \in \mathbb{R}^{2d} : u \in \text{Dom}\left(\partial\phi\left(u\right)\right) \quad \text{and} \quad u^* \in \partial\phi\left(u\right)\right\}$$

We check that ϕ is a convex, s.c.i. function, proper with $\text{Dom}\left(\partial\phi\left(u\right)\right) = \overline{\boldsymbol{\Theta}}$ and that

$$\partial\phi\left(x\right) = \left\{y^* \in \mathbb{R}^d : \langle y, x-z\rangle \geq 0\ ,\ \forall, z \in \overline{\boldsymbol{\Theta}} \text{ for } x \in \overline{\boldsymbol{\Theta}}\right\}$$

For each fixed $y \in \mathbb{R}$, let set $\hat{e}$ be the solution of the Poisson equation :

$$L\hat{e}(x,y) + e(x,y) = 0,\ x \in \mathbb{T}^d,\ y \in \mathbb{R} \tag{24}$$

More precisely by (2), $\hat{e}$ is centred with respect to the invariant measure m and is given by the formula

$$\hat{e}(x,y)=\int_0^{\infty}\mathbb{E}^x e\left(\overline{X}_t,y\right)dt. \tag{25}$$

Note that, see [10], $\hat{e}\in C^{0,2}\left(\mathbb{T}^d,\mathbb{R}\right)$ and $\hat{e}(.,y),\dfrac{\partial}{\partial y}\hat{e}(.,y),\dfrac{\partial^2}{\partial y^2}\hat{e}(.,y)\in W^{2,p}\left(\mathbb{T}^d\right)$, for any $p\geq 1$ there exists K' such that for all $y\in\mathbb{R}$

$$\|\hat{e}(.,y)\|_{W^{2,p}\left(\mathbb{T}^d\right)}+\left\|\frac{\partial}{\partial y}\hat{e}(.,y)\right\|_{W^{2,p}\left(\mathbb{T}^d\right)}+\left\|\frac{\partial^2}{\partial y^2}\hat{e}(.,y)\right\|_{W^{2,p}\left(\mathbb{T}^d\right)}\leq K' \tag{26}$$

$$0\in\mathbf{\Theta}. \tag{27}$$

Our objective is to show the family $\left(X^{\varepsilon},M^{X^{\varepsilon}},\varphi^{\varepsilon},Y^{\varepsilon},M^{\varepsilon},K^{\varepsilon}\right)$ converge in loi on $\left(X,M^X,\varphi,Y,M,K\right)$ such that

$$\begin{aligned}X_t&=x+c_0t+\int_0^t b_0(Y_s)ds\\&\quad+\sqrt{a_0}B_t+\gamma_0\varphi_t\\Y_t&=g(X_T)+\int_t^T f_0(Y_s)ds\\&\quad+\int_t^T h_0(Y_s)d\varphi_s+M_t-M_T+K_T-K_t\end{aligned}$$

where

$$\begin{aligned}b_0(y)&=\int_{\mathbb{T}^d}\left[\left(I+\nabla\hat{b}\right)a\frac{\partial^2\hat{e}}{\partial x\partial y}(.,y)\right](x)m(dx),\\h_0(y)&=\int_{\mathbb{T}^{d-1}}\left(h(.,y)+\left\langle\frac{\partial\hat{e}}{\partial x}(.,y),\gamma\right\rangle\right)(x)m_0(dx)\\f_0(y)&=\int_{\mathbb{T}^d}\left(f(.,y)+\left[\left\langle\frac{\partial\hat{e}}{\partial x}(.,y),c\right\rangle-\left(\frac{\partial\hat{e}}{\partial y}(.,y)\times e(.,y)\right)\right.\right.\\&\quad\left.\left.+\frac{\partial^2\hat{e}}{\partial x\partial y}(.,y)a\left(\frac{\partial\hat{e}}{\partial x}\right)^*(.,y)\right]\right)(x)m(dx).\end{aligned},$$

Choose that Z_t^{ε} such that $Z_t^{\varepsilon}=H_t^{\varepsilon}\sigma\left(X_t^{\varepsilon}/\varepsilon\right)$ and we introduce the notation

$$M_t^{\varepsilon}=\int_0^t H_r^{\varepsilon}dM_r^{X^{\varepsilon}}\quad;\quad M_t=\int_0^t H_r dM_r^X,\quad 0\leq t\leq T$$

$$K_t^{\varepsilon}=\int_0^t U_r^{\varepsilon}dr\quad;\quad K_t=\int_0^t U_r dr,\quad 0\leq t\leq T$$

3. MAIN RESULT

Theorem 3.1. *Under the condition on the space $C\Big([0,t],\mathbb{R}^{2d+1}\Big)\times D\Big([0,t],\mathbb{R}^{2}\Big)$ where we equip the first factor with the sup-norm topology and the second factor with the S-topology of Jakubowski ,*

$$\big(X^\varepsilon, M^{X^\varepsilon}, \varphi^\varepsilon, Y^\varepsilon, M^\varepsilon, K^\varepsilon\big) \;\Rightarrow\; \big(X, M^X, \varphi, Y, M, K\big)$$

Moreover ,

$$Y_0^\varepsilon \longrightarrow Y_0 \;\; in \;\; \mathbb{R}.$$

Proof. Let us proceed step by step.
$*$ *Step 1: Transformation of the BSDE*
Let $\widetilde{Y}_s^\varepsilon = Y_s^\varepsilon - \varepsilon\hat{e}(\overline{X}_s^\varepsilon, Y_s^\varepsilon)$ and $\overline{X_t^\varepsilon} = \dfrac{1}{\varepsilon}X_t^\varepsilon$.
We deduce from the Itô's formula that:

$$\begin{aligned}
&\widetilde{Y}_s^\varepsilon + \varepsilon\hat{e}(\overline{X}_t^\varepsilon, Y_t^\varepsilon) = g(X_t^\varepsilon) + \int_s^t \bigg(< \nabla_x\hat{e}, c > - \frac{\partial \hat{e}}{\partial y}.e\bigg)(\overline{X}_r^\varepsilon, Y_r^\varepsilon)dr \\
&+ \int_s^t \bigg(1 - \varepsilon\frac{\partial \hat{e}}{\partial y}(\overline{X}_r^\varepsilon, Y_r^\varepsilon)\bigg)\bigg(h(\overline{X}_r^\varepsilon, Y_r^\varepsilon)d\varphi_r^\epsilon + h(\overline{X}_r^\varepsilon, Y_r^\varepsilon)dr\bigg) \\
&+ \int_s^t \bigg(1 - \varepsilon\frac{\partial \hat{e}}{\partial y}(\overline{X}_r^\varepsilon, Y_r^\varepsilon)\bigg)U_r^\varepsilon dr + \frac{\varepsilon}{2}\int_s^t \frac{\partial^2 \hat{e}}{\partial y^2}(\overline{X}_r^\epsilon, Y_r^\varepsilon)|Z_r^\varepsilon|^2 dr \\
&+ \frac{1}{2}\int_s^t \frac{\partial^2 \hat{e}}{\partial x\partial y}(\overline{X}_r^\varepsilon, Y_r^\varepsilon)Z_r^\varepsilon\sigma(\overline{X}_r^\varepsilon)dr + \varepsilon\int_s^t \frac{\partial \hat{e}}{\partial y}(\overline{X}_r^\varepsilon, Y_r^\varepsilon)Z_r^\varepsilon dB_r \\
&+ \int_s^t \bigg(\nabla_x\hat{e}(\overline{X}_r^\varepsilon, Y_r^\varepsilon)\sigma(\overline{X}_r^\varepsilon) - Z_r^\varepsilon\bigg)dB_r \qquad .
\end{aligned}$$

We now define :

$$\tilde{Z}_s^\varepsilon = Z_s^\varepsilon - \nabla_x\hat{e}(\overline{X}_s^\varepsilon, Y_s^\varepsilon)\sigma(\overline{X}_s^\varepsilon)\;, 0 \le s \le t$$

and note that the difference between $\tilde{Z}_s^\varepsilon$ and Z_s^ε is a uniformly bounded process.
It then follows that

$$\begin{aligned}
&Y_s^\varepsilon + \varepsilon\Big(\hat{e}(\overline{X}_t^\varepsilon, Y_t^\varepsilon) - \hat{e}(\overline{X}_s^\varepsilon, Y_s^\varepsilon)\Big) = g(X_t^\varepsilon) \\
&+ \int_s^t \bigg(1 - \varepsilon\frac{\partial \hat{e}}{\partial y}(\overline{X}_r^\varepsilon, Y_r^\varepsilon)\bigg)\bigg(h(\overline{X}_r^\varepsilon, Y_r^\varepsilon)d\varphi_r^\varepsilon + f(\overline{X}_r^\varepsilon, Y_r^\varepsilon)dr\bigg) \\
&+ \int_s^t \bigg(< \nabla_x\hat{e}, c > - \frac{\partial \hat{e}}{\partial y}.e + \frac{\partial^2 \hat{e}}{\partial x\partial y}a\nabla_x\hat{e}\bigg)(\overline{X}_r^\varepsilon, Y_r^\varepsilon)dr \\
&- \int_s^t \tilde{Z}_r^\varepsilon\bigg(dB_r - \frac{\partial^2 \hat{e}}{\partial x\partial y}\sigma(\overline{X}_r^\varepsilon, Y_r^\varepsilon)dr\bigg) + \frac{\varepsilon}{2}\int_s^t \frac{\partial^2 \hat{e}}{\partial y^2}(\overline{X}_r^\varepsilon, Y_r^\varepsilon)|Z_r^\varepsilon|^2 dr \\
&+ \varepsilon\int_s^t \frac{\partial \tilde{e}}{\partial y}(\overline{X}_r^\varepsilon, Y_r^\varepsilon)Z_r^\varepsilon dB_r + \int_s^t \bigg(1 - \varepsilon\frac{\partial \hat{e}}{\partial y}(\overline{X}_r^\varepsilon, Y_r^\varepsilon)\bigg)U_r^\varepsilon dr.
\end{aligned}$$

We next let

$$\tilde{B}_s = B_s - \int_0^s \frac{\partial^2 \hat{e}}{\partial x \partial y} \sigma(\overline{X}^\varepsilon_r, Y^\varepsilon_r) dr.$$

It follows from Girsanov's theorem that there exists a new probability measure $\tilde{\mathbb{P}}$ equivalent to $\mathbb{P}$ under which $\tilde{B}_s$ is a martingale.

Let $\widetilde{X}^\epsilon_s = X^\epsilon_s + \varepsilon \hat{b}(\overline{X}^\varepsilon_s)$, we have ($c_\varepsilon = c(\overline{X}^\varepsilon_s)$, and similary for $(I + \nabla \hat{b}), a, \sigma, \gamma)$

$$d\widetilde{X}^\varepsilon_s = \Big(I + \nabla \hat{b}\Big)_\varepsilon \Big(c_\varepsilon ds + \gamma_\varepsilon d\varphi^\varepsilon_s + \sigma_\varepsilon d\widetilde{B}_s + a_\varepsilon \frac{\partial^2 \hat{e}}{\partial x \partial y} (\overline{X}^\varepsilon_s, Y^\varepsilon_s) ds\Big)$$
$$\widetilde{X}^\varepsilon_0 = x + \varepsilon \hat{b}(\frac{x}{\varepsilon}).$$

and

$$\begin{aligned}
&Y^\varepsilon_s + \varepsilon\Big(\hat{e}(\overline{X}^\varepsilon_t, Y^\varepsilon_t) - \hat{e}(\overline{X}^\varepsilon_s, Y^\varepsilon_s)\Big) = g(X^\varepsilon_t) \\
&+ \int_s^t \Big(1 - \varepsilon \frac{\partial \hat{e}}{\partial y}(\overline{X}^\varepsilon_r, Y^\varepsilon_r)\Big)\Big(h(\overline{X}^\varepsilon_r, Y^\varepsilon_r) d\varphi^\epsilon_r + f(\overline{X}^\varepsilon_r, Y^\varepsilon_r) dr\Big) \\
&\quad + \int_s^t \Big[< \nabla_x \hat{e}, c > - \frac{\partial \hat{e}}{\partial y}.e + \frac{\partial^2 \hat{e}}{\partial x \partial y} a \nabla_x \hat{e}^* \Big] (\overline{X}^\varepsilon_r, Y^\varepsilon_r) dr \\
&\quad - \int_s^t \tilde{Z}^\varepsilon_r d\tilde{B}_r + \frac{\varepsilon}{2} \int_s^t \frac{\partial^2 \hat{e}}{\partial y^2}(\overline{X}^\varepsilon_r, Y^\varepsilon_r) dr + \int_s^t \Big(1 - \varepsilon \frac{\partial \hat{e}}{\partial y}(\overline{X}^\varepsilon_r, Y^\varepsilon_r)\Big) U^\varepsilon_r dr \\
&\quad + \varepsilon \int_s^t \frac{\partial \tilde{e}}{\partial y}(\overline{X}^\varepsilon_r, Y^\varepsilon_r) Z^\varepsilon_r \Big[d\tilde{B}_r + \frac{\partial^2 \hat{e}}{\partial x \partial y} \sigma(\overline{X}^\varepsilon_r, Y^\varepsilon_r) dr \Big].
\end{aligned}$$

Since $\frac{\partial^2 \widehat{e}}{\partial x \partial y}$ is bounded, then for any $p > 0$, $\sup_\varepsilon \widetilde{\mathbb{E}} \|X^\varepsilon_t\|^p < +\infty$, hence for any $k > 0$, $\sup_\varepsilon \widetilde{\mathbb{E}} \|g(X^\varepsilon_t)\|^k < +\infty$.

$*$ *Step 2: A priori estimate for* $(Y^\varepsilon, Z^\varepsilon)$
We need to bound appropriate moments Y^ε and Z^ε under $\widetilde{\mathbb{P}}$.
We first go back to our unperturbed BSDE under the mew martingale $\widetilde{B}_t$.

$$\begin{aligned}
Y^\varepsilon_s =& g(X^\varepsilon_s) + \int_s^t \Big[\frac{1}{\varepsilon} e(\overline{X}^\varepsilon_r, Y^\varepsilon_r) + f(r, \overline{X}^\varepsilon_r, Y^\varepsilon_r)\Big] dr - \int_s^t Z^\varepsilon_r \frac{\partial^2 \widehat{e}}{\partial x \partial y} \sigma(\overline{X}^\varepsilon_r, Y^\varepsilon_r) dr \\
&+ \int_s^t h(\overline{X}^\varepsilon_r, Y^\varepsilon_r) d\varphi^\varepsilon_r - \int_s^t Z^\varepsilon_r d\widetilde{B}_r - \int_s^t U^\varepsilon_r dr.
\end{aligned}$$

Applying Itô's formula to develop $e^{\nu s}|Y_s^\varepsilon|^3$, with $\nu > 0$ we have

$$\begin{aligned}
&e^{\nu s}|Y_s^\varepsilon|^3 + \int_s^t e^{\nu r}\left(\nu|Y_r^\varepsilon|^3 + 3|Y_r^\varepsilon| \times |Z_r^\varepsilon|^2\right) dr\\
=&e^{\nu s}|g\left(X_s^\varepsilon\right)|^3 + \frac{3}{\varepsilon}\int_s^t e^{\nu r}|Y_r^\varepsilon|Y_r^\varepsilon e\left(\overline{X}_r^\varepsilon, Y_r^\varepsilon\right) dr + 3\int_s^t e^{\nu r}|Y_r^\varepsilon|Y_r^\varepsilon f\left(\overline{X}_r^\varepsilon, Y_r^\varepsilon\right) dr\\
&+ 3\int_s^t e^{\nu r}|Y_r^\varepsilon|Y_r^\varepsilon h\left(\overline{X}_r^\varepsilon, Y_r^\varepsilon\right) d\varphi_r^\varepsilon - 3\int_s^t e^{\nu r}|Y_r^\varepsilon|Y_r^\varepsilon Z_r^\varepsilon \frac{\partial^2\widehat{e}}{\partial x\partial y}\sigma(\overline{X}_r^\varepsilon, Y_r^\varepsilon)dr\\
&- 3\int_s^t e^{\nu r}|Y_r^\varepsilon|Y_r^\varepsilon Z_r^\varepsilon d\widetilde{B}_r - 3\int_s^t e^{\nu r}|Y_r^\varepsilon|Y_r^\varepsilon U_r^\varepsilon dr.
\end{aligned}$$

It follows from an argument in Pardoux and Peng ([9]) that the expectation of the above stochastic integral is zero. Moreover from (4 to 8), we have

$$\varepsilon\tilde{\mathbb{E}}\int_s^t |Y_r^\varepsilon||Z_r^\varepsilon|^2 dr \leq c\left[\varepsilon + \tilde{\mathbb{E}}\left(\int_s^t |Y_r^\varepsilon|^2 dr + \varepsilon\left(\varphi_t - \varphi_s\right)\right)\right]. \tag{28}$$

Let us assume for the moment the following lemma :

Lemma 3.2. *Under the conditions of the (2.1) ,we have*

$$\sup_\varepsilon \tilde{\mathbb{E}}\int_s^t |U_r^\varepsilon|^2 dr < +\infty.$$

Let $\widetilde{Y}_t^\varepsilon = Y_t^\varepsilon - \varepsilon\Big(\hat{e}(\overline{X}_s^\varepsilon, Y_s^\varepsilon)\Big)$.
We deduce from Itô's formula and the last relation in the step 1:

$$\begin{aligned}
&\left|\widetilde{Y}_t^\varepsilon\right|^2 + \int_s^t \left|\widetilde{Z}_r^\varepsilon - \varepsilon Z_r^\varepsilon\frac{\partial\hat{e}}{\partial y}(\overline{X}_r^\varepsilon, Y_r^\varepsilon)\right|^2 dr = \left|g(X_s^\varepsilon) - \varepsilon\hat{e}(\overline{X}_s^\varepsilon, Y_s^\varepsilon)\right|^2\\
&+2\int_s^t \widetilde{Y}_r^\varepsilon\big[< \nabla_x\hat{e}, c > -\frac{\partial\hat{e}}{\partial y}.e + (1-\varepsilon\frac{\partial\widehat{e}}{\partial y})f + \frac{\partial^2\hat{e}}{\partial x\partial y}a\nabla_x\hat{e}^*\big](\overline{X}_r^\varepsilon, Y_r^\varepsilon)dr\\
&-2\int_s^t \widetilde{Y}_r^\varepsilon\widetilde{Z}_r^\varepsilon d\widetilde{B}_r + \varepsilon\int_s^t \frac{\partial^2\hat{e}}{\partial y^2}(\overline{X}_r^\varepsilon, Y_r^\varepsilon)\widetilde{Y}_r^\varepsilon\left|\widetilde{Z}_r^\varepsilon\right|^2 dr\\
&- 2\int_s^t \left(1-\varepsilon\frac{\partial\hat{e}}{\partial y}(\overline{X}_r^\varepsilon, Y_r^\varepsilon)\right)U_r^\varepsilon dr\\
&+ 2\int_s^t \widetilde{Y}_r^\varepsilon\left[(1-\varepsilon\frac{\partial\widehat{e}}{\partial y})h + \frac{\partial\widehat{e}}{\partial x}\gamma\right](\overline{X}_r^\varepsilon, Y_r^\varepsilon)d\varphi_r^\varepsilon\\
&+2\varepsilon\int_s^t \widetilde{Y}_r^\epsilon Z_r^\varepsilon\frac{\partial\hat{e}}{\partial y}(\overline{X}_r^\varepsilon, Y_r^\varepsilon)\Big[d\widetilde{B}_r + \frac{\partial^2\hat{e}}{\partial x\partial y}\sigma(\overline{X}_r^\varepsilon, Y_r^\varepsilon)dr\Big]
\end{aligned}$$

Exploiting estimations (28) and (4 to 8)) together with the fact that $1-\varepsilon\frac{\partial\widehat{e}}{\partial y} \geq \dfrac{1}{2}$ for ε small enough, and standard inequalities, there exists $\lambda > 0$ such that

$$\sup_{0\leq s\leq t}\tilde{\mathbb{E}}\,|Y_s^\varepsilon|^2 + \frac{1}{2}\int_s^t \left|\tilde{Z}_r^\varepsilon\right|^2 dr + \lambda\int_s^t \left|\tilde{Y}_r^\varepsilon\right|^2 d\varphi_r^\varepsilon \leq C.$$

and, finally, from this last inequality, the above identity and the David-Burkholder Gundy's inequality is strict.

$$\sup_{0\leq\varepsilon\leq\varepsilon_0}\widetilde{\mathbb{E}}\left(\sup_{0\leq s\leq t}|Y_s^\varepsilon|^2+\int_0^t\left|\tilde{Z}_r^\varepsilon\right|^2dr+\int_0^t\left|\tilde{Y}_r^\varepsilon\right|^2d\varphi_r^\varepsilon\right)\leq\infty$$

$*$ *Step 3: Tightness*

Let $\mathcal{F}_t^\varepsilon=\mathcal{F}_t^{X^\varepsilon}$

We write our reflected BSDE in the form

$$Y_s^\varepsilon=g(X_t^\varepsilon)+A_t^\varepsilon-A_s^\varepsilon+M_t^\varepsilon-M_s^\varepsilon+D_t^\varepsilon-D_s^\varepsilon$$

Where

$$\begin{aligned}
A_s^\varepsilon=&\int_0^s\left[<\nabla_x\hat{e},c>-\frac{\partial\hat{e}}{\partial y}.e+\frac{\partial^2\hat{e}}{\partial x\partial y}a\nabla_x\hat{e}^*\right](\overline{X}_r^\varepsilon,Y_r^\varepsilon)dr\\
&+\int_0^s\left[h+\frac{\partial\widehat{e}}{\partial x}\gamma\right](\overline{X}_r^\varepsilon,Y_r^\varepsilon)d\varphi_r^\varepsilon+\int_0^s f(\overline{X}_r^\varepsilon,Y_r^\varepsilon)dr\\
M_s^\varepsilon=&\int_0^s\widetilde{Z}_r^\varepsilon d\widetilde{B}_r\quad,\quad K_s^\varepsilon=-\int_0^s U_r^\varepsilon dr\\
D_s^\varepsilon=&\varepsilon\Big(\hat{e}(\overline{X}_s^\varepsilon,Y_s^\varepsilon)-\hat{e}(\overline{X}_t^\varepsilon,Y_t^\varepsilon)\Big)+\frac{\varepsilon}{2}\int_0^s\frac{\partial^2\hat{e}}{\partial y^2}(\overline{X}_r^\varepsilon,Y_r^\varepsilon)dr\\
&+\varepsilon\int_0^s\frac{\partial\tilde{e}}{\partial y}(\overline{X}_r^\varepsilon,Y_r^\varepsilon)Z_r^\varepsilon\left[d\tilde{B}_r+\frac{\partial^2\hat{e}}{\partial x\partial y}\sigma(\overline{X}_r^\varepsilon,Y_r^\varepsilon)dr\right]\\
&+\varepsilon\int_0^s\frac{\partial\hat{e}}{\partial y}(\overline{X}_r^\varepsilon,Y_r^\varepsilon)U_r^\varepsilon dr\\
&+\varepsilon\int_0^s\frac{\partial\hat{e}}{\partial y}(\overline{X}_r^\varepsilon,Y_r^\varepsilon)f(\overline{X}_r^\varepsilon,Y_r^\varepsilon)dr+\varepsilon\int_0^s\frac{\partial\hat{e}}{\partial y}(\overline{X}_r^\varepsilon,Y_r^\varepsilon)h(\overline{X}_r^\varepsilon,Y_r^\varepsilon)d\varphi_r^\varepsilon
\end{aligned}$$

It is easy to check that

$$\widetilde{\mathbb{E}}\sup_{0\leq s\leq t}|D_s^\varepsilon|\longrightarrow 0$$

hence , $\sup_{0\leq s\leq t}|D_s^\varepsilon|$ tends to zero in $\widetilde{\mathbb{P}}$,probability, or equivalently in law.

According to the lemma (3.2) we have

$$\tilde{\mathbb{E}}\left[|K_s^\varepsilon-K_t^\varepsilon|^2\right]\leq C|s-t|\,,\quad\forall\,0\leq s\leq t\tag{29}$$

and consequently the tension of the family $\{K^\varepsilon:\varepsilon>0\}$ is obtained by using the Aldous criterion ([1]) (see also [4]).

In order to treat the other terms, we adopt the point of view of the S-topology. We define the conditional variation of the process A^ε on the interval [0, t] as the quantity

$$C\mathcal{V}_t(A^\epsilon)=\sup\widetilde{\mathbb{E}}\left(\sum_i\left\|\widetilde{\mathbb{E}}\Big(A_{t_{i+1}}^\epsilon-A_{t_i}^\varepsilon/\mathcal{F}_{t_i}^\varepsilon\Big)\right\|\right),$$

where the sup is over all the partitions of the interval [0, t] .
Clearly,

$$C\mathcal{V}_t(A^\varepsilon) \leq \widetilde{\mathbb{E}}\bigg(\int_0^t f(X_s^\varepsilon, Y_s^\varepsilon)ds + \int_0^t h(X_s^\varepsilon, Y_s^\varepsilon)d\varphi_s^\varepsilon \bigg)$$

and it follows from step 2 that

$$\sup_{\epsilon}\left(C\mathcal{V}_t(A^\varepsilon) + \sup_{0\leq s\leq t} \widetilde{\mathbb{E}}\,|Y_s^\varepsilon| + \sup_{0\leq s\leq t} \widetilde{\mathbb{E}}\left| \int_0^t \widetilde{Z}_s^\varepsilon d\widetilde{B}_s \right| \right) < \infty$$

hence, the sequences $\left\{ \left(Y_s^\varepsilon, \int_0^t \widetilde{Z}_s^\varepsilon d\widetilde{B}_s \right), 0 \leq s \leq t \right\}$ satisfy Meyer-Zheng's tightness criterion for quasimartingales under $\widetilde{\mathbb{P}}$.

∗ *Step 4: Passage to the limit*

After extraction of a suitable subsequence, which we omit as an abuse of notations, we have that

$$\left\{ \left(X^\varepsilon, M^\varepsilon, \varphi^\varepsilon, Y^\varepsilon, \int_0^t \widetilde{Z}_s^\varepsilon d\widetilde{B}_s, K^\varepsilon \right), 0 \leq s \leq t \right\} \Rightarrow \big\{ \big(X, M, \varphi, Y, \overline{M}, K \big) \big\}$$

weakly on $C\left([0,t];\mathbb{R}^{d+1}\right) \times D\left([0,t];\mathbb{R}^2\right)$ equipped with the product of the topology of uniform convergence on the first factor, and the S-topology on the second factor.
It remains to recall the two next results:

Lemma 3.3. $\varphi : \mathbb{R}^d \times \mathbb{R} \longrightarrow \mathbb{R}$ *be measurable and locally bounded, periodic of period one in each direction, continuous with respect to its second argument, uniformly with respect to the first. Then the sequence of processes* $\left\{ \int_0^s \varphi\left(\frac{X_r^\varepsilon}{\varepsilon}, Y_r^\varepsilon \right) dr; 0 \leq r \leq s \right\}$ *converges in law under* $\widetilde{\mathbb{P}}$ *to* $\left\{ \int_0^s \varphi_0\left(Y_r\right) dr; 0 \leq r \leq s \right\}$, *where*

$$\varphi_0(y) = \int_{\mathbb{T}^d} \varphi(x,y)m(dx)$$

Proof. (see [8]). □

Lemma 3.4. $\phi : \mathbb{R}^d \times \mathbb{R} \longrightarrow \mathbb{R}$ *be measurable and locally bounded, periodic of period one in each direction with respect to its first argument, continuous with respect to its second argument, uniformly with respect to the first. Then*

$$\sup_{0\leq s\leq T}\left| \int_0^s \phi\left(\frac{X_r^\varepsilon}{\varepsilon}, Y_r^\varepsilon \right) d\phi_r^\varepsilon - \int_0^s \phi_0\left(Y_r\right) d\phi_r \right| \to 0$$

in $\widetilde{\mathbb{P}}$ *probability as* $\varepsilon \to 0$ *,where*

$$\phi_0(y) = \int_{\mathbb{T}^{d-1}} \phi(x,y) m_0(dx).$$

$*$ *Step 3: Identification of the limit*

Let $\left(\overline{Y}, \overline{U}\right)$ denote the unique solution of the BSDE

$$\begin{cases} \overline{Y}_t = & g\left(X_T\right) + \int_t^T f_0\left(\overline{Y}_s\right) ds + \int_t^T h_0\left(\overline{Y}_s\right) d\varphi_s \\ & - \int_t^T \overline{Z}_s dM_s^X - \int_t^T \overline{U}_s ds \\ & \left(\overline{Y}, \overline{U}\right) \in Gr\left(\partial\phi\right) \end{cases} \tag{30}$$

satisfying ,

$$\mathbb{E}Tr \int_0^t \overline{Z}_r d < M^X >_r \overline{Z}_r^* < \infty$$

It follows from the lemma (3.2) that $\{K^\varepsilon : \varepsilon > 0\}$ the process is bounded in $\mathbb{L}^2\left(\Omega, H^1\left([0,t], \mathbb{R}^d\right)\right)$ with $H^1\left([0,t], \mathbb{R}^d\right)$ is the Sobolev space of absolutely continuous functions with derivatives in $\mathbb{L}^2\left([0,t]\right)$. Therefore $\left\{\overline{U}_s, 0 \le s \le t\right\}$ the process is boundedly continuous with derivative such that

$$\int_0^t |\overline{U}_r|^2 dr < +\infty$$

and let $M_s = \int_0^s \overline{U} dM_r^X$, $K_t = -\int_0^t U_s ds$.

Since $\overline{Y}$ and $\overline{U}$ are $\mathcal{F}_t^X$ adapted and M_r^X is a $\mathcal{F}_t^{X,Y}$-martingale , so is also $\widetilde{M}$. It follows from Itô's formula for possibly discontinuous semimartingales that

$$\begin{aligned} & \mathbb{E}\left|Y_s - \overline{Y}_s\right|^2 + \mathbb{E}\left(\left[M - \overline{M}\right]_t - \left[M - \overline{M}\right]_s\right) \\ = & 2\mathbb{E}\int_s^t \left\langle Y_r - \overline{Y}_r, f(Y_r) - f(\overline{Y}_r)\right\rangle dr \\ + & 2\mathbb{E}\int_s^t \left\langle Y_r - \overline{Y}_r, h(Y_r) - h(\overline{Y}_r)\right\rangle d\phi_r \\ & + 2\mathbb{E}\int_s^t \left\langle Y_r - \overline{Y}_r, dK_r - dK_r\right\rangle \\ \le & \, 2\mu \int_s^t \mathbb{E}\left|Y_r - \overline{Y}_r\right|^2 dr. \end{aligned}$$

Therefore $\displaystyle\int_s^t \left\langle Y_r - \overline{Y}_r, dK_r - dK_r \right\rangle \leq 0$ (we use the fact that $\beta \leq 0$).
Hence from Gronwall's lemma

$$Y_s = \overline{Y}_s \quad , \quad M_s = \overline{M}_s \quad \text{and} \quad U_s = \overline{U}_s \;\; 0 \leq s \leq t$$

Before proving the lemma (3.2), let us recall some results concerning the method of penalty method (see Menaldi [5])

$$\alpha(x) = \frac{1}{2} grad \left(min \left\{ |x - y|, y \in \overline{\mathbf{\Theta}} \right\} \right)$$

Note that there exists $a \in \mathbb{R}^d$ and $\beta > 0$ such that

$$(x - a)\, \alpha(x) \geq \beta |\alpha|, \forall x \in \mathbb{R}^d,$$

As a result, the

$$\langle A_n(x), x - a \rangle \geq \delta |A_n(x)|. \tag{31}$$

Let's say $A_n\left(Y_s^\varepsilon\right) = n\left(Y_s^\varepsilon - Pr_{\overline{\mathbf{\Theta}}}\left(Y_s^\varepsilon\right)\right)$, where $Pr_{\mathbf{\Theta}}$ is the projection of $\mathbf{\Theta}$. By an approximation technique(Gegout [6]), we can assume that $\mathbf{\Theta}$ is bounded convex and regular ie

$$\rho(x) = d^2\left(x, \mathbf{\Theta}\right) = |x - Pr(x)|^2$$

, is convex twice differentiable and $\mathbf{\Theta} = \left\{ x \in \mathbb{R}^d \;,\; \rho > 0 \right\}$; $\partial\mathbf{\Theta} = \left\{ x \in \mathbb{R}^d \;,\; \rho = 0 \right\}$,note that

$$\nabla \rho(x) = 2\alpha(x) = 2\left(x - Pr(x)\right)^* .$$

Proof of the lemma (3.2):
Let $(Y_r^{\varepsilon,n}, Z_r^{\varepsilon,n})$ is unique solution of the BSDE

$$\begin{aligned} Y_s^{\varepsilon,n} =& g(X_s^\varepsilon) + \int_s^t \left[\frac{1}{\varepsilon} e(\overline{X}_r^\varepsilon, Y_r^{\varepsilon,n}) + f(r, \overline{X}_r^\varepsilon, Y_r^{\varepsilon,n}) \right] dr \\ & - \int_s^t Z_r^{\varepsilon,n} \frac{\partial^2 \widehat{e}}{\partial x \partial y} \sigma(\overline{X}_r^\varepsilon, Y_r^{\varepsilon,n}) dr - \int_s^t A_n\left(Y_r^{\varepsilon,n}\right) dr \\ & + \int_s^t h(\overline{X}_r^\varepsilon, Y_r^{\varepsilon,n}) d\varphi_r^\varepsilon - \int_s^t Z_r^{\varepsilon,n} d\widetilde{B}_r \end{aligned} .$$

let $a \in \mathbb{R}^d$ verifying (31), it follows from Ito's formula that

$$\begin{aligned}
& e^{\nu s}|Y_s^{\varepsilon,n}-a|^3+\int_s^t e^{\nu r}\left(\nu|Y_r^{\varepsilon,n}-a|^3+3|Y_r^{\varepsilon,n}-a|\times|Z_r^{\varepsilon,n}|^2\right)dr\\
=& e^{\nu s}|g\left(X_s^{\varepsilon}\right)|^3+\frac{3}{\varepsilon}\int_s^t e^{\nu r}|Y_r^{\varepsilon,n}-a|\left(Y_r^{\varepsilon,n}-a\right)e\left(\overline{X}_r^{\varepsilon},Y_r^{\varepsilon}\right)dr\\
& +3\int_s^t e^{\nu r}|Y_r^{\varepsilon,n}-a|Y_r^{\varepsilon,n}f\left(\overline{X}_r^{\varepsilon},Y_r^{\varepsilon,n}\right)dr\\
& +3\int_s^t e^{\nu r}|Y_r^{\varepsilon,n}-a|Y_r^{\varepsilon,n}h\left(\overline{X}_r^{\varepsilon},Y_r^{\varepsilon,n}\right)d\varphi_r^{\varepsilon}\\
& -3\int_s^t e^{\nu r}|Y_r^{\varepsilon,n}-a|\left(Y_r^{\varepsilon,n}-a\right)Z_r^{\varepsilon,n}\frac{\partial^2\widehat{e}}{\partial x\partial y}\sigma(\overline{X}_r^{\varepsilon},Y_r^{\varepsilon,n})dr\\
& -3\int_s^t e^{\nu r}|Y_r^{\varepsilon,n}-a|\left(Y_r^{\varepsilon,n}-a\right)Z_r^{\varepsilon,n}d\widetilde{B}_r\\
& -3\int_s^t e^{\nu r}|Y_r^{\varepsilon,n}-a|\left(Y_r^{\varepsilon,n}-a,A_n\left(Y_r^{\varepsilon,n}\right)\right)dr
\end{aligned}$$

Based on the assumptions made above,

$$\begin{aligned}
|Y_r^{\varepsilon,n}-a|\left(Y_r^{\varepsilon,n}-a\right)f\left(\overline{X}_r^{\varepsilon},Y_r^{\varepsilon,n}\right)&\leq|Y_r^{\varepsilon,n}-a|^3+c\\
-|Y_r^{\varepsilon,n}-a|\left(Y_r^{\varepsilon,n}-a,A_n\left(Y_r^{\varepsilon,n}\right)\right)&\leq-\delta|A_n\left(Y_r^{\varepsilon,n}\right)||Y_r^{\varepsilon,n}-a|\leq0 \quad .
\end{aligned}$$

Let $\widetilde{Y}_t^{\varepsilon,n}=Y_t^{\varepsilon,n}-\varepsilon\Big(\hat{e}(\overline{X}_s^{\varepsilon},Y_s^{\varepsilon,n})\Big)$.
We deduce from Itô's formulla

$$\begin{aligned}
& \left|\widetilde{Y}_t^{\varepsilon,n}-a\right|^2+\int_s^t\left|\widetilde{Z}_r^{\varepsilon,n}-\varepsilon Z_r^{\varepsilon,n}\frac{\partial\hat{e}}{\partial y}(\overline{X}_r^{\varepsilon},Y_r^{\varepsilon,n})\right|^2dr\\
=& \left|g(X_s^{\varepsilon})-\varepsilon\hat{e}(\overline{X}_s^{\varepsilon},Y_s^{\varepsilon,n})\right|^2-2\int_s^t\left(\widetilde{Y}_r^{\varepsilon,n}-a\right)\widetilde{Z}_r^{\varepsilon,n}d\widetilde{B}_r\\
& +2\int_s^t\left(\widetilde{Y}_r^{\varepsilon,n}-a\right)\Big[<\nabla_x\hat{e},c>-\frac{\partial\hat{e}}{\partial y}.e\\
& \quad+(1-\varepsilon\frac{\partial\widehat{e}}{\partial y})f+\frac{\partial^2\hat{e}}{\partial x\partial y}a\nabla_x\hat{e}^*\Big](\overline{X}_r^{\varepsilon},Y_r^{\varepsilon,n})dr\\
& \quad+\varepsilon\int_s^t\frac{\partial^2\hat{e}}{\partial y^2}(\overline{X}_r^{\varepsilon},Y_r^{\varepsilon,n})\left(\widetilde{Y}_r^{\varepsilon,n}-a\right)\left|\widetilde{Z}_r^{\varepsilon,n}\right|^2dr\\
& \quad-2\int_s^t\left(1-\varepsilon\frac{\partial\hat{e}}{\partial y}(\overline{X}_r^{\varepsilon},Y_r^{\varepsilon,n})\right)\left(\widetilde{Y}_r^{\varepsilon,n}-a,A_n\left(\widetilde{Y}_r^{\varepsilon,n}\right)\right)dr\\
& +2\varepsilon\int_s^t\left(\widetilde{Y}_r^{\varepsilon,n}-a\right)Z_r^{\varepsilon,n}\frac{\partial\hat{e}}{\partial y}(\overline{X}_r^{\varepsilon},Y_r^{\varepsilon,n})\Big[d\widetilde{B}_r+\frac{\partial^2\hat{e}}{\partial x\partial y}\sigma(\overline{X}_r^{\varepsilon},Y_r^{\varepsilon,n})dr\Big]\\
& \quad+2\int_s^t\left(\widetilde{Y}_r^{\varepsilon,n}-a\right)\Big[(1-\varepsilon\frac{\partial\widehat{e}}{\partial y})h+\frac{\partial\widehat{e}}{\partial x}\gamma\Big](\overline{X}_r^{\varepsilon},Y_r^{\varepsilon,n})d\varphi_r^{\varepsilon}
\end{aligned}$$

as (27), $A_n\left(\widetilde{Y}_t^{\varepsilon,n}\right)=A_n\left(Y_t^{\varepsilon,n}\right)$ moreover

$$\left\langle\widetilde{Y}_t^{\varepsilon,n}-a,A_n\left(\widetilde{Y}_t^{\varepsilon,n}\right)\right\rangle\geq\delta|A_n\left(\widetilde{Y}_t^{\varepsilon,n}\right)|=\delta|A_n\left(Y_t^{\varepsilon,n}\right)|.$$

Exploiting estimations (28) together with the fact that
$1-\varepsilon\frac{\partial\widehat{e}}{\partial y}\geq\dfrac{1}{2}$ for ε small enough, standard inequalities and the Gronwall lemma we have

$$\begin{aligned}\sup_{0\leq s\leq t}\sup_{\varepsilon,n}\tilde{\mathbb{E}}\Bigg(|Y_s^{\varepsilon,n}-a|^2+\int_s^t\left|\tilde{Z}_r^{\varepsilon,n}\right|^2dr+\int_s^t\left|\tilde{Y}_r^{\varepsilon,n}\right|^2d\varphi_r^\varepsilon\Bigg)\\ +\delta\int_s^t|A_n\left(Y_r^{\varepsilon,n}\right)|^2dr\leq+\infty.\end{aligned}\tag{32}$$

Now by Itô's formula , the convex of the function ρ ,
$\widetilde{Y}_t^{\varepsilon,n}=Y_t^{\varepsilon,n}-\varepsilon\hat{e}(\overline{X}_s^\varepsilon,Y_s^{\varepsilon,n})$, we have

$$\begin{aligned}&\rho\left(Y_s^{\varepsilon,n}-\varepsilon\hat{e}(\overline{X}_s^\varepsilon,Y_s^{\varepsilon,n})\right)\leq\rho\left(g(X_t^\varepsilon)-\varepsilon\hat{e}(\overline{X}_t^\varepsilon,Y_t^{\varepsilon,n})\right)\\&+\frac{2}{n}\int_s^t\widehat{Y}_r^{\varepsilon,n}\Bigg[<\nabla_x\hat{e},c>-\frac{\partial\hat{e}}{\partial y}.e\\&+(1-\varepsilon\frac{\partial\widehat{e}}{\partial y})f+\frac{\partial^2\hat{e}}{\partial x\partial y}a\nabla_x\hat{e}^*\Bigg](\overline{X}_r^\varepsilon,Y_r^{\varepsilon,n})dr\\&-\frac{2}{n}\int_s^t\widehat{Y}_r^{\varepsilon,n}\widetilde{Z}_r^{\varepsilon,n}d\widetilde{B}_r+\frac{2\varepsilon}{n}\int_s^t\frac{\partial^2\hat{e}}{\partial y^2}(\overline{X}_r^\varepsilon,Y_r^{\varepsilon,n})\widehat{Y}_r^{\varepsilon,n}\left|\widetilde{Z}_r^{\varepsilon,n}\right|^2dr\\&-\frac{2}{n}\int_s^t\left(1-\varepsilon\frac{\partial\hat{e}}{\partial y}(\overline{X}_r^\varepsilon,Y_r^{\varepsilon,n})\right)A_n\left(\widehat{Y}_r^{\varepsilon,n}\right)^*A_n\left(Y_r^{\varepsilon,n}\right)dr\\&+\frac{2\varepsilon}{n}\int_s^tA_n\left(\widehat{Y}_r^{\varepsilon,n}\right)Z_r^{\varepsilon,n}\frac{\partial\hat{e}}{\partial y}(\overline{X}_r^\varepsilon,Y_r^{\varepsilon,n})\Big[d\widetilde{B}_r+\frac{\partial^2\hat{e}}{\partial x\partial y}\sigma(\overline{X}_r^\varepsilon,Y_r^{\varepsilon,n})dr\Big]\\&+2\int_s^t\left(\widehat{Y}_r^{\varepsilon,n}\right)\Bigg[(1-\varepsilon\frac{\partial\widehat{e}}{\partial y})h+\frac{\partial\widehat{e}}{\partial x}\gamma\Bigg](\overline{X}_r^\varepsilon,Y_r^{\varepsilon,n})d\varphi_r^\varepsilon\qquad.\end{aligned}$$

Together with the fact that $1-\varepsilon\frac{\partial\widehat{e}}{\partial y}\geq\dfrac{1}{2}$ and (27),we have

$$\begin{aligned}&n\rho\left(\widehat{Y}_s^{\varepsilon,n}\right)+\int_s^t|A_n\left(Y_r^{\varepsilon,n}\right)|^2dr\\&\leq+2\int_s^tA_n\left(Y_r^{\varepsilon,n}\right)\Bigg[<\nabla_x\hat{e},c>-\frac{\partial\hat{e}}{\partial y}.e\\&+(1-\varepsilon\frac{\partial\widehat{e}}{\partial y})f+\frac{\partial^2\hat{e}}{\partial x\partial y}a\nabla_x\hat{e}^*\Bigg](\overline{X}_r^\varepsilon,Y_r^{\varepsilon,n})dr\\&-2\int_s^tA_n\left(Y_r^{\varepsilon,n}\right)\widetilde{Z}_r^{\varepsilon,n}d\widetilde{B}_r+2\varepsilon\int_s^t\frac{\partial^2\hat{e}}{\partial y^2}(\overline{X}_r^\varepsilon,Y_r^{\varepsilon,n})A_n\left(Y_r^{\varepsilon,n}\right)\left|\widetilde{Z}_r^{\varepsilon,n}\right|^2dr\\&+2\varepsilon\int_s^tA_n\left(\widehat{Y}_r^{\varepsilon,n}\right)Z_r^{\varepsilon,n}\frac{\partial\hat{e}}{\partial y}(\overline{X}_r^\varepsilon,Y_r^{\varepsilon,n})\Big[d\widetilde{B}_r+\frac{\partial^2\hat{e}}{\partial x\partial y}\sigma(\overline{X}_r^\varepsilon,Y_r^{\varepsilon,n})dr\Big]\\&+2\int_s^tA_n\left(Y_r^{\varepsilon,n}\right)\Bigg[(1-\varepsilon\frac{\partial\widehat{e}}{\partial y})h+\frac{\partial\widehat{e}}{\partial x}\gamma\Bigg](\overline{X}_r^\varepsilon,Y_r^{\varepsilon,n})d\varphi_r^\varepsilon\qquad.\end{aligned}$$

If we want to find a majoration of this last integral, we have

$$\rho\left(\widehat{Y}_s^{\varepsilon,n}\right)+\frac{1}{2}\int_s^t trace\left(Z^{\varepsilon,n}Z^{\varepsilon,n*}Hess\left(\rho\left(Y^{\varepsilon,n}\right)\right)\right)dr$$

$$=\int_s^t\left(\frac{1}{\varepsilon}e\left(\overline{X}_r^{\varepsilon},Y_r^{\varepsilon}\right)+f\left(\overline{X}_r^{\varepsilon},Y_r^{\varepsilon}\right)-Z_r^{\varepsilon}\frac{\partial^2\widehat{e}}{\partial x\partial y}\sigma(\overline{X}_r^{\varepsilon},Y_r^{\varepsilon})\right)\nabla\rho\left(Y_r^{\varepsilon,n}\right)dr$$

$$+\int_s^t\nabla\rho\left(Y_r^{\varepsilon,n}\right)h\left(\overline{X}_r^{\varepsilon},Y_r^{\varepsilon}\right)d\varphi_r^{\varepsilon}-\int_s^t\nabla\rho\left(Y_r^{\varepsilon,n}\right)Z_r^{\varepsilon,n}d\widetilde{B}_r$$

$$-\int_s^t\nabla\rho\left(Y_r^{\varepsilon,n}\right)A_n\left(Y_r^{\varepsilon,n}\right)dr$$

It follows from Itô's formula that

$$e^{\nu s}\rho^{3/2}\left(Y_s^{\varepsilon,n}\right)+\frac{3}{4}\int_s^t trace\left(Z^{\varepsilon,n}Z^{\varepsilon,n*}Hess\left(\rho^{1/2}\left(Y^{\varepsilon,n}\right)\right)\right)\rho\left(Y_r^{\varepsilon,n}\right)dr$$

$$+\int_s^t\nu e^{\nu r}\rho^{3/2}\left(Y^{\varepsilon,n}\right)dr+\frac{3}{8}\int_s^t e^{\nu r}|Z^{\varepsilon,n}|^2|\nabla\rho\left(Y_r^{\varepsilon,n}\right)|^2\rho^{-1/2}\left(Y^{\varepsilon,n}\right)dr$$

$$=\frac{3}{2}\int_s^t e^{\nu r}\left(\frac{1}{\varepsilon}e\left(\overline{X}_r^{\varepsilon},Y_r^{\varepsilon,n}\right)+f\left(\overline{X}_r^{\varepsilon},Y_r^{\varepsilon,n}\right)\right.$$

$$\left.-Z_r^{\varepsilon,n}\frac{\partial^2\widehat{e}}{\partial x\partial y}\sigma(\overline{X}_r^{\varepsilon},Y_r^{\varepsilon,n})\right)\nabla\rho\left(Y_r^{\varepsilon,n}\right)\rho^{1/2}\left(Y_r^{\varepsilon,n}\right)dr$$

$$-\frac{3}{2}\int_s^t\nabla\rho\left(Y_r^{\varepsilon,n}\right)\rho^{1/2}\left(Y_r^{\varepsilon,n}\right)h\left(\overline{X}_r^{\varepsilon},Y_r^{\varepsilon,n}\right)d\varphi_r^{\varepsilon}$$

$$-\frac{3}{2}\int_s^t\nabla\rho\left(Y_r^{\varepsilon,n}\right)Z_r^{\varepsilon,n}\rho^{1/2}\left(Y_r^{\varepsilon,n}\right)d\widetilde{B}_r$$

$$-\frac{3}{2}\int_s^t\nabla\rho\left(Y_r^{\varepsilon,n}\right)A_n\left(Y_r^{\varepsilon,n}\right)\rho^{1/2}\left(Y_r^{\varepsilon,n}\right)dr$$

Furthermore,$|A_n\left(Y_r^{\varepsilon,n}\right)|=\dfrac{n}{2}\nabla\rho\left(Y_r^{\varepsilon,n}\right)$ and $\rho^{1/2}\left(x\right)=|x-Pr(x)|$ we have

$$ne^{\nu s}\rho^{3/2}\left(Y_s^{\varepsilon,n}\right)+\frac{3}{2}\int_s^t trace\left(Z^{\varepsilon,n}Z^{\varepsilon,n*}Hess\left(\rho^{1/2}\left(Y^{\varepsilon,n}\right)\right)\right)\rho\left(Y_r^{\varepsilon,n}\right)dr$$

$$+n\int_s^t\nu e^{\nu r}\rho^{3/2}\left(Y^{\varepsilon,n}\right)dr+\frac{3}{2}\int_s^t e^{\nu r}|Z^{\varepsilon,n}|^2|A_n\left(Y_r^{\varepsilon,n}\right)|dr$$

$$=\frac{3n}{2}\int_s^t e^{\nu r}\left(\frac{1}{\varepsilon}e\left(\overline{X}_r^{\varepsilon},Y_r^{\varepsilon,n}\right)+f\left(\overline{X}_r^{\varepsilon},Y_r^{\varepsilon,n}\right)\right.$$

$$\left.-Z_r^{\varepsilon,n}\frac{\partial^2\widehat{e}}{\partial x\partial y}\sigma(\overline{X}_r^{\varepsilon},Y_r^{\varepsilon,n})\right)\nabla\rho\left(Y_r^{\varepsilon,n}\right)\rho^{1/2}\left(Y_r^{\varepsilon,n}\right)dr$$

$$-\frac{3}{2}\int_s^t\nabla\rho\left(Y_r^{\varepsilon,n}\right)\rho^{1/2}\left(Y_r^{\varepsilon,n}\right)h\left(\overline{X}_r^{\varepsilon},Y_r^{\varepsilon,n}\right)d\varphi_r^{\varepsilon}$$

$$-n\int_s^t\nabla\rho\left(Y_r^{\varepsilon,n}\right)Z_r^{\varepsilon,n}\rho^{1/2}\left(Y_r^{\varepsilon,n}\right)d\widetilde{B}_r-\frac{3}{n}\int_s^t|A_n\left(Y_r^{\varepsilon,n}\right)|^3dr$$

We have,

$$
\begin{aligned}
& Z_r^{\varepsilon,n}\frac{\partial^2\widehat{e}}{\partial x\partial y}\sigma(\overline{X}_r^{\varepsilon},Y_r^{\varepsilon,n})\nabla\rho\left(Y_r^{\varepsilon,n}\right)\rho^{1/2}\left(Y_r^{\varepsilon,n}\right)\\
& \leq \frac{n}{2}|Z_r^{\varepsilon,n}|\left\|\frac{\partial^2\widehat{e}}{\partial x\partial y}\right\|_\infty|\nabla\rho\left(Y_r^{\varepsilon,n}\right)|^2\\
& \leq \frac{n}{2}\left\|\frac{\partial^2\widehat{e}}{\partial x\partial y}\right\|_\infty\left(\frac{|Z_r^{\varepsilon,n}|^2|\nabla\rho\left(Y_r^{\varepsilon,n}\right)|}{\varrho}+\varrho|\nabla\rho\left(Y_r^{\varepsilon,n}\right)|^3\right)\\
& \leq \frac{1}{4\varrho}\left\|\frac{\partial^2\widehat{e}}{\partial x\partial y}\right\|_\infty|Z_r^{\varepsilon,n}|^2|A_n\left(Y_r^{\varepsilon,n}\right)|+\frac{n\varrho}{4}\left\|\frac{\partial^2\widehat{e}}{\partial x\partial y}\right\|_\infty|\nabla\rho\left(Y_r^{\varepsilon,n}\right)|^3\\
=& \frac{1}{4\varrho}\left\|\frac{\partial^2\widehat{e}}{\partial x\partial y}\right\|_\infty|Z_r^{\varepsilon,n}|^2|A_n\left(Y_r^{\varepsilon,n}\right)|+\frac{2\varrho}{n^2}\left\|\frac{\partial^2\widehat{e}}{\partial x\partial y}\right\|_\infty|A_n\left(Y_r^{\varepsilon,n}\right)|^3
\end{aligned}
$$

By choosing ϱ larger so that $\frac{1}{4\varrho}\left\|\frac{\partial^2\widehat{e}}{\partial x\partial y}\right\|_\infty\leq\frac{3}{8}$ and $\frac{\varrho}{n^2}\left\|\frac{\partial^2\widehat{e}}{\partial x\partial y}\right\|_\infty\leq\frac{3}{2n}$

$$
\begin{aligned}
& nf\left(\overline{X}_r^{\varepsilon},Y_r^{\varepsilon,n}\right)\nabla\rho\left(Y_r^{\varepsilon,n}\right)\rho^{1/2}\left(Y_r^{\varepsilon,n}\right)\\
\leq & nK\left(\frac{|\nabla\rho\left(Y_r^{\varepsilon,n}\right)|^2}{2}+\frac{|\rho\left(Y_r^{\varepsilon,n}\right)|}{2}\right)\\
& \leq\left(\frac{K}{n}|A_n\left(Y_r^{\varepsilon,n}\right)|^2+\frac{nK|\rho\left(Y_r^{\varepsilon,n}\right)|}{2}\right)
\end{aligned}
$$

$$
\begin{aligned}
& nh\left(\overline{X}_r^{\varepsilon},Y_r^{\varepsilon,n}\right)\nabla\rho\left(Y_r^{\varepsilon,n}\right)\rho^{1/2}\left(Y_r^{\varepsilon,n}\right)\\
\leq & nK\left(\frac{|\nabla\rho\left(Y_r^{\varepsilon,n}\right)|^2}{2}+\frac{|\rho\left(Y_r^{\varepsilon,n}\right)|}{2}\right)\\
& \leq\left(\frac{K}{n}\left(|A_n\left(Y_r^{\varepsilon,n}\right)|^2+1\right)+\frac{nK|\rho\left(Y_r^{\varepsilon,n}\right)|}{2}\right)
\end{aligned}
$$

Finally, for ν well chosen, n large enough and the relation (32), we have

$$
\varepsilon\tilde{\mathbb{E}}\int_s^t|A_n\left(Y_r^{\varepsilon,n}\right)|Z_r^{\varepsilon,n}|^2dr\leq C\left(1+\frac{1}{n}\tilde{\mathbb{E}}\int_s^t\left(Y_r^{\varepsilon,n}\right)|^2dr+\frac{1}{n}\left(\varphi_t^{\varepsilon}-\varphi_s^{\varepsilon}\right)\right) \tag{33}
$$

For n large enough and ε small enough taking into account the inequalities and (32) we have

$$
\sup_{\varepsilon,n}\tilde{\mathbb{E}}\int_0^t|A_n\left(Y_r^{\varepsilon,n}\right)|^2dr<\infty.
$$

Now available at

$$
\lim_{n\to\infty}\mathbb{E}\left(\sup_{0\leq s\leq t}|K_s^{\varepsilon,n}-K_s^{\varepsilon}\right)=0.
$$

where $K_s^{\varepsilon,n}=-\int_0^s A_n\left(Y_r^{\varepsilon,n}\right)dr$ (see [7]).

Inequality (33) and Fatou's lemma gives the result. $\square$

4. HOMOGENIZATION OF SEMILINEAR PDES

Let u be the IVS solution

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) - L_0 u(t,x) - f_0\left(u(t,x)\right) - b_0\left(u(t,x)\right)\nabla u(t,x) \in \partial\phi\left(u^{\varepsilon}(t,x)\right),\ x \in \boldsymbol{D},\ 0 < t \\ \Gamma_0 u(t,x) + h_0\left(u(t,x)\right) \in \partial\phi\left(u(t,x)\right),\ x \in \partial\boldsymbol{D},\ 0 \leq t \\ u(0,x) = g(x), \qquad x \in \partial\boldsymbol{D} \end{cases} \tag{34}$$

Where L_0 , Γ_0 , b_0; f_0 and h_0 as defined in section(2). We deduce from the smoothness of the solution u and uniqueness of limiting BSDE that $\overline{Y} = u(t,x)$.

Theorem 4.1. *Under the above assumptions,* $\forall (t,x) \in \mathbb{R}_+ \times \overline{\boldsymbol{D}}$ *we have*

$$u^{\varepsilon}(t,x) \longrightarrow u(t,x) \ \textit{ in } \ \mathbb{R} \quad \textit{as} \quad \varepsilon \to 0.$$

Acknowledgment

This research was supported by the mathematics and applications laboratory of Assane Seck University.

We thank the anonymous referees for comments and suggestions which will allowed us to improve the final version of this article.

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