

On MDS Symbol-Pair γ -Constacyclic Codes of Length p^s over $\mathbb{F}_{p^m}[u]/\langle u^2 \rangle$

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ABSTRACT

MDS symbol-pair codes form an optimal class of symbol-pair codes for their high error-correction capability. Let $\mathcal{R} = \mathbb{F}_{p^m}[u]/\langle u^2 \rangle$ be the finite commutative chain ring with unity. γ -constacyclic codes of length p^s over $\mathcal{R}$, where γ is a nonzero element of the field $\mathbb{F}_{p^m}$, had been classified into four types. Let $\mathcal{C}_3 = \langle (x - \gamma_0)^\delta + u(x - \gamma_0)^t h(x) \rangle$ be a code of Type 3, where $h(x)$ is a unit, $0 \leq t < \delta$, and $1 \leq \delta \leq p^s - 1$. In this paper, we derive necessary and sufficient conditions under which $\mathcal{C}_3$ is a maximum distance separable (MDS) code with respect to the symbol-pair metric.

Keywords: Repeated-root codes; constacyclic codes; MDS codes; symbol-pair distance; finite chain rings.

1 INTRODUCTION

Let us consider the code alphabet Σ with q elements, whose elements are called symbols. In symbol-pair read channels, a codeword $\mathbf{x} = (x_0, x_1, \dots, x_{n-1})$ is represented as

$$\pi(\mathbf{x}) = ((x_0, x_1), (x_1, x_2), \dots, (x_{n-1}, x_0)) \in (\Sigma^2)^n. \quad (1)$$

Every vector $\mathbf{x} \in \Sigma^n$ has a unique pair representation $\pi(\mathbf{x})$. For any two symbol pairs (a, b) and (f, g) , say $(a, b) = (f, g)$ if both $a = f$ and $b = g$. An important parameter of symbol-pair codes is the symbol-pair distance. In 2010, Cassuto and Blaum [1] gave the definition of the symbol-pair distance as the Hamming distance over the alphabet

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$(\mathcal{L}, \mathcal{L})$. Given $\mathbf{x} = (x_0, x_1, \dots, x_{n-1})$, $\mathbf{y} = (y_0, y_1, \dots, y_{n-1})$, the symbol-pair distance between $\mathbf{x}$ and $\mathbf{y}$ is defined as

$$d_{sp}(\mathbf{x}, \mathbf{y}) = d_H(\pi(\mathbf{x}), \pi(\mathbf{y})) = |\{i \in \mathbb{Z}_n \mid (x_i, x_{i+1}) \neq (y_i, y_{i+1})\}|, \quad (2)$$

where $\mathbb{Z}_n$ denotes the ring $\mathbb{Z}/n\mathbb{Z}$. The symbol-pair distance of a symbol-pair code C is defined as

$$d_{sp}(C) = \min\{d_{sp}(\mathbf{x}, \mathbf{y}) \mid \mathbf{x}, \mathbf{y} \in C, \mathbf{x} \neq \mathbf{y}\}. \quad (3)$$

The symbol-pair weight of a vector $\mathbf{x}$ is defined as the Hamming weight of its symbol-pair vector $\pi(\mathbf{x})$:

$$wt_{sp}(\mathbf{x}) = wt_H(\pi(\mathbf{x})) = |\{i \in \mathbb{Z}_n \mid (x_i, x_{i+1}) \neq (0, 0)\}|. \quad (4)$$

If the code C is linear, its symbol-pair distance is equal to the minimum symbol-pair weight of nonzero codewords of C :

$$d_{sp}(C) = \min\{wt_{sp}(\mathbf{x}) \mid 0 \neq \mathbf{x} \in C\}. \quad (5)$$

Inspired by reading restrictions in high-density data storage systems, symbol-pair codes desired to protect against a certain number of pair-errors were first introduced by Cassuto and Blaum [1]. They provided constructions and decoding methods of symbol-pair codes. Again, they established the relationship between the symbol-pair distance and the Hamming distance by [2]. In 2011, by using algebraic methods, Cassuto and Litsyn [3] constructed cyclic symbol-pair codes. Recently, in [4], Yaakobi et al. considered and gave a lower bound on the symbol-pair distances for binary cyclic codes. By giving the definition of parity-check matrix for symbol-pair codes, a new syndrome decoding algorithm of symbol-pair codes is given by Hirotomo et al. [5]. In particular, in 2015, Kai et al. [6] extended the result of Cassuto and Litsyn [3] for the case of simple-root constacyclic codes.

It is well-known that, for any fixed code of length n and dimension k , the maximum distance separable (briefly, MDS) codes have the best possible error correcting capability. Thus, constructing MDS codes has become one of the central topics in coding theory. MDS symbol-pair codes can be considered as a generalization of MDS codes for the classical case. This is the reason why many researchers evaluated MDS symbol-pair codes (for example, [6; 7; 8; 9; 10]).

In engineering, constacyclic codes are most preferred since they can be efficiently encoded using shift registers, due to their rich algebraic structure. For a unit element λ in a commutative ring R , a code C of length n over R is said to be λ -constacyclic if $(c_0, c_1, \dots, c_{n-1}) \in C$ implies that $(\lambda c_{n-1}, c_0, \dots, c_{n-2}) \in C$. A λ -constacyclic code of length n over R can be classified as an ideal in $R[x]/\langle x^n - \lambda \rangle$. When $\lambda = 1$, the class of λ -constacyclic codes are known as cyclic codes. The case when the code length n is divisible by the characteristic p of the residue field R , the constacyclic codes are called

repeated-root constacyclic codes. Let γ be a nonzero element of the field $\mathbb{F}_{p^m}$. All γ -constacyclic code of length p^s over $\mathcal{R} = \mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$ are classified into 4 distinct types and their detailed structures are also established in [11]. Let $\mathcal{C}_3 = \langle (x - \gamma_0)^\delta + u(x - \gamma_0)^t h(x) \rangle$ be of Type 3, as in [11], where $h(x)$ is a unit, $0 \leq t < \delta$, and $1 \leq \delta \leq p^s - 1$.

In [12], Dinh et al. obtained MDS symbol-pair γ -constacyclic codes of prime power length over $\mathbb{F}_{p^m}$, by satisfying the Singleton bound of symbol-pair codes. To continue the line of study, in this paper, we derive necessary and sufficient conditions under which $\mathcal{C}_3$ is a MDS symbol-pair code.

The organization of this paper is as follows. In Section 2, some preliminary results that are needed to derive our main results are discussed. In Section 3, we give the symbol-pair distance $d_{sp}(\mathcal{C}_3)$ of the code $\mathcal{C}_3$ obtained in [13]. In Section 4, we derive necessary and sufficient conditions under which $\mathcal{C}_3$ is a MDS symbol-pair code. We mention a brief conclusion and discuss some interesting open problems in this direction in Section 5.

2 PRELIMINARIES

All rings are commutative rings with identity. A ring R is called principal ideal ring if its ideals are principal. R is called a local ring if R has a unique maximal ideal. Finally, R is called a chain ring if the set of all ideals of R is linearly ordered under set-theoretic inclusion.

The following result is well-known for the class of finite commutative rings (see [14]).

Proposition 2.1. If R is a finite commutative ring, then the following conditions are equivalent:

- (i) R is a local ring and the maximal ideal of R is principal,
- (ii) R is a local principal ideal ring,
- (iii) R is a chain ring.

If we denote by $\langle \mathfrak{a} \rangle$ the maximal ideal of the finite chain ring R , then $\mathfrak{a}$ is nilpotent with nilpotency index some integer e and the ideal of R from the following chain:

$$\langle 0 \rangle = \langle \mathfrak{a}^e \rangle \subsetneq \langle \mathfrak{a}^{e-1} \rangle \subsetneq \cdots \subsetneq \langle \mathfrak{a} \rangle \subsetneq \langle \mathfrak{a}^0 \rangle = R. \quad (6)$$

In this paper, let $\mathbb{F}_{p^m}$ be a finite field of p^m elements, where p is a prime number, and denote

$$\mathcal{R} = \mathbb{F}_{p^m}[u]/\langle u^2 \rangle. \quad (7)$$

The ring $\mathcal{R}$ is a finite commutative ring with p^{2m} elements, and can be expressed as $\mathcal{R} = \mathbb{F}_{p^m} + u\mathbb{F}_{p^m} = \{a + bu \mid a, b \in \mathbb{F}_{p^m}\}$. It is easy to check that $\mathcal{R}$ is a local ring

with maximal ideal $\langle u \rangle = u\mathbb{F}_{p^m}$. Therefore, by proposition 2.1, it is a chain ring. The ring $\mathcal{R}$ has precisely $p^m(p^m - 1)$ units and every invertible element in $\mathcal{R}$ is of the form: $a + bu$ where $a, b \in \mathbb{F}_{p^m}$ and $a \neq 0$.

For any invertible element γ of $\mathbb{F}_{p^m}$, γ -constacyclic codes of length p^s over a finite field $\mathbb{F}_{p^m}$ are precisely the ideals of the finite chain ring $\mathbb{F}_{p^m}[x]/\langle x^{p^s} - \gamma \rangle$. Since γ is a nonzero element of the field $\mathbb{F}_{p^m}$, there exists $\gamma_0 \in \mathbb{F}_{p^m}$ such that $\gamma_0^{p^s} = \gamma$. In [15], [12], the algebraic structure and symbol-pair distances of γ -constacyclic codes of length p^s over $\mathbb{F}_{p^m}$ were established and given by the following theorem.

Theorem 2.2 (cf. [12]) Let $\mathcal{C}$ be a γ -constacyclic code of length p^s over $\mathbb{F}_{p^m}$. Then $\mathcal{C} = \langle (x - \gamma_0)^i \rangle \subseteq \mathbb{F}_{p^m}[x]/\langle x^{p^s} - \gamma \rangle$, for $i \in \{0, 1, \dots, p^s\}$, and its symbol-pair distance $d_{sp}(\mathcal{C})$ is completely determined by:

$$d_{sp}(\mathcal{C}) = \begin{cases} \bullet 2, & \text{if } i = 0, \\ \bullet 3p^k, & \text{if } i = p^s - p^{s-k} + 1, \\ & \text{where } 0 \leq k \leq s-2, \\ \bullet 4p^k, & \text{if} \\ & p^s - p^{s-k} + 2 \leq i \leq p^s - p^{s-k} + p^{s-k-1}, \\ & \text{where } 0 \leq k \leq s-2, \\ \bullet 2(\sigma + 2)p^k, & \text{if} \\ & p^s - pr + \sigma r + 1 \leq i \leq p^s - pr + (\sigma + 1)r, \\ & \text{where } r = p^{s-k-1}, 0 \leq k \leq s-2 \\ & \text{and } 1 \leq \sigma \leq p-2, \\ \bullet (\sigma + 2)p^{s-1}, & \text{if } i = p^s - p + \sigma, \\ & \text{where } 1 \leq \sigma \leq p-2, \\ \bullet p^s, & \text{if } i = p^s - 1, \\ \bullet 0, & \text{if } i = p^s. \end{cases}$$

3 SYMBOL-PAIR DISTANCE

Let γ be a nonzero element of the field $\mathbb{F}_{p^m}$. γ -constacyclic code of length p^s over $\mathcal{R}$ are precisely the ideals of the ring

$$\mathcal{R}_\gamma = \mathcal{R}[x]/\langle x^{p^s} - \gamma \rangle. \quad (8)$$

All γ -constacyclic code of length p^s over $\mathcal{R}$ are classified into 4 distinct types and their detailed structures are also established in [11].

Let $\mathcal{C}_3$ be a γ -constacyclic code of Type 3 (principal ideals with monic polynomial generators) as in [11]:

Here, we have $\mathcal{C}_3 = \langle (x - \gamma_0)^\delta + u(x - \gamma_0)^t h(x) \rangle$, where $1 \leq t \leq \delta \leq p^s - 1$, $0 \leq$

$t < \delta$, and either $h(x)$ is 0 or a unit in $\mathbb{F}_{p^m}[x]/\langle x^{p^s} - \gamma \rangle$, $\deg h(x) \leq T - t - 1$, and T is the smallest integer satisfying $u(x - \gamma_0)^T \in \mathcal{C}_3$, i.e.,

$$T = \begin{cases} \delta, & \text{if } h(x) = 0, \\ \min\{\delta, p^s - \delta + t\}, & \text{if } h(x) \neq 0. \end{cases} \quad (9)$$

Moreover, we have

$$|\mathcal{C}_3| = p^{m(2p^s - \delta - T)}. \quad (10)$$

Note that $\mathbb{F}_{p^m}$ is a subring of $\mathcal{R}$, for a code $\mathcal{C}$ over $\mathcal{R}$, we denote $d_{sp}(\mathcal{C}_{\mathbb{F}})$ as the symbol-pair distance of $\mathcal{C}|_{\mathbb{F}_{p^m}}$.

In [16, Theorem 4.3(Type 3)], Dinh et al. stated that: $d_{sp}(\mathcal{C}_3) = d_{sp}(\langle (x - \gamma_0)^\delta \rangle_{\mathbb{F}})$. This result is not always true, which we illustrate in the following example.

Example 3.1. Consider the γ -constacyclic code $\mathcal{C}_3 = \langle (x - \gamma_0)^4 + u(x - \gamma_0)h(x) \rangle$ of length 5 over $\mathcal{R} = \mathbb{F}_{5^m} + u\mathbb{F}_{5^m}$, where $h(x) \neq 0$. Here $p = 5$, $s = 1$, $\delta = 4$ and $t = 1$. Then $T = 2$, which implies that $\langle u(x - \gamma_0)^2 \rangle \subseteq \mathcal{C}_3$. This implies that $d_{sp}(\langle u(x - \gamma_0)^2 \rangle) = d_{sp}(\langle (x - \gamma_0)^2 \rangle_{\mathbb{F}}) \geq d_{sp}(\mathcal{C}_3)$. By Theorem 2.2, we see that $d_{sp}(\langle (x - \gamma_0)^2 \rangle_{\mathbb{F}}) = 4$, which implies that

$$d_{sp}(\mathcal{C}_3) \leq 4. \quad (11)$$

By using Theorem 2.2 again, we see that

$$d_{sp}(\langle (x - \gamma_0)^4 \rangle_{\mathbb{F}}) = 5 \quad (12)$$

Now by (11) and (12), we see that $d_{sp}(\mathcal{C}_3) \neq d_{sp}(\langle (x - \gamma_0)^4 \rangle_{\mathbb{F}})$. This example shows that [16, Theorem 4.3(Type 3)] is incorrect in general.

By the above Example, we see that Theorem 4.3(Type 3) in [16] is not entirely true. In [13], Laaouine had added some assumptions to correct it completely. We list the correct result in the following, and it will play an important role in our main result.

Theorem 3.2 ([13, Theorem 6.]) Let $\mathcal{C}_3 = \langle (x - \gamma_0)^\delta + u(x - \gamma_0)^t h(x) \rangle$ be a γ -constacyclic code of length p^s over $\mathcal{R}$ of Type 3, where $1 \leq T \leq \delta \leq p^s - 1$, $0 \leq t < \delta$, either $h(x)$ is 0 or $h(x)$ is a unit and T is the smallest integer satisfying $u(x - \gamma_0)^T \in \mathcal{C}_3$, i.e.,

$$T = \begin{cases} \delta, & \text{if } h(x) = 0, \\ \min\{\delta, p^s - \delta + t\}, & \text{if } h(x) \neq 0. \end{cases}$$

Then the symbol-pair distance $d_{sp}(\mathcal{C}_3)$ of the code $\mathcal{C}_3$ is given by

$$\begin{aligned}
d_{sp}(\mathcal{C}_3) &= d_{sp}(\langle (x - \gamma_0)^T \rangle_{\mathbb{F}}) \\
&= \begin{cases} \bullet 3p^k, & \text{if } T = p^s - p^{s-k} + 1, \\ & \text{where } 0 \leq k \leq s-2, \\ \bullet 4p^k, & \text{if} \\ & p^s - p^{s-k} + 2 \leq T \leq p^s - p^{s-k} + p^{s-k-1}, \\ & \text{where } 0 \leq k \leq s-2, \\ \bullet 2(\alpha + 2)p^k, & \text{if} \\ & p^s - pr + \alpha r + 1 \leq T \leq p^s - pr + (\alpha + 1)r, \\ & \text{where } r = p^{s-k-1}, \ 0 \leq k \leq s-2 \\ & \text{and } 1 \leq \alpha \leq p-2, \\ \bullet (\alpha + 2)p^{s-1}, & \text{if } T = p^s - p + \alpha, \\ & \text{where } 1 \leq \alpha \leq p-2, \\ \bullet p^s, & \text{if } T = p^s - 1. \end{cases}
\end{aligned}$$

Proof. We first observe that

$$wt_{sp}(a(x) + ub(x)) \geq \max\{wt_{sp}(a(x)), wt_{sp}(b(x))\}, \quad (13)$$

where $a(x), b(x) \in \mathbb{F}_{p^m}[x]$.

Since $\langle u(x - \gamma_0)^T \rangle \subseteq \mathcal{C}_3$, it follows that

$$d_{sp}(\mathcal{C}_3) \leq d_{sp}(\langle u(x - \gamma_0)^T \rangle) = d_{sp}(\langle (x - \gamma_0)^T \rangle_{\mathbb{F}}). \quad (14)$$

Now, let $c(x)$ be an arbitrary nonzero element of $\mathcal{C}_3$. That means there exist $f(x), g(x) \in \mathbb{F}_{p^m}[x]$ such that

$$\begin{aligned}
c(x) &= [f(x) + ug(x)][(x - \gamma_0)^\delta + u(x - \gamma_0)^t h(x)] \\
&= f(x)(x - \gamma_0)^\delta + u[g(x)(x - \gamma_0)^\delta + f(x)(x - \gamma_0)^t h(x)] \\
&= f(x)(x - \gamma_0)^\delta + ur(x).
\end{aligned} \quad (15)$$

where $r(x) = g(x)(x - \gamma_0)^\delta + f(x)(x - \gamma_0)^t h(x)$. Thus, by (13), we obtain that

$$\begin{aligned}
wt_{sp}(c(x)) &\geq \max\{wt_{sp}(f(x)(x - \gamma_0)^\delta), wt_{sp}(r(x))\} \\
&\geq \max\{wt_{sp}(f(x)(x - \gamma_0)^\delta), wt_{sp}(g(x)(x - \gamma_0)^\delta)\} \\
&\geq d_{sp}(\langle (x - \gamma_0)^\delta \rangle_{\mathbb{F}}) \\
&\geq d_{sp}(\langle (x - \gamma_0)^T \rangle_{\mathbb{F}}) \quad (\text{because } \langle (x - \gamma_0)^\delta \rangle \subseteq \langle (x - \gamma_0)^T \rangle).
\end{aligned} \quad (16)$$

Hence, $d_{sp}(c(x)) \geq d_{sp}(\langle (x - \gamma_0)^T \rangle_{\mathbb{F}})$. From this, we get

$$d_{sp}(\mathcal{C}_3) = d_{sp}(\langle (x - \gamma_0)^T \rangle_{\mathbb{F}}). \quad (17)$$

The rest of the proof follows from Theorem 2.2 and the discussion above.

When $h(x) = 0$ or $h(x) \neq 0$ and $1 \leq \delta \leq \frac{p^s+t}{2}$, we have $T = \delta$ and when $h(x) \neq 0$

and $\frac{p^s+t}{2} < \delta \leq p^s - 1$, then $T = p^s - \delta + t$.

Corollary 3.3 ([13, Corollary 2.]) Following the same notations as in Theorem 3.2, we have the following results:

Case 1) If $h(x)$ is 0 or $h(x)$ is a unit and $1 \leq \delta \leq \frac{p^s+t}{2}$, then

$$d_{sp}(\mathcal{C}_3) = d_{sp}(\langle (x - \gamma_0)^\delta \rangle_{\mathbb{F}})$$

$$= \begin{cases} \bullet 3p^k, & \text{if } \delta = p^s - p^{s-k} + 1, \\ & \text{where } 0 \leq k \leq s-2, \\ \bullet 4p^k, & \text{if} \\ & p^s - p^{s-k} + 2 \leq \delta \leq p^s - p^{s-k} + p^{s-k-1}, \\ & \text{where } 0 \leq k \leq s-2, \\ \bullet 2(\alpha+2)p^k, & \text{if} \\ & p^s - pr + \alpha r + 1 \leq \delta \leq p^s - pr + (\alpha+1)r, \\ & \text{where } r = p^{s-k-1}, \ 0 \leq k \leq s-2 \\ & \text{and } 1 \leq \alpha \leq p-2, \\ \bullet (\alpha+2)p^{s-1}, & \text{if } \delta = p^s - p + \alpha, \\ & \text{where } 1 \leq \alpha \leq p-2, \\ \bullet p^s, & \text{if } \delta = p^s - 1. \end{cases}$$

Case 2) If $h(x)$ is a unit and $\frac{p^s+t}{2} < \delta \leq p^s - 1$, then

$$d_{sp}(\mathcal{C}_3) = d_{sp}(\langle (x - \gamma_0)^{p^s-\delta+t} \rangle_{\mathbb{F}})$$

$$= \begin{cases} \bullet 3p^k, & \text{if } \delta = t + p^{s-k} - 1, \\ & \text{where } 0 \leq k \leq s-2, \\ \bullet 4p^k, & \text{if} \\ & t + p^{s-k} - p^{s-k-1} \leq \delta \leq t + p^{s-k} - 2, \\ & \text{where } 0 \leq k \leq s-2, \\ \bullet 2(\alpha+2)p^k, & \text{if} \\ & t + pr - (\alpha+1)r \leq \delta \leq t + pr - \alpha r - 1, \\ & \text{where } r = p^{s-k-1}, \ 0 \leq k \leq s-2 \\ & \text{and } 1 \leq \alpha \leq p-2, \\ \bullet (\alpha+2)p^{s-1}, & \text{if } \delta = t + p - \alpha, \\ & \text{where } 1 \leq \alpha \leq p-2, \\ \bullet p^s, & \text{if } \delta = t + 1. \end{cases}$$

4 MDS SYMBOL-PAIR CONSTACYCLIC CODES.

To get MDS symbol-pair codes, we need to determine the singleton bound for symbol-pair codes first. Singleton bound for symbol-pair code $\mathcal{C}$ of length n over a finite commutative ring R with symbol-pair distance $d_{sp}(\mathcal{C})$ is as follows: $|\mathcal{C}| \leq$

$|R|^{(n-d_{sp}(\mathcal{C})+2)}$ (see [17]). A symbol-pair code $\mathcal{C}$ is called an MDS symbol-pair code if it attains the singleton bound for symbol-pair codes, i.e.,

$$|\mathcal{C}| = |R|^{(n-d_{sp}(\mathcal{C})+2)}. \quad (18)$$

Let $\mathcal{C} = \langle (x - \gamma_0)^i \rangle$ be a γ -constacyclic code of length p^s over $\mathbb{F}_{p^m}$, where $0 \leq i \leq p^s$. It is well known that $|\mathcal{C}| = p^{m(p^s-i)}$. So, the dimension of code $\mathcal{C}$ is $p^s - i$. By (18), $\mathcal{C}$ is an MDS symbol-pair code if and only if $p^s - i = p^s - d_{sp}(\mathcal{C}) + 2$, i.e., $i = d_{sp}(\mathcal{C}) - 2$. In [12], Dinh et al. identified all the MDS symbol-pair constacyclic codes of length p^s over $\mathbb{F}_{p^m}$.

Theorem 4.1 (cf. [12]) Let $\mathcal{C} = \langle (x - \gamma_0)^i \rangle$ be a γ -constacyclic code of length p^s over $\mathbb{F}_{p^m}$, for $i \in \{0, 1, \dots, p^s\}$. Then $\mathcal{C}$ is a MDS symbol-pair code if and only if one of the following conditions holds:

- If $s = 1$, then $i = \eta$, for $0 \leq \eta \leq p - 2$. Then $d_{sp}(\mathcal{C}) = \eta + 2$.
- If $s \geq 2$, then
 - $i = 0$, $d_{sp}(\mathcal{C}) = 2$,
 - $i = 1$, $d_{sp}(\mathcal{C}) = 3$,
 - $i = 2$, $d_{sp}(\mathcal{C}) = 4$,
 - $s = 2$, $p = 3$, $i = 4$, $d_{sp}(\mathcal{C}) = 6$,
 - $i = p^s - 2$, $d_{sp}(\mathcal{C}) = p^s$.

Now, let $\mathcal{C}_3 = \langle (x - \gamma_0)^\delta + u(x - \gamma_0)^t h(x) \rangle$ be a γ -constacyclic code of length p^s over $\mathcal{R}$ of Type 3, where $h(x)$ is a unit, $1 \leq \delta \leq p^s - 1$ and $0 \leq t < \delta$. In this section, we shall determine necessary and sufficient conditions for $\mathcal{C}_3$ to be MDS symbol-pair code. For this, the following two cases arise.

Case1: When $1 \leq \delta \leq \frac{p^s+t}{2}$ then, $|\mathcal{C}_3| = p^{2m(p^s-\delta)}$. Thus by (18), $\mathcal{C}_3$ is a symbol-pair MDS code if and only if $p^s - \delta = p^s - d_{sp}(\mathcal{C}_3) + 2$, i.e., $\delta = d_{sp}(\mathcal{C}_3) - 2$. Hence, follows the theorem.

Theorem 4.2. Let $\mathcal{C}_3 = \langle (x - \gamma_0)^\delta + u(x - \gamma_0)^t h(x) \rangle$ be of Type 3, where $h(x)$ is a unit, $0 \leq t < \delta$ and $1 \leq \delta \leq \frac{p^s+t}{2}$, then $\mathcal{C}_3$ is a MDS symbol-pair code if and only if one of the following conditions holds:

- If $s = 1$, $\delta = n$, $1 \leq n \leq p - 2$ and $\max\{2n - p, 0\} \leq t < n$, then $d_{sp}(\mathcal{C}_3) = n + 2$.
- If $s \geq 2$,
 - $\delta = 1$ and $t = 0$, then $d_{sp}(\mathcal{C}_3) = 3$,
 - $\delta = 2$ and $0 \leq t \leq 1$, then $d_{sp}(\mathcal{C}_3) = 4$,

- $s = 2, p = 3, \delta = 4$ and $0 \leq t \leq 3$, then $d_{sp}(\mathcal{C}_3) = 6$,
- $\delta = p^s - 2$ and $p^s - 4 \leq t \leq p^s - 3$, then $d_{sp}(\mathcal{C}_3) = p^s$.

Proof. We get MDS code for $\delta = d_{sp}(\mathcal{C}_3) - 2$, which is similar to the MDS γ -constacyclic symbol-pair codes over $\mathbb{F}_{p^m}$ (see Theorem 4.1). But we have $1 \leq \delta \leq \frac{p^s+t}{2}$ and $0 \leq t < \delta$, which implies that $\max\{2\delta - p^s, 0\} \leq t < \delta$. This completes the proof of the theorem.

Remark 4.3. Let $\mathcal{C}_3 = \langle (x - \gamma_0)^\delta + u(x - \gamma_0)^t h(x) \rangle$, where $h(x)$ is a unit, $0 \leq t < \delta$ and $1 \leq \delta \leq p^{s-1} + \lfloor \frac{t}{2} \rfloor$. Then by (18), $\mathcal{C}_3$ is a MDS symbol-pair code if and only if $\delta = d_{sp}(\mathcal{C}_3) - 2$. In [17, Theorem 8], Dinh et al. stated that: When $s \geq 2$, then $\delta \geq 2$. Unfortunately, this result is not true. For example, taking $s = 4, p = 5$ and $\delta = 1$, then $t = 0$. By Corollary 3.3 (case 1), we have $d_{sp}(\mathcal{C}_3) = 3$, then $d_{sp}(\mathcal{C}_3) - 2 = 3 - 2 = 1 = \delta$, which implies that $\mathcal{C}_3$ is a MDS symbol-pair code. This case shows that is incomplete.

Case2: When $\frac{p^s+t}{2} < \delta \leq p^s - 1$. In this case, $|\mathcal{C}_3| = p^{m(p^s-t)}$. Applying (18), $\mathcal{C}_3$ is an MDS symbol-pair constacyclic code if and only if $p^s - t = 2p^s - 2d_{sp}(\mathcal{C}_3) + 4$, i.e., $t = 2d_{sp}(\mathcal{C}_3) - p^s - 4$. Hence, follows the theorem.

Theorem 4.4. Let $\mathcal{C}_3 = \langle (x - \gamma_0)^\delta + u(x - \gamma_0)^t h(x) \rangle$ be of Type 3, where $h(x)$ is a unit, $0 \leq t < \delta$ and $\frac{p^s+t}{2} < \delta \leq p^s - 1$. Then no MDS symbol-pair constacyclic code exists.

Proof. Let $\mathcal{C}_3 = \langle (x - \gamma_0)^\delta + u(x - \gamma_0)^t h(x) \rangle$, where $h(x)$ is a unit. If $\frac{p^s+t}{2} < \delta \leq p^s - 1$, i.e., $2\delta > p^s + t$, we get MDS code if and only if $t = 2d_{sp}(\mathcal{C}_3) - p^s - 4$. The symbol-pair distance $d_{sp}(\mathcal{C}_3)$ is established in Corollary 3.3(case 2). In the following, we are going to discuss the case when $t = 2d_{sp}(\mathcal{C}_3) - p^s - 4$ and $2\delta > p^s + t$.

Case 1: $\delta = t + p^{s-k} - 1$, where $0 \leq k \leq s - 2$. Then $d_{sp}(\mathcal{C}_3) = 3p^k$, and $2\delta = 2t + 2p^{s-k} - 2 > p^s + t$. Hence,

$$\begin{aligned}
 t &> p^s - 2p^{s-k} + 2 \\
 &= 2p^{s-k}(p^k - 1) - p^s + 2 \\
 &\geq 2p^2(p^k - 1) - p^s + 2 \\
 &\geq 8(p^k - 1) - p^s + 2 \\
 &= 6p^k - p^s + 2p^k - 2 - 4 \\
 &\geq 6p^k - p^s - 4 \\
 &= 2d_{sp}(\mathcal{C}_3) - p^s - 4.
 \end{aligned} \tag{19}$$

Since, $t > 2d_{sp}(\mathcal{C}_3) - p^s - 4$, no MDS symbol-pair constacyclic code exists in this case.

Case 2: $t + p^{s-k} - p^{s-k-1} \leq \delta \leq t + p^{s-k} - 2$, where $0 \leq k \leq s - 2$. So

$d_{sp}(\mathcal{C}_3) = 4p^k$. We consider $\delta = t + p^{s-k} - r$, where $2 \leq r \leq p^{s-k-1}$. Then $2\delta = 2t + 2p^{s-k} - 2r > p^s + t$. Hence,

$$\begin{aligned}
 t &> p^s - 2p^{s-k} + 2r \\
 &= 2p^{s-k}(p^k - 1) - p^s + 2r \\
 &\geq 2p^2(p^k - 1) - p^s + 4 \\
 &\geq 8(p^k - 1) - p^s + 4 \\
 &= 8p^k - p^s - 4 \\
 &= 2d_{sp}(\mathcal{C}_3) - p^s - 4.
 \end{aligned} \tag{20}$$

Since, $t > 2d_{sp}(\mathcal{C}_3) - p^s - 4$, no MDS symbol-pair constacyclic code exists in this case.

Case 3: $t + p^{s-k} - (\alpha + 1)p^{s-k-1} \leq \delta \leq t + p^{s-k} - \alpha p^{s-k-1} - 1$, where $0 \leq k \leq s - 2$ and $1 \leq \alpha \leq p - 2$. Then $d_{sp}(\mathcal{C}_3) = 2(\alpha + 2)p^k$. We consider $\delta = t + p^{s-k} - \alpha p^{s-k-1} - r$, where $1 \leq r \leq p^{s-k-1}$. Then $2\delta = 2t + 2p^{s-k} - 2\alpha p^{s-k-1} - 2r > p^s + t$. Hence,

$$\begin{aligned}
 t &> p^s - 2p^{s-k} + 2\alpha p^{s-k-1} + 2r \\
 &= 2p^{s-k}(p^k - 1) - p^s + 2\alpha p^{s-k-1} + 2r \\
 &\geq 2p^2(p^k - 1) - p^s + 2\alpha p + 2 \\
 &\geq 2(\alpha + 2)^2(p^k - 1) - p^s + 2\alpha(\alpha + 2) + 2 \\
 &= 2(\alpha + 1)(\alpha + 2)p^k - p^s + 2(\alpha + 2)(p^k - 2) + 2 \\
 &\geq 4(\alpha + 2)p^k - p^s + 6(p^k - 2) + 2 \\
 &\geq 4(\alpha + 2)p^k - p^s - 4 \\
 &= 2d_{sp}(\mathcal{C}_3) - p^s - 4.
 \end{aligned} \tag{21}$$

Since, $t > 2d_{sp}(\mathcal{C}_3) - p^s - 4$, no MDS symbol-pair constacyclic code exists in this case.

Case 4: $\delta = t + p - \alpha$, where $1 \leq \alpha \leq p - 2$. Then $d_{sp}(\mathcal{C}_3) = (\alpha + 2)p^{s-1}$, and $2\delta = 2t + 2p - 2\alpha > p^s + t$. Hence,

$$\begin{aligned}
 t &> p^s - 2p + 2\alpha \\
 &= 2p(p^{s-1} - 1) - p^s + 2\alpha \\
 &\geq 2(\alpha + 2)(p^{s-1} - 1) - p^s + 2\alpha \\
 &= 2(\alpha + 2)p^{s-1} - p^s - 4 \\
 &= 2d_{sp}(\mathcal{C}_3) - p^s - 4.
 \end{aligned} \tag{22}$$

Since, $t > 2d_{sp}(\mathcal{C}_3) - p^s - 4$, no MDS symbol-pair constacyclic code exists in this case.

Case 5: $\delta = t + 1$. Then $d_{sp}(\mathcal{C}_3) = p^s$, and $2\delta = 2t + 2 > p^s + t$. Hence,

$$\begin{aligned}
t &> p^s - 2 \\
&= 2p^s - p^s - 2 \\
&> 2p^s - p^s - 4 \\
&= 2d_{sp}(\mathcal{C}_3) - p^s - 4.
\end{aligned} \tag{23}$$

Since, $t > 2d_{sp}(\mathcal{C}_3) - p^s - 4$, no MDS symbol-pair constacyclic code exists in this case.

Therefore, MDS symbol-pair code does not exist for γ -constacyclic codes of Type 3, when $h(x)$ is a unit and $\frac{p^s+t}{2} < \delta \leq p^s - 1$. This completes the proof.

Remark 4.5. Let $\mathcal{C}_3 = \langle (x - \gamma_0)^\delta + u(x - \gamma_0)^t h(x) \rangle$, where $h(x)$ is a unit, $0 \leq t < \delta$ and $p^{s-1} + \lfloor \frac{t}{2} \rfloor < \delta \leq p^s - 1$. Then by (18), $\mathcal{C}_3$ is a MDS symbol-pair code if and only if $t = 2d_{sp}(\mathcal{C}_3) - p^s - 4$. In [17, Theorem 9], Dinh et al. stated that:

1. $\mathcal{C}_3 = \langle (x - \gamma_0)^{p^s-1} + u(x - \gamma_0)^t h(x) \rangle$ is a MDS symbol-pair code, where $h(x)$ is a unit, $s \geq 1$, $p \geq 5$ and $t = p^s - 4$. Unfortunately, this result is not true. For example, taking $s = 1$ and $p = 7$, then $t = 3$. By Corollary 3.3 (case 2), we have $d_{sp}(\mathcal{C}_3) = 6$, then $2d_{sp}(\mathcal{C}_3) - p^s - 4 = 12 - 7 - 4 = 1 \neq t = 3$, which implies that $\mathcal{C}_3$ is not a MDS symbol-pair code.
2. $\mathcal{C}_3 = \langle (x - \gamma_0)^{2^s-1} + u(x - \gamma_0)^t h(x) \rangle$ is a MDS symbol-pair code, where $h(x)$ is a unit, $s \geq 2$ and $t = 2^s - 4$. This result is not true. For example, taking $s = 3$, then $t = 4$. By Corollary 3.3(case 2), we have $d_{sp}(\mathcal{C}_3) = 6$, then $2d_{sp}(\mathcal{C}_3) - 2^s - 4 = 12 - 8 - 4 = 0 \neq t = 4$, which implies that $\mathcal{C}_3$ is not a MDS symbol-pair code.
3. $\mathcal{C}_3 = \langle (x - \gamma_0)^{3^s-1} + u(x - \gamma_0)^t h(x) \rangle$ is a MDS symbol-pair code, where $h(x)$ is a unit, $s \geq 2$ and $t = 3^s - 4$. This result is not true. For example, taking $s = 2$, then $t = 5$. By Corollary 3.3(case 2), we have $d_{sp}(\mathcal{C}_3) = 6$, then $2d_{sp}(\mathcal{C}_3) - 3^s - 4 = 12 - 9 - 4 = -1 \neq t = 5$, which implies that $\mathcal{C}_3$ is not a MDS symbol-pair code.
4. $\mathcal{C}_3 = \langle (x - \gamma_0)^{2^s-3} + u(x - \gamma_0)^t h(x) \rangle$ is a MDS symbol-pair code, where $h(x)$ is a unit, $s \geq 3$ and $t = 2^{s-1} - 4$. This result is not true. For example, taking $s = 3$, then $t = 0$. By Corollary 3.3(case 2), we have $d_{sp}(\mathcal{C}_3) = 4$, then $2d_{sp}(\mathcal{C}_3) - 2^s - 4 = 8 - 8 - 4 = -4 \neq t = 0$, which implies that $\mathcal{C}_3$ is not a MDS symbol-pair code.
5. $\mathcal{C}_3 = \langle (x - \gamma_0)^{3^s-5} + u(x - \gamma_0)^t h(x) \rangle$ is a MDS symbol-pair code, where $h(x)$ is a unit, $s \geq 3$ and $t = 3^{s-1} - 4$. This result is not true. For example, taking $s = 3$, then $t = 5$. By Corollary 3.3(case 2), we have $d_{sp}(\mathcal{C}_3) = 6$, then $2d_{sp}(\mathcal{C}_3) - 3^s - 4 = 12 - 27 - 4 = -19 \neq t = 5$, which implies that $\mathcal{C}_3$ is not a MDS symbol-pair code.

These examples shows that [17, Theorem 9] is incorrect.

To conclude this section, we provide some cases study of constacyclic codes to illustrate our results.

Case study1: γ -constacyclic codes of length 7 over the chain ring $\mathcal{R} = \mathbb{F}_7 + u\mathbb{F}_7$ are precisely the ideals of $\mathcal{R}[x]/\langle x^7 - \gamma \rangle$, where $\gamma \in \{1, 2, 3, 4, 5, 6\}$. In the following, we list all distinct γ -constacyclic codes of length 7 over the chain ring $\mathbb{F}_7 + u\mathbb{F}_7$ of Type 3, where $h(x)$ is a unit, $0 \leq t < \delta$ and $1 \leq \delta \leq 6$. There are 1242 distinct γ -constacyclic codes listed below. In all codes we have $h_0 \in \{1, 2, \dots, 6\}$, $h_1, h_2 \in \{0, 1, \dots, 6\}$.

- $\delta = 1$ and $t = 0$: $\langle (x - \gamma) + h_0 u \rangle$,
- $\delta = 2$ and $t = 0$: $\langle (x - \gamma)^2 + h_0 u + h_1 u(x - \gamma) \rangle$,
- $\delta = 2$ and $t = 1$: $\langle (x - \gamma)^2 + h_0 u(x - \gamma) \rangle$,
- $\delta = 3$ and $t = 0$: $\langle (x - \gamma)^3 + h_0 u + h_1 u(x - \gamma) + h_2 u(x - \gamma)^2 \rangle$,
- $\delta = 3$ and $t = 1$: $\langle (x - \gamma)^3 + h_0 u(x - \gamma) + h_1 u(x - \gamma)^2 \rangle$,
- $\delta = 3$ and $t = 2$: $\langle (x - \gamma)^3 + h_0 u(x - \gamma)^2 \rangle$,
- $\delta = 4$ and $t = 0$: $\langle (x - \gamma)^4 + h_0 u + h_1 u(x - \gamma) + h_2 u(x - \gamma)^2 \rangle$,
- $\delta = 4$ and $t = 1$: $\langle (x - \gamma)^4 + h_0 u(x - \gamma) + h_1 u(x - \gamma)^2 + h_2 u(x - \gamma)^3 \rangle$,
- $\delta = 4$ and $t = 2$: $\langle (x - \gamma)^4 + h_0 u(x - \gamma)^2 + h_1 u(x - \gamma)^2 \rangle$,
- $\delta = 4$ and $t = 3$: $\langle (x - \gamma)^4 + h_0 u(x - \gamma)^3 \rangle$,
- $\delta = 5$ and $t = 0$: $\langle (x - \gamma)^5 + h_0 u + h_1 u(x - \gamma) \rangle$,
- $\delta = 5$ and $t = 1$: $\langle (x - \gamma)^5 + h_0 u(x - \gamma) + h_1 u(x - \gamma)^2 \rangle$,
- $\delta = 5$ and $t = 2$: $\langle (x - \gamma)^5 + h_0 u(x - \gamma)^2 + h_1 u(x - \gamma)^3 \rangle$,
- $\delta = 5$ and $t = 3$: $\langle (x - \gamma)^5 + h_0 u(x - \gamma)^3 + h_1 u(x - \gamma)^4 \rangle$,
- $\delta = 5$ and $t = 4$: $\langle (x - \gamma)^5 + h_0 u(x - \gamma)^4 \rangle$,
- $\delta = 6$ and $t = 0$: $\langle (x - \gamma)^6 + h_0 u \rangle$,
- $\delta = 6$ and $t = 1$: $\langle (x - \gamma)^6 + h_0 u(x - \gamma) \rangle$,
- $\delta = 6$ and $t = 2$: $\langle (x - \gamma)^6 + h_0 u(x - \gamma)^2 \rangle$,
- $\delta = 6$ and $t = 3$: $\langle (x - \gamma)^6 + h_0 u(x - \gamma)^3 \rangle$,
- $\delta = 6$ and $t = 4$: $\langle (x - \gamma)^6 + h_0 u(x - \gamma)^4 \rangle$,
- $\delta = 6$ and $t = 5$: $\langle (x - \gamma)^6 + h_0 u(x - \gamma)^5 \rangle$

Using the results in Sections 3 and 4, we list all symbol-pair distances d_{sp} of such codes. We also give all MDS and non-MDS symbol-pair codes (Table 1). Among these 1242 codes, 786 of them are MDS symbol-pair codes.

Table 1. γ -constacyclic codes of length 7 over the chain ring $\mathbb{F}_7 + u\mathbb{F}_7$ of Type 3 ($h(x)$ is a unit).

Ideal	d_{sp}	MDS code
$\langle(x - \gamma) + h_0u\rangle$	3	Yes
$\langle(x - \gamma)^2 + h_0u + h_1u(x - \gamma)\rangle$	4	Yes
$\langle(x - \gamma)^2 + h_0u(x - \gamma)\rangle$	4	Yes
$\langle(x - \gamma)^3 + h_0u + h_1u(x - \gamma) + h_2u(x - \gamma)^2\rangle$	5	Yes
$\langle(x - \gamma)^3 + h_0u(x - \gamma) + h_1u(x - \gamma)^2\rangle$	5	Yes
$\langle(x - \gamma)^3 + h_0u(x - \gamma)^2\rangle$	5	Yes
$\langle(x - \gamma)^4 + h_0u + h_1u(x - \gamma) + h_2u(x - \gamma)^2\rangle$	5	No
$\langle(x - \gamma)^4 + h_0u(x - \gamma) + h_1u(x - \gamma)^2 + h_2u(x - \gamma)^3\rangle$	6	Yes
$\langle(x - \gamma)^4 + h_0u(x - \gamma)^2 + h_1u(x - \gamma)^2\rangle$	6	Yes
$\langle(x - \gamma)^4 + h_0u(x - \gamma)^3\rangle$	6	Yes
$\langle(x - \gamma)^5 + h_0u + h_1u(x - \gamma)\rangle$	4	No
$\langle(x - \gamma)^5 + h_0u(x - \gamma) + h_1u(x - \gamma)^2\rangle$	5	No
$\langle(x - \gamma)^5 + h_0u(x - \gamma)^2 + h_1u(x - \gamma)^3\rangle$	6	No
$\langle(x - \gamma)^5 + h_0u(x - \gamma)^3 + h_1u(x - \gamma)^4\rangle$	7	Yes
$\langle(x - \gamma)^5 + h_0u(x - \gamma)^4\rangle$	7	Yes
$\langle(x - \gamma)^6 + h_0u\rangle$	3	No
$\langle(x - \gamma)^6 + h_0u(x - \gamma)\rangle$	4	No
$\langle(x - \gamma)^6 + h_0u(x - \gamma)^2\rangle$	5	No
$\langle(x - \gamma)^6 + h_0u(x - \gamma)^3\rangle$	6	No
$\langle(x - \gamma)^6 + h_0u(x - \gamma)^4\rangle$	7	No
$\langle(x - \gamma)^6 + h_0u(x - \gamma)^5\rangle$	7	No

Case study2: Consider cyclic codes of length 4 over the chain ring $\mathcal{R} = \mathbb{F}_2 + u\mathbb{F}_2$ which are the ideals of $\mathcal{R}[x]/\langle x^4 - 1 \rangle$. In the following, we list all distinct cyclic codes of length 4 over the chain ring $\mathbb{F}_2 + u\mathbb{F}_2$ of Type 3, where $h(x)$ is a unit, $0 \leq t < \delta$ and $1 \leq \delta \leq 3$. There are 7 distinct cyclic codes listed below.

- $\delta = 1$ and $t = 0$: $\langle(x - 1) + u\rangle$,
- $\delta = 2$ and $t = 0$: $\langle(x - 1)^2 + u + h_1(x - 1)\rangle$, where $h_1 \in \{0,1\}$,

- $\delta = 2$ and $t = 1$: $\langle (x - 1)^2 + u(x - 1) \rangle$,
- $\delta = 3$ and $t = 0$: $\langle (x - 1)^3 + u \rangle$,
- $\delta = 3$ and $t = 1$: $\langle (x - 1)^3 + u(x - 1) \rangle$,
- $\delta = 3$ and $t = 2$: $\langle (x - 1)^3 + u(x - 1)^2 \rangle$.

Applying Theorem 3.2, we can compute all symbol-pair distances of cyclic codes d_{sp} . Using Theorems 4.2 and 4.4, all MDS and non-MDS symbol-pair codes are determined in Table 2. Among these 7 codes, 4 of them are MDS symbol-pair codes.

Table 2. Cyclic codes of length 4 over the chain ring $\mathbb{F}_2 + u\mathbb{F}_2$ of Type 3 ($h(x)$ is a unit).

Ideal	d_{sp}	MDS code
$\langle (x - 1) + u \rangle$	3	Yes
$\langle (x - 1)^2 + u + h_1(x - 1) \rangle$	4	Yes
$\langle (x - 1)^2 + u(x - 1) \rangle$	4	Yes
$\langle (x - 1)^3 + u \rangle$	3	No
$\langle (x - 1)^3 + u(x - 1) \rangle$	4	No
$\langle (x - 1)^3 + u(x - 1)^2 \rangle$	4	No

5 CONCLUSION

Let γ be a nonzero element of the finite field $\mathbb{F}_{p^m}$. Determining the symbol-pair distances of constacyclic codes and obtaining MDS symbol-pair constacyclic codes are very important in coding theory. Motivated by this, in this paper, we completed the problem of determining MDS symbol-pair γ -constacyclic codes of length p^s over $\mathcal{R} = \mathbb{F}_{p^m} + u\mathbb{F}_{p^m} (u^2 = 0)$.

For future work, it would be interesting to determine the symbol-pair distances of γ -constacyclic codes of length p^s over $R = \mathbb{F}_{p^m} + u\mathbb{F}_{p^m} + u^2\mathbb{F}_{p^m} (u^3 = 0)$, and to determine MDS symbol-pair γ -constacyclic codes of length p^s over R .

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