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Generalized Growth of Entire Solutions of Generalized Bi-axially Symmetric Helmholtz Equation

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Abstract. For an entire solution u of the generalized bi-axially symmetric helmholtz equation

$$\frac{\partial^2 u}{\partial t^2} + \frac{2\mu}{t}\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial r^2} + \frac{2\nu}{r}\frac{\partial u}{\partial r} + c^2 u = 0, \mu, \nu > 0,$$

formulae involving the generalized growth characteristics of u and its Bessel-Jacobi series expansion coefficients have been obtained by using function theoretic method.

Keywords: Entire GBSHE functions, bi-axially symmetric helmholtz equation, generalized order and type, Jacobi polynomials and Bessel-Jacobi series expansion.

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1. INTRODUCTION

The partial differential equation

$$\frac{\partial^2 u}{\partial t^2} + \frac{2\mu}{t}\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial r^2} + \frac{2\nu}{r}\frac{\partial u}{\partial r} + c^2 u = 0, \mu, \nu > 0, \tag{1.1}$$

is called generalized bi-axially symmetric helmholtz equation $(GBSHE)$ and the solutions of (1.1) subject to the initial conditions

$$(i)\frac{\partial}{\partial t}u(r,0) = 0, \quad (ii)u(r,0) = \sum_{n=0}^{\infty} a_n r^{2n}, \tag{1.2}$$

are called $GBSHE$ functions. It is known that [6] a $GBSHE$ function satisfying the condition $(1.2)(i)$, has the following Bessel-Jacobi series expansion

$$u(r,t) \equiv u(\rho,\theta) = \rho^{-(\mu+\nu)} \sum_{n=0}^{\infty} A_n J_{2n+\mu+\nu}(c\rho) P^{(\mu-\frac{1}{2},\nu-\frac{1}{2})}(\cos 2\theta), \tag{1.3}$$

where $r = \rho\cos\theta, t = \rho\sin\theta, J_{2n+\mu+\nu}$ are Bessel functions of first kind and $P^{(\mu-\frac{1}{2},\nu-\frac{1}{2})}(\cos 2\theta)$ are Jacobi polynomials.
The coefficients A_n are determined from the condition $(1.2)(ii)$ and given as

$$A_n = \frac{(2n+\mu+\nu)\Gamma(\mu+\frac{1}{2})n!}{\Gamma(n+\mu+\frac{1}{2})}(\frac{2}{c})^{(2n+\mu+\nu)} \sum_{s=0}^{n} (\frac{c}{2})^{2s} \frac{\Gamma(2n-s+\mu+\nu)}{s!} a_{n-s}.$$

A $GBSHE$ function u is said to be entire if the series (1.2) of u converges absolutely and uniformly on the compact subsets of the whole complex plane. Parihar [6, Thm.11.2] proved the following theorem:

Theorem A. For $\mu, \nu \geq 0$, consider the initial value problem (1.1)-(1.2). Let R be the radius of convergence of the series in $(1.2)(ii)$. Then the series

$$u(r,t) = \sum_{n=0}^{\infty} b_n(\mu,\nu) V_n(\mu,\nu;r,t),$$

where

$$\begin{aligned} b_n(\mu,\nu) &= \sum_{s=0}^{n} (\frac{c}{2})^{2s} \frac{\Gamma(2n-s+\mu+\nu)}{s!} a_{n-s}, \\ V_n(\mu,\nu;r,t) &= S_n(\mu,\nu;r,t) B_n(\mu,\nu;r,t), \\ S_n(\mu,\nu;r,t) &= \frac{\Gamma(2n+\mu+\nu+1)}{(\frac{c}{2}\sqrt{r^2+t^2})^{(2n+\mu+\nu)}} J_{2n+\mu+\nu}(c\sqrt{r^2+t^2}), \\ B_n(\mu,\nu;r,t) &= \frac{n!\Gamma(\mu+\frac{1}{2})}{\Gamma(n+\mu+\frac{1}{2})}(r^2+t^2)^n P_n^{(\mu-\frac{1}{2},\nu-\frac{1}{2})}(\frac{r^2-t^2}{r^2+t^2}), \end{aligned}$$

converges to a solution $u(r,t)$ of (1.1)-(1.2) in the region $r^2+t^2 < R^2$ and diverges if $r^2+t^2 > R^2$.

The radii of convergence of the series in $(1.2)(ii)$ and (1.3) are identical.

For the purpose of motivation, it is significant to mention, that the study of $GBSHE$ functions have a bearing on the study of the scattering of particles in quantum mechanics. The solutions of equation (1.1), which satisfy a suitable radiation condition, correspond to scattered waves, and their singularities are related to the quantum states of the scattered particles.

The limit $\mu \downarrow \nu$ corresponding to $c = 0$, produces the generalized axisymmetric potential equation. Reduction of $GASP$ equation to the harmonic function follows from the limit $\mu \downarrow 0$ that also reduces the zonal harmonics to thes harmonic, respectively analytic functions, regular at the origin. The $GBSHE$ functions, then, are natural extensions of harmonic or analytic functions.

The growth of an entire function $f(z)$ can be measured in terms of the order ζ defined by

$$\zeta = \limsup_{\rho \to \infty} \frac{\log \log M(\rho, f)}{\log \rho}$$

where $M(\rho, f) := \sup_{|z| \leq \rho} |f(z)|$. If the order is a positive real number the type T of the function is defined by

$$T = \limsup_{\rho \to \infty} \frac{\log M(\rho, f)}{\rho^{\zeta}}.$$

For entire harmonic functions of one variable (identifying $\mathbb{R}^2$ with $\mathbb{C}$) there is a large literature concerning the growth of this topic. For entire harmonic functions elementary characterizations of the growth order and type in terms of Taylor coefficients can be found in the work of Temliakow [10], Fryant [1], Fugard [3], Kapoor and Nautiyal [5], Fryant and Shankar [2] and Veselovska [12].

The concept of order and type was generalized in the literature (see e.g. Seremeta [7]). Here one replaces the $\log$ function in the above formulae by more general functions α, β defined on an interval (r_0, ∞), which are assumed to be positive, strictly increasing and tending to infinity as $\rho \to \infty$, and satisfying properties of class L^0 and Λ defined below:

Let a function $h(\xi)$ is defined on $[a, \infty)$ for some $a \geq 0$ and it is strongly monotonically increasing and tends to ∞ as $\xi \to \infty$. According to Seremeta [7], this function belongs to the class L^0 if, for any real function ϕ such that $\phi(\xi) \to \infty$ as $\xi \to \infty$, the equality

$$\lim_{\xi \to \infty} \frac{h\{(1 + \frac{1}{\phi(\xi)})\xi\}}{h(\xi)} = 1.$$

It belongs to the class Λ if, for all $c, 0 < c < \infty$,

$$\lim_{\xi \to \infty} \frac{h(c\xi)}{h(\xi)} = 1.$$

By using functions α and β from the classes L^0 and Λ, by analogy with [7], define the generalized order $\zeta_{\alpha,\beta}(u)$ and generalized lower order $\lambda_{\alpha,\beta}(u)$ of an entire $GBSHE$ function u by the formulas:

$$\zeta_{\alpha,\beta}(u) = \limsup_{\rho\to\infty} \frac{\alpha(\log M(\rho,u))}{\beta(\rho)}, \lambda_{\alpha,\beta}(u) = \liminf_{\rho\to\infty} \frac{\alpha(\log M(\rho,u))}{\beta(\rho)}.$$

where $M(\rho,u) = \max_{0\leq\theta\leq 2\pi} |u(\rho,\theta)|$.

Gilbert and Howard [4] had studied the order $\zeta(u)$ of an entire $GASHE$ (corresponding to $\nu \downarrow 0$ in (1.1)) function u in terms of the coefficients occurring in the Bessel-Gegenbauer series expansion. In the present paper, using a different technique, we define formulae for the generalized growth characteristics of entire $GBSHE$ functions in terms of Bessel-Jacobi series expansion coefficients.

2. AUXILLIARY RESULTS

To prove our main results the following lemmas are required.

Lemma 2.1. Let u be an entire $GBSHE$ function given by (1.2), then

$$\frac{|A_n|}{n!}(\frac{c\rho}{2\rho_*})^{2n} \leq K_{\mu,\nu}K_*M(\rho,u),$$

where constants $\rho_* > 1$ and $K_* < \infty$.

Proof. Using the orthogonality property of Jacobi polynomials [9, p. 68] and the uniform convergence of the series (1.2), we have

$$\frac{\rho^{(\mu+\nu)}A_nJ_{2n+\mu+\nu}(c\rho)}{(2n+\mu+\nu)A(n,\mu,\nu)} = 2\int_0^{\frac{\pi}{2}} u(\rho,\theta)P_n^{(\mu-\frac{1}{2},\nu-\frac{1}{2})}(\cos 2\theta)\sin^{2\mu}\theta\cos^{2\nu}\theta d\theta, \quad (2.1)$$

where $A(n,\mu,\nu) = \frac{\Gamma(n+1)\Gamma(n+\mu+\nu)}{\Gamma(n+\mu+\frac{1}{2})\Gamma(n+\nu+\frac{1}{2})}$.
Further from the well known series expansion of $J_{\delta+n}(c\rho)$ we have

$$\begin{aligned} J_{\delta+n}(c\rho) =& (\frac{c\rho}{2})^{(\delta+n)}\sum_{m=0}^{\infty}\frac{(-1)^m(c\rho)^{2m}}{2^{2m}m!\Gamma(n+\delta+m+1)}, \\ =& (\frac{c\rho}{2})^{(\delta+n)}\frac{1}{\Gamma(n+\delta+1)}\sum_{m=0}^{\infty}\frac{(-1)^m(c\rho)^{2m}\Gamma(n+\delta+1)}{2^{2m}m!\Gamma(n+\delta+m+1)}, \end{aligned}$$

and so $n \geq [(c\rho)^2]$, where [x] denotes the integral part of x, we have

$$J_{\delta+n}(c\rho) \geq \frac{1}{2\Gamma(n+\delta+1)}(\frac{c\rho}{2})^{(\delta+n)}. \quad (2.2)$$

From (2.1) and (2.2) for $n \geq [(c\rho)^2]$, and using Schwartz's inequality with orthogonality of Jacobi polynomials we obtain

$$\frac{|A_n|}{n!}(\frac{c\rho}{2})^{2n} \leq \frac{2^{\mu+\nu}\Gamma(2n+\mu+\nu+1)((2n+\mu+\nu)A(n,\mu,\nu)A(\mu,\nu))^{\frac{1}{2}}}{c^{\mu+\nu}n!}M(\rho,u)$$

or

$$\frac{|A_n|}{n!}(\frac{c\rho}{2})^{2n} \leq K_{\mu,\nu}[\frac{(2n+\mu+\nu)\sqrt{A(n,\mu,\nu)}}{\Gamma(n+1)}]\Gamma(2n+\mu+\nu)M(\rho,u), \tag{2.3}$$

where

$$K_{\mu,\nu} = \frac{2^{\mu+\nu+1}\sqrt{A(\alpha,\beta)}}{c^{\mu+\nu}}, \quad A(\mu,\nu) = \frac{\Gamma(\mu+\frac{1}{2})\Gamma(\nu+\frac{1}{2})}{\Gamma\mu+\nu+1}.$$

Since $[\frac{(2n+\mu+\nu)\sqrt{A(n,\mu,\nu)}}{n}]^{\frac{1}{n}} \to 1$ as $n \to \infty$, we can choose constants $K_* < \infty$ and $\rho_* > 1$ such that $\frac{\Gamma(2n+\mu+\nu)(2n+\mu+\nu)\sqrt{A(n,\mu,\nu)}}{n\Gamma n} \leq K_*\rho_*^{2n}$ for $n \geq 1$. Thus for $n \geq [(c\rho)^2]$, (2.3) gives that

$$\frac{|A_n|}{n!}(\frac{c\rho}{2\rho_*})^{2n} \leq K_{\mu,\nu}K_*M(\rho,u).$$

Lemma 2.2. Let u be an entire $GBSHE$ function represented by (1.2), then for $k \in \mathbb{N}$

$$M(\rho,u) \leq K\sum_{n=0}^{\infty}\frac{A_n\Gamma(n+k+1)}{n!\Gamma(2n+\mu+\nu+1)}(\frac{c\rho}{2})^{2n},$$

where K is a constant independent of n and ρ.

Proof. Since $J_\delta(\rho) \leq \frac{(\frac{\rho}{2})^\delta}{\Gamma(\delta+1)}$ and

$$\max_{0\leq\theta\leq 2\pi}|P_n^{\mu-\frac{1}{2},\nu-\frac{1}{2}}(\cos 2\theta)| \leq \frac{\Gamma(n+k+1)}{\Gamma(q+1)\Gamma(n+1)}, \quad q = \max(\mu,\nu).$$

Now we obtain from (1.2)

$$\begin{aligned} M(\rho,u) \leq& \rho^{-(\mu+\nu)}\sum_{n=0}^{\infty}\frac{|A_n|(\frac{c\rho}{2})^{(2n+\mu+\nu)}\Gamma(n+k+1)}{\Gamma(2n+\mu+\nu+1)\Gamma(q+1)\Gamma(n+1)} \\ \leq& \frac{(\frac{c}{2})^{\mu+\nu}}{\Gamma(q+1)}\sum_{n=0}^{\infty}\frac{|A_n|(\frac{c\rho}{2})^{2n}\Gamma(n+k+1)}{\Gamma(2n+\mu+\nu+1)\Gamma(n+1)}. \end{aligned} \tag{2.4}$$

Hence the proof is completed.

3. MAIN RESULTS

In this section we will prove our main theorems.

Theorem 3.1. The $GBSHE$ function u continues to entire $GBSHE$ function if and only if the following equality holds

$$\limsup_{n\to\infty}\{\frac{|A_n|\Gamma(n+k+1)}{n!\Gamma(2n+\mu+\nu+1)}\}^{\frac{1}{2n}}=0. \tag{3.1}$$

Proof. Assume that the $GBSHE$ function u continues to the entire $GBSHE$ function which will also be denoted by u. The equality (3.1) directly follows from Lemma 2.1. To prove only if part, using Lemma 2.2, on the basis of (3.1) a uniform convergence of the series in the right hand side of equality (1.2) on compact subsets of whole complex plane $\mathbb{C}$ follows. Therefore, setting the $GBSHE$ function u by series (1.2), we shall continue it over the whole complex plane $\mathbb{C}$.

Theorem 3.2. Let $u(r,t)$ be an entire $GBSHE$ function. If for all $c, 0<c<\infty$, one of the following conditions is satisfied

$$(i)\alpha,\beta\in\Lambda,\frac{d\log F(x,c)}{d\log x}=O(1), x\to\infty;\;\; (ii)\alpha,\beta\in L^0,\lim_{x\to\infty}\frac{d\log F(x,c)}{d\log x}=p,$$

where $0<p<\infty$ and the function $F(x,c)=\beta^{-1}(c\alpha(x))$, then the generalized order $\zeta_{\alpha,\beta}(u)$ of the entire $GBSHE$ function u is given by

$$\zeta_{\alpha,\beta}(u)=\limsup_{n\to\infty}\frac{\alpha(2pn)}{\beta([\frac{|A_n|}{n!}]^{-\frac{1}{2n}})}.$$

Proof. Let us consider the entire functions of complex variable z:

$$f_1(z)=\sum_{n=0}^{\infty}\frac{|A_n|}{n!(\rho_*)^{2n}K_{\alpha,\beta}K_*}z^{2n},$$
$$f_2(z)=\sum_{n=0}^{\infty}\frac{|A_n|\Gamma(n+k+1)}{n!\Gamma(2n+\mu+\nu+1)}z^{2n}.$$

By Lemma 2.1 and inequality (2.4), we get

$$m(\rho,f_1)\le M(\rho,u)\le M(\rho,f_2), \tag{3.2}$$

where $m(\rho,f_1)$ is the maximum term of power series of entire function $f_1(z)$ on circle $\{z:|z|=\rho\}$, and $M(\rho,f_2)=\max_{|z|=\rho}|f_2(z)|$. From (3.2) we obtain

$$\log m(\rho,f_1)\le\log M(\rho,u)\le\log M(\rho,f_2). \tag{3.3}$$

Since $\alpha, \beta \in \Lambda$ or L^0, are monotonically increasing functions, therefore from (3.3) we get

$$\frac{\alpha(\log m(\rho, f_1))}{\beta(\rho)} \leq \frac{\alpha(\log M(\rho, u))}{\beta(\rho)} \leq \frac{\alpha(\log M(\rho, f_2))}{\beta(\rho)}.$$

Using a result of Valiron [11] on maximum term $m(\rho, f_1)$, we get

$$\log M(\rho, f_1) \simeq \log m(\rho, f_1) \ \ as \ \ \rho \to \infty.$$

Hence

$$\zeta_{\alpha,\beta}(f_1) \leq \zeta_{\alpha,\beta}(u) \leq \zeta_{\alpha,\beta}(f_2). \tag{3.4}$$

Now applying a formula of the generalized order of an entire function of one complex variable in terms of its power series expansion coefficients [7] and using the fact that $\alpha, \beta \in \Lambda$ or L^0, we obtain the following equality

$$\zeta_{\alpha,\beta}(f_1) = \zeta_{\alpha,\beta}(f_2) = \lim_{n\to\infty} \sup \frac{\alpha(2pn)}{\beta([\frac{|A_n|}{n!}]^{-\frac{1}{2n}})}. \tag{3.5}$$

On combining (3.4) and (3.5) the proof is completed.

Shah [8] obtained the coefficient characterization of the generalized lower order of entire function $f(z)$ in terms of Taylor coefficients. Following the technique of Shah we obtain the following theorem:

Theorem B. Let $f(z) = \sum_{n=0}^{\infty} a_n(f) z^n$ be an entire function of one complex variable with generalized lower order of growth $\lambda_{\alpha,\beta}(f)$, where $\alpha, \beta \in \Delta$ or L^0, for $c = 1$, a function $F(x, 1) = F(x) = \beta^{-1}(\alpha(x))$, where β^{-1} is a function inverse to β, satisfies the condition

(a) For some function $\varphi(x) \to \infty$ (howsoever slowly) as $x \to \infty$,

$$\frac{\beta(x\varphi(x))}{\beta(e^x)} \to 0 \ \text{ as } \ x \to \infty,$$

(b)

$$\frac{d(\log F(x))}{d(\log x)} = O(1) \ \text{ as } \ x \to \infty,$$

(c) $|\frac{a_n(f)}{a_{n+1}(f)}|$ is ultimately a non decreasing function of n. Then

$$\lambda_{\alpha,\beta}(f) = \liminf_{n\to\infty} \frac{\alpha(|n|)}{\beta(|a_n|^{\frac{-1}{n}})}.$$

Theorem 3.3. Let u be a $GBSHE$ function. If condition (3.1) is satisfied, then $GBSHE$ function u can be continued to entire $GBSHE$ function, for which

$$\lambda_{\alpha,\beta}(u) = \liminf_{n\to\infty} \frac{\alpha(2pn)}{\beta([\frac{|A_n|}{n!}]^{\frac{-1}{2n}})}. \tag{3.6}$$

If, in addition the ratio $|\frac{A_n}{A_{n+1}}(n+1)|$ forms a nondecreasing function of n and one of the conditions (i), (ii) of Theorem 3.2 with condition (a) of Theorem B are satisfied, then inequality (3.6) becomes equality.

Proof. The proof follows on the lines of proof of Theorem 3.2 with Theorem B.

Remark 3.1. For $\alpha(x) = \beta(x) = \log x$, Theorem 3.2 gives the formula for the classical order $\zeta(u)$ as:

$$\zeta(u) = \limsup_{n\to\infty} \frac{2n \log n}{\log[\frac{|A_n|}{n!}]^{-1}}.$$

Remark 3.2. For $\alpha(x) = x, \beta(x) = x^{\zeta}, p = \frac{1}{\zeta}$, where ζ is the order of $GBSHE$ function u, the formula for the classical type $T(u)$ is obtained from Theorem 3.2 as:

$$(\frac{T(u)\zeta e}{2})^{\frac{1}{\zeta}} = \limsup_{n\to\infty} n^{\frac{1}{\zeta}}[\frac{|A_n|}{n!}]^{\frac{1}{2n}}.$$

Remark 3.3. For $\alpha(x) = x, \beta(x) = x^{\zeta(x)}$, where $\zeta(x)$ is the proximate order of $GBSHE$ function u, the formula for the generalized type $T^*(u)$ (same orders but infinite type) with respect to proximate order $\zeta(x)$:

$$(\frac{T^*(u)\zeta e}{2})^{\frac{1}{\zeta}} = \limsup_{n\to\infty} \phi(n)[\frac{|A_n|}{n!}]^{\frac{1}{2n}},$$

where $x = \phi(\tau)$ is the function, inverse to $\tau = x^{\tau(x)}$.

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