Controllability of Fractional Descriptor Linear System

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Abstract

The objective of this paper is to find conditions for controllability of fractional descriptor linear system in state spaces. The system is decomposed into a slow subsystem and a fast subsystem. We derive state response of the two subsystems based on Laplace transformation and convolution formula. The sufficient and necessary conditions for controllability of the slow subsystem are obtained. A criterion of controllability for the fast subsystem is also established by using solution formula. The results obtained extend some existing results of controllability and observability for fractional descriptor systems.

AMS subject classification: 93B05.
Keywords: Fractional descriptor linear systems, state response, controllability.

1. Introduction

Descriptor systems, which are also referred to as singular systems or differential-algebraic systems, can be considered as generalization of dynamical systems. Descriptor linear systems provide a suitable mathematical model for many mechanical systems and circuit systems, which have been widely studied over the past few decades. One can see the monograph [1] and the references therein. On the other hand, fractional calculus have attracted much attention of researchers during the past three decades or so [2, 3]. In control theory, fractional calculus also has a wide range of applications, see for example [4, 5, 6].

Controllability plays an important role in the development of modern control theory. Many results have been obtained on controllability of fractional order systems, such as robust controllability [7], finite time controllability [8] and delay system controllability [9].
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However, there are very few contributions regarding controllability of descriptor linear systems with fractional order. The sufficient and necessary conditions of controllability for fractional singular dynamical systems with control delay are established based on the algebraic approach in [10]. The results are obtained by analysing the state response and reachable set in the paper, but the state response of fast subsystem does not contain impulse term which is important for impulsive control design. While Huang Q. et al. [11] obtained the controllability criterion for fractional singular dynamical systems. Nevertheless, state response of fast subsystem has not been derived specifically. In this paper, we investigated controllability and of fractional descriptor linear systems. By using Laplace transform and convolution theorem, we derive state response of the slow subsystem and fast subsystem. Controllability criteria for such systems are established based on the state response.

The paper is organized as follows. Section 2 formulates the problem and presents the preliminary definitions and results in fractional calculus. Section 3 contains the main results which consists of state response and controllability criteria of the systems. Finally, some conclusions are drawn in Section 4.

2. Preliminaries

We first give some definitions about fractional calculus. For more details, see [5, 12, 13].

Definition 2.1. The \( \alpha \)th-order Riemann-Liouville fractional integral of a function \( h : (0, \infty) \to R \) is defined as

\[
_0D_t^{-\alpha} h(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds,
\]

where \( \Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt \) is a Gamma function and \( \alpha \) is a positive real number.

Definition 2.2. The \( \alpha \)th-order Riemann-Liouville fractional derivative of a function \( h : (0, \infty) \to R \) is defined as

\[
_0D_t^\alpha h(t) = \frac{d^n}{dt^n} (\_0D_t^{-(n-\alpha)} h(t)) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-s)^{n-\alpha-1} h(s) ds,
\]

where \( n-1 < \alpha \leq n \), \( n \) is a positive integer.

Definition 2.3. The \( \alpha \)th-order Caputo fractional derivative of a function \( h : (0, \infty) \to R \) is defined

\[
_0^C D_t^\alpha h(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} h^{(n)}(s) ds,
\]

where \( n-1 < \alpha \leq n \), \( n \) is a positive integer.
Laplace transform is an important tool to solve fractional order systems. Denote $H(s) = L[h(t); s]$ as Laplace transform of a function $h(t)$. The following equalities hold.

$$L \left[ C_0^D t^\alpha h(t); s \right] = s^\alpha H(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} h^{(k)}(0), \quad n - 1 < \alpha \leq n,$$

$$L \left[ \delta^{(k)}(t); s \right] = s^k, \quad k \in N^+,$$

$$L \left[ t^{\alpha-1}; s \right] = \Gamma(\alpha)s^{-\alpha}, \quad \alpha > 0,$$

where $\delta(t)$ is the Dirac function and $N^+$ is the set of all positive integers.

### 3. Main results

Consider the following descriptor fractional linear systems with the Caputo fractional derivative operator

$$E^C_0 D_t^\alpha x(t) = Ax(t) + Bu(t), \quad x(0) = x_0 \quad 0 < \alpha < 1, \quad t \geq 0 \quad (1)$$

where $E^C_0 D_t^\alpha$ is the Caputo fractional derivative operator, $x(t) \in \mathbb{R}^n$ is the state variable, $E$ is $n \times n$ singular constant matrix, $A$ and $B$ are the known $n \times n, n \times m$ constant matrices, $u(t) \in \mathbb{R}^{m \times n}$ is the control input matrix.

Assume the matrix pair $(E, A)$ is regular, then there exists a constant scalar $\gamma \in \mathbb{R}$ such that $\det(\gamma^\alpha E - A) \neq 0$. Therefore, there exist two nonsingular matrix $Q, P$ such that

$$QEP = \begin{bmatrix} I_{n_1} & 0 \\ 0 & N \end{bmatrix}, \quad QAP = \begin{bmatrix} A_1 & 0 \\ 0 & I_{n_2} \end{bmatrix}, \quad QB = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$$

where $n_1 + n_2 = n, A_1 \in \mathbb{R}^{n_1 \times n_1}, N \in \mathbb{R}^{n_2 \times n_2}$, and the matrix $N$ is nilpotent with nilpotent index $h$. Hence, the system (1) is equivalent to the following canonical system

$$E^C_0 D_t^\alpha x_1(t) = A_1 x_1(t) + B_1 u(t), \quad x_1(0) = x_{10} \quad (2)$$

$$N^C_0 D_t^\alpha x_2(t) = x_2(t) + B_2 u(t), \quad x_2(0) = x_{20} \quad (3)$$

where $x_1 \in \mathbb{R}^{n_1}, x_2 \in \mathbb{R}^{n_2}$. We also call (2) the slow subsystem, (3) the fast subsystem.

In this section, we derive the solution formula of the slow subsystem and the fast subsystem. Based on the solutions, some controllability criteria are obtained.

### 3.1. State Response

**Theorem 3.1.** The solution of slow subsystem (2) is given by

$$x_1(t) = \sum_{k=0}^{\infty} \frac{A_1^{k+1} t^k}{\Gamma(ak + 1)} x_{10} + \int_0^t \sum_{k=0}^{\infty} \frac{(t - \tau)^{(k+1)\alpha-1}}{\Gamma((k + 1)\alpha)} A_1^k B_1 u(\tau) d\tau$$

(4)
Proof. Taking the Laplace transform on both sides of the slow subsystem (2), we obtain

\[ s^\alpha X_1(s) - s^{\alpha-1}x_{10} = A_1X_1(s) + B_1U(s), \]

where \( X_1(s) \) and \( U(s) \) are the Laplace transform of \( x_1(t) \) and \( u(t) \), respectively.

Thus, we can obtain

\[ X_1(s) = (s^\alpha I_{n_1} - A_1)^{-1}(s^{\alpha-1}x_{10} + B_1U(s)). \]  

(5)

In addition, the following series expansion holds

\[ (s^\alpha I_{n_1} - A_1)^{-1} = \sum_{k=0}^{\infty} s^{-k\alpha-\alpha} A_1^k. \]  

(6)

Substituting (6) into (5), yields

\[ X_1(s) = \sum_{k=0}^{\infty} s^{-k\alpha-\alpha} A_1^k(s^{\alpha-1}x_{10} + B_1U(s)) \]

\[ = \sum_{k=0}^{\infty} s^{-k\alpha-1} A_1^k x_{10} + \sum_{k=0}^{\infty} s^{-k\alpha-\alpha} A_1^k B_1 U(s). \]  

(7)

Taking the inverse Laplace transform on both sides of (7), it follows from convolution theorem that

\[ x_1(t) = \sum_{k=0}^{\infty} \frac{A_1^k t^{k\alpha}}{\Gamma(ak+1)} x_{10} + \int_0^t \sum_{k=0}^{\infty} \frac{(t-\tau)^{(k+1)\alpha-1}}{\Gamma((k+1)\alpha)} A_1^k B_1 U(\tau) d\tau \]

The proof is thus completed. \[ \blacksquare \]

**Theorem 3.2.** The solution of the fast subsystem (3) with consistent initial condition

\[ x_{20} = -\sum_{i=0}^{h-1} N^i B_2 u^{(\alpha i)}(0) \] is given by

\[ x_2(t) = -\sum_{i=0}^{h-1} \frac{N^{i+1}}{\Gamma(m_i - ai - \alpha + 1)} \int_0^t \delta^{(m_i)}(\tau)(t-\tau)^{m_i-ai-\alpha} d\tau x_{20} - \sum_{i=0}^{h-1} N^i B_2 u^{(\alpha i)}(t) \]

\[ - \sum_{i=0}^{h-1} \sum_{k=0}^{n_i-1} \frac{N^i B_2 u^{(k)}(0)}{\Gamma(n_k - ai + k + 1)} \int_0^t \delta^{(n_k)}(\tau)(t-\tau)^{n_k-ai+k} d\tau \]  

(8)

**Proof.** Taking the Laplace transform on both sides of the fast subsystem (3), we obtain

\[ s^{\alpha} N X_2(s) -Ns^{\alpha-1}x_{20} = X_2(s) + B_2U(s), \]
it follows that
\[ X_2(s) = (s^\alpha N - I_{n_2})^{-1}Ns^{\alpha-1}x_{20} + (s^\alpha N - I_{n_2})^{-1}B_2U(s), \] (9)

Considered that the following series expansion holds
\[ (s^\alpha N - I_{n_2})^{-1} = -\sum_{i=0}^{h-1} N^i s^{\alpha i}. \] (10)

Substituting (10) into (9), yields
\[ X_2(s) = -\sum_{i=0}^{h-1} s^{\alpha i + \alpha - 1 - m_i} N^{i+1}x_{20} - \sum_{i=0}^{h-1} s^{\alpha i} N^i B_2 U(s). \] (11)

There exist positive inter numbers \( m_i \) and \( n_i \) such that
\[ m_i - 1 < \alpha i + \alpha - 1 \leq m_i, \quad n_i - 1 < \alpha i \leq n_i. \] (12)

Therefore, the Eq. (11) can be rewritten as
\[ X_2(s) = -\sum_{i=0}^{h-1} s^{m_i s^{\alpha i + \alpha - 1 - m_i}} N^{i+1}x_{20} - \sum_{i=0}^{h-1} s^{n_i s^{\alpha i - n_i}} N^i B_2 U(s). \] (13)

Applying convolution theorem and taking the inverse Laplace transform on both sides of (13), we have
\[ x_2(t) = -\sum_{i=0}^{h-1} \frac{N^{i+1}}{\Gamma(m_i - \alpha i - \alpha + 1)} \int_0^t \delta^{(m_i)}(\tau)(t - \tau)^{m_i - \alpha i - \alpha} d\tau x_{20} - \sum_{i=0}^{h-1} N^i B_2 u^{(\alpha i)}(t) \\
-\sum_{i=0}^{h-1} \sum_{k=0}^{n_i - 1} \frac{N^i B_2 u^{(k)}(0)}{\Gamma(n_k - \alpha i + k + 1)} \int_0^t \delta^{(n_k)}(\tau)(t - \tau)^{n_k - \alpha i + k} d\tau, \]

where \( n_k \) is a positive inter number and satisfies \( n_k - 1 < \alpha i - k - 1 \leq n_k \). Moreover, the solution obviously satisfies the consistent initial condition \( x_{20} = -\sum_{i=0}^{h-1} N^i B_2 u^{(\alpha i)}(0) \).

This therefore completes the proof.

3.2. State controllability

State controllability of the slow and fast subsystem are considered based on previous definitions and results in this section. Similar to the concepts of controllability for general fractional linear systems, the definition of controllability for fractional descriptor linear systems is given as follows.
**Theorem 3.4.** The slow subsystem (2) is controllable on \([0, t_1]\) if and only if the controllability Gramian matrix

\[
W_c(0, t_1) = \int_0^{t_1} \sum_{k=0}^{\infty} \frac{(1 - \tau)^{(k+1)\alpha-1}}{\Gamma((k+1)\alpha)} A_1^k B_1^T \sum_{k=0}^{\infty} \frac{(1 - \tau)^{(k+1)\alpha-1}}{\Gamma((k+1)\alpha)} (A_1^T)^k d\tau
\]

is non-singular.

**Proof.** If the matrix \(W_c(0, t_1)\) is non-singular, then its inverse is well-defined. For an initial state \(x_1(0) = x_{10} \neq 0\), we choose

\[
u(t) = B_1^T \sum_{k=0}^{\infty} \frac{(1 - t)^{(k+1)\alpha-1}}{\Gamma((k+1)\alpha)} (A_1^T)^k W_c^{-1}(0, t_1) \left[ x_{1(t_1)} - \sum_{k=0}^{\infty} \frac{A_1^{k,t_1} x_{10}}{\Gamma(\alpha k + 1)} \right],
\]

it follows from the solution formula (4) that

\[
x_{1(t_1)} = \sum_{k=0}^{\infty} \frac{A_1^{k,t_1}}{\Gamma(\alpha k + 1)} x_{10} + \int_0^{t_1} \sum_{k=0}^{\infty} \frac{(1 - \tau)^{(k+1)\alpha-1}}{\Gamma((k+1)\alpha)} A_1^k B_1 \nu(\tau) d\tau
\]

\[
= \sum_{k=0}^{\infty} \frac{A_1^{k,t_1}}{\Gamma(\alpha k + 1)} x_{10} + \int_0^{t_1} \sum_{k=0}^{\infty} \frac{(1 - \tau)^{(k+1)\alpha-1}}{\Gamma((k+1)\alpha)} A_1^k B_1 B_1^T \sum_{k=0}^{\infty} \frac{(1 - \tau)^{(k+1)\alpha-1}}{\Gamma((k+1)\alpha)} (A_1^T)^k d\tau
\]

\[
\times (A_1^T)^k W_c^{-1}(0, t_1) \left[ x_{1(t_1)} - \sum_{k=0}^{\infty} \frac{A_1^{k,t_1}}{\Gamma(\alpha k + 1)} x_{10} \right] d\tau
\]

\[
= \sum_{k=0}^{\infty} \frac{A_1^{k,t_1}}{\Gamma(\alpha k + 1)} x_{10} + \int_0^{t_1} \sum_{k=0}^{\infty} \frac{(1 - \tau)^{(k+1)\alpha-1}}{\Gamma((k+1)\alpha)} A_1^k B_1 B_1^T \sum_{k=0}^{\infty} \frac{(1 - \tau)^{(k+1)\alpha-1}}{\Gamma((k+1)\alpha)} (A_1^T)^k d\tau
\]

\[
\times (A_1^T)^k d\tau W_c^{-1}(0, t_1) \left[ x_{1(t_1)} - \sum_{k=0}^{\infty} \frac{A_1^{k,t_1}}{\Gamma(\alpha k + 1)} x_{10} \right]
\]

\[
= \sum_{k=0}^{\infty} \frac{A_1^{k,t_1}}{\Gamma(\alpha k + 1)} x_{10} + W_c(0, t_1) W_c^{-1}(0, t_1) \left[ x_{1(t_1)} - \sum_{k=0}^{\infty} \frac{A_1^{k,t_1}}{\Gamma(\alpha k + 1)} x_{10} \right]
\]

Thus, the system (2) is controllable on \([0, t_1]\).
On the other hand, suppose that the system (2) is controllable on \([0, t_1]\) but the matrix 
\(W_c(0, t_1)\) is singular. Then there exists an \(n_1 \times 1\) nonzero vector \(v\) such that

\[
0 = v^T W_c(0, t_1)v = \int_0^{t_1} v^T \frac{(t_1 - \tau)^{(k+1)\alpha-1}}{\Gamma((k+1)\alpha)} A_1^k B_1^T \frac{(t_1 - \tau)^{(k+1)\alpha-1}}{\Gamma((k+1)\alpha)} (A_1^T)^k v d\tau
\]

which implies

\[
v^T \frac{(t_1 - \tau)^{(k+1)\alpha-1}}{\Gamma((k+1)\alpha)} A_1^k B_1 \equiv 0
\]

for all \(\tau \in [0, t_1]\). Since the system (2) is controllable, there exists an input that transfers
the initial \(x_1(0) = x_{10}\) to \(x_1(t_1) = 0\). We choose \(x_{10} = -\left[ \sum_{k=0}^{\infty} \frac{A_1^k t_1^{\alpha}}{\Gamma(\alpha k + 1)} \right]^{-1} v\), then
there exists a input such that

\[
x_1(t_1) = -v + \int_0^{t_1} \frac{(t_1 - \tau)^{(k+1)\alpha-1}}{\Gamma((k+1)\alpha)} A_1^k B_1 u(\tau) d\tau = 0,
\]

i.e.

\[
v = \int_0^{t_1} \frac{(t_1 - \tau)^{(k+1)\alpha-1}}{\Gamma((k+1)\alpha)} A_1^k B_1 u(\tau) d\tau.
\]

Its pre-multiplication by \(v^T\) yields

\[
v^T v = \int_0^{t_1} v^T \frac{(t_1 - \tau)^{(k+1)\alpha-1}}{\Gamma((k+1)\alpha)} A_1^k B_1 u(\tau) d\tau = 0,
\]

which contradicts \(v \neq 0\). So the matrix \(W_c(0, t_1)\) is non-singular. The proof is thus
completed.

For convenience, we rewrite the solution to fast subsystem (3) as

\[
x_2(t) = \Phi_0(t)x_{20} - W U^\alpha(t) - \Phi(t) U^1(0)
\]
where

\[
\Phi_0(t) = -\sum_{i=0}^{h-1} \frac{N^{i+1}}{\Gamma(m_i - \alpha i - \alpha + 1)} \int_0^t \delta^{(m_i)}(\tau)(t - \tau)^{m_i - \alpha i - \alpha} d\tau,
\]

\[
W = \begin{bmatrix}
B_2 & N B_2 & \cdots & N^{h-1} B_2
\end{bmatrix},
\]

\[
U^\alpha(t) = \begin{bmatrix}
u(t) & u^{(\alpha)}(t) & \cdots & u^{(h\alpha - \alpha)}(t)
\end{bmatrix}^T,
\]

\[
\Phi(t) = -\sum_{i=0}^{h-1} \begin{bmatrix}
\xi_0 & \xi_1 & \cdots & \xi_{n_i-1}
\end{bmatrix},
\]

\[
\xi_k = \frac{N^i B_2}{\Gamma(n_k - \alpha i + k + 1)} \int_0^t \delta^{(n_k)}(\tau)(t - \tau)^{n_k - \alpha i + k} d\tau, k = 0, 1, \cdots, n_i - 1.
\]

It is noted that (16) is only a algebraic equation. The inversion of matrix \( W \) plays a key role for controllability of the system (3). If \( W \) is inverse, we can find a input \( U^\alpha(t) \) such that the system state can transfer \( x_{20} \) to \( x_{2t_1} \) at infinite time. Then, on certain condition, we can find \( u(t) \) such that the system is controllable. However, the matrix \( W \) may not be square. Hence, pseudo inverse of a matrix is used here to investigate the controllability of the system (3). So the following theorem can be proved.

**Theorem 3.5.** Assume the fast subsystem (3) satisfies consistent initial condition \( x_{20} = -W u^{(\alpha i)}(0) \). If the matrix \( W = \sum_{i=0}^{h-1} N^i B_2 \) is row full rank, then the system (3) is controllable on \([0, t_1]\).

**Proof.** Since the matrix \( M \) is row full rank, the pseudo inverse of it is uniquely determined as \( W^+ = W^H (W W^H)^{-1} \) where the superscript \( H \) indicates complex conjugate transpose. We choose input \( U^\alpha(t) = W^+ \left[ \Phi_0(t)x_{20} + \Phi(t)U^1(0) + x_{2t_1} \right] \) with initial condition \( U^\alpha(0) = x_{2t_1} \). It follows from (16) that

\[
x_{2(t_1)} = \Phi_0(t)x_{20} - WW^+ \left[ \Phi_0(t)x_{20} + \Phi(t)U^1(0) + x_{2t_1} \right] - \Phi(t)U^1(0)
\]

\[
= \Phi_0(t)x_{20} - \left[ \Phi_0(t)x_{20} + \Phi(t)U^1(0) + x_{2t_1} \right] - \Phi(t)U^1(0)
\]

\[
= x_{2t_1},
\]

which implies that the system state can transfer \( x_{20} \) to \( x_{2t_1} \) by choosing \( U^\alpha(t) \). According to the function expression of \( U^\alpha(t) \), \( u(t) \) can be determined by \( U^\alpha(t) \) and the initial condition \( U^\alpha(0) = x_{2t_1} \). Thus the system (3) is controllable on \([0, t_1]\). The proof is thus completed.

4. Conclusions

The descriptor linear system with fractional order is decomposed into a slow subsystem and a fast subsystem in state space. By using Laplace transform and convolution theorem,
state response of the two subsystems are obtained. Controllability criteria for such systems are established based on the state response. The results obtained will be useful in the analysis and synthesis of fractional descriptor systems. Extending the results of this paper to multi-terms fractional descriptor systems is a future work.

Acknowledgments

The author would like to thank the editors and reviewers for their helpful suggestions. This research is partially supported by the National Natural Science Foundation of China (61463002) and the Yunnan provincial Natural Science Foundation of China (2012FB175).

References


