Closed Form Solutions of a Chaotic Equation Using the Method of Elliptic Truncation

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Abstract
Hunting for any analytic exact solutions in closed form of any nonlinear ordinary differential equation is always a challenge to mathematicians. The equation considered in this paper is the wave traveling reduction to the Kuramoto–Sivashinsky equation. This equation has attracted considerable attention, due to its occurrence in many areas, in physics and in chemistry. We introduce a method which involves a truncation procedure in relation to elliptic functions. We call this method as Elliptic Truncation Method. We then attempt to establish some exact solutions by this method.

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1. Introduction
The ordinary differential equation considered in this paper is the traveling wave reduction of a very famous PDE, namely, the Kuramoto–Sivashinsky (KS) equation. The equation is an important dynamical model. In short, we simply present the traveling wave reduction and call it the KS ODE.

\[ \nu u''' + bu'' + \mu u' + \frac{u^2}{2} + A = 0, \quad (\nu, b, \mu) \text{ real constants}, \quad \nu \neq 0, \quad (1) \]

where \( A \) is a constant. The special feature of this KS ODE is its chaotic behavior.

The chaotic structure of eqn. (1) does not prohibit us from hunting for analytic solutions in closed form. Although the analytic solutions cannot be general solution, it
is still worth to look for any solutions of such an ordinary differential equation and we also like to see what is the chaotic contribution to the solution.

In Sect. 2, we shall explore the analytical structure of the ordinary differential equation and to see the list of known particular solutions of this equation. In Sect. 3 we shall introduce a method, namely the Elliptic Truncation Method, to find and generate some explicit solutions of the nonlinear equation. The success of applying this method to KS ODE should allow us to apply the same method to other non-integrable, chaotic equations.

2. Analytical structure in the chaotic equation

Given the KS ODE (1), we can formally find its Laurent series. The Laurent series solution will be dependent on one free parameter $x_0$:

$$u^{(0)} = 120v \chi^{-3} - 15b \chi^{-2} + \left( \frac{60\mu}{19} - \frac{15b^2}{76v} \right) \chi^{-1} + \cdots, \quad \chi = x - x_0.$$ 

If the Laurent series is closely investigated, you will find that no more arbitrary constants can be generated and $x_0$ is the only one that can be incorporated in the Laurent series. Around $u^{(0)}$, the linearized equation has two complex Fuchs indices: $j = -1, (13 \pm i\sqrt{71})/2$ with $u^{(1)} = \chi^{j-3}$. The irrational indices indeed represent the chaotic nature of the equation. The KS ODE is assumed to be analytic only if two other (irrational) Fuchs indices do not contribute to the solution.

The main result of this paper is to look for any closed form analytic solutions by assuming a rational expression of $(\wp, \wp')$. Generically, this is not an easy way to make a proper assumption of the rational expression. We need to explore the singularity structure of the target equation. The solutions which can be found by the truncation method may involve a free parameter. The analytic solutions can be justified by performing numerical calculations.

Fournier, Spiegel and Thual [4] found the exact solution of (1) for $b = \mu = 0$: $u = -60v \wp'(x - x_0), 0, g_3)$, where $\wp$ is the Weierstrass elliptic function and $g_2 = 0, g_3 = A\nu^{-2}/1080$, which was later extrapolated to $b^2 = 16\nu \mu$ by Kudryashov [6] in 1989 (i.e. (C5) shown below). The following five cases constitute all the analytic solutions currently known on the KS ODE.

Known particular cases:

\[\begin{array}{cccccc}
  b^2 & k^2 & g_2 & g_3 & A \\
  \text{C1a} & 0 & \frac{11}{19} \left( \frac{\mu}{v} \right) & \frac{k^4}{12} & -\frac{k^6}{216} & -\frac{450}{19^2} \mu^2 k^2 \\
  \text{C1b} & 0 & -\frac{1}{19} \left( \frac{\mu}{v} \right) & \frac{k^4}{12} & -\frac{k^6}{216} & -\frac{450}{19^2} \mu^2 k^2 \\
  \text{C2} & \frac{144}{47} & \frac{1}{47} \left( \frac{\mu}{v} \right) & \frac{k^4}{12} & -\frac{k^6}{216} & -\frac{1800}{47^3} \left( \frac{\mu^3}{v} \right) \\
\end{array}\]
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C3 \( \frac{256}{73} \mu v \frac{1}{73} \left( \frac{\mu}{v} \right) \frac{k^4}{12} - \frac{k^6}{216} - \frac{4050}{733} \left( \frac{\mu^3}{v} \right) \)

C4a \( 16 \mu v \frac{\mu}{v} \frac{k^4}{12} - \frac{k^6}{216} - 18 \left( \frac{\mu^3}{v} \right) \)

C4b \( 16 \mu v - \frac{\mu}{v} \frac{k^4}{12} - \frac{k^6}{216} - 8 \left( \frac{\mu^3}{v} \right) \)

C5 \( 16 \mu v \frac{1}{12} \left( \frac{\mu}{v} \right)^2 \text{arb} \ 1080 g_3 v^2 - 13 \left( \frac{\mu^3}{v} \right) \)

(C1,C2,C3,C4):

\[
\begin{align*}
C1, C2, C3, C4 : \\
\quad & u = \sum_{j=0}^{3} c_j \left( \frac{k}{2} \coth \frac{k}{2} (x - x_0) \right)^j, \ k^2 = \text{fixed}, \ A = \text{numerical} \neq \text{arbitrary}; \\
C5 : \\
\quad & u = -60v \varphi' - 15b \varphi - \frac{b\mu}{4v}, \ A \text{ is arbitrary}.
\end{align*}
\]

3. Closed form analytic solutions in terms of \( \varphi \) and \( \varphi' \)

In the literature, there is a very well-known truncation method namely the Weiss–Tabor–Carnevale (WTC) method [2, 6]. This method enables us to find all the known trigonometric solutions presented in (C1)–(C4) easily. In the following we would like to present a method with which we can generate a number of closed form explicit solutions. The idea of the method is to assume the probable solution as a special expression which depends on \( \varphi \) and \( \varphi' \) with some undetermined coefficients. We believe that further improvement could be made to find other closed form analytic solutions of more chaotic equations, the Kuramoto–Sivashinsky ordinary differential equation is one particular example. Please remind that the method we are going to present is not necessarily the best method to generate all possible analytic solution but it should give rise to some explicit solutions that other methods may also succeeded in finding those solutions in other forms. In particular, for the Kuramoto-Sivashinsky ordinary differential equation we expect the existence of solutions of zero co-dimension (i.e. without constraint on the fixed parameters \( (\nu, b, \mu, A) \)) but in fact no one can find the solution either numerically or analytically. We are ready to show the closed form solutions of the KS equation.

Consider the KS-equation (1) with the assumption of analytic exact solution

\[
u = \text{rational function in } (\varphi, \varphi').
\]

We confine our attention to the assumption of the exact solution \( u \), and we would expect to hunt for any solutions of such form.
Based on the duplication formulas for Weierstrass elliptic function:

\[
\wp(2x) = -2\wp(x) + \left[\frac{\wp''(x)}{2\wp'(x)}\right]^2,
\]

\[
\wp'(2x) = -4\wp'^4(x) + 12\wp(x)\wp'^3(x)\wp''(x) - \wp''^3(x)
\]

\[
\frac{4\wp'^3(x)}{4\wp'^3(x)}
\]

and note also that

\[
\wp''(x) = 6\wp^2(x) - g_2/2,
\]

\[
\wp'^2(x) = 4\wp^3(x) - g_2\wp(x) - g_3,
\]

\[
\wp''(x) = 12\wp(x)\wp'(x),
\]

we may assume the exact solution is of the form

\[
u = \left(\frac{1}{\wp'} \sum_{k=0}^{6} a_k \wp^k + \sum_{l=0}^{4} b_l \wp^l\right) / \sum_{m=0}^{3} c_m \wp^m, \quad (5)
\]

where \(\wp\) itself is a function in \((x, g_2, g_3)\), or more precisely,

\[
A(\wp) = a_0 + a_1\wp + a_2\wp^2 + a_3\wp^3 + a_4\wp^4 + a_5\wp^5 + a_6\wp^6,
\]

\[
B(\wp) = b_0 + b_1\wp + b_2\wp^2 + b_3\wp^3 + b_4\wp^4,
\]

\[
C(\wp) = c_0 + c_1\wp + c_2\wp^2 + c_3\wp^3.
\]

In the following we shall introduce the method of elliptic truncation and also illustrate the direct deduction of elliptic solutions by this method. By making use of the elliptic equation one could easily deduce that

\[
\wp'^2 = 4\wp^3 - g_2\wp - g_3,
\]

\[
\wp'' = 6\wp^2 - \frac{g_2}{2},
\]

\[
\wp''' = 12\wp\wp',
\]

\[
\wp^{(4)} = 12(\wp')^2 + 12\wp\wp'' = 120\wp^3 - 18\wp - 12g_3, \quad (6)
\]
and by considering the numerator of (4) with (5) substituted we obtain accordingly

\[ 0 = G(\wp, \wp', \wp'', \wp''', \wp^{(4)}) \]

\[ = \sum \left( \wp^{k_0} (\wp')^{k_1} (\wp'')^{k_2} (\wp''')^{k_3} (\wp^{(4)})^{k_4} \right) \]

\[ = \sum \left( \wp^{k_0} (\wp')^{k_1} (\wp^2 + 1)^{k_2} (\wp \wp')^{k_3} (\wp^3 + \wp + 1)^{k_4} \right) \]

\[ = \sum \left( \wp^\alpha (\wp')^\beta \right) \]

\[ = G_1(\wp) + \wp' G_2(\wp), \]

where \( G, G_1, G_2 \) are polynomials. In fact, the truncation of the two polynomial equations of \( \wp \) give rise to the determining equations. Hence by the analytical assumption (5) we eventually obtain

\[ \sum_{j=0}^{21} c_1^j \wp^j + \wp' \sum_{j=0}^{19} c_2^j \wp^j = 0, \]

where \( c_{1,2}^j = \text{algebraic in } (a_k, b_l, c_m; \nu_0 = 1, b, \mu, A) = 0, \forall j. \)

However, it could be very troublesome and difficult to solve the equations \( c_{1,2}^j = 0 \) for \( a_k, b_l, c_m. \) With the aid of MAPLE we can recover the analytic solutions by solving all equations. In the following we intend to recover the analytic solutions for the special case of \( b^2 = 16\mu \nu. \)

**Elliptic Solution I**

This is the well-known elliptic solution found by Kudryashov in 1989.

\[ u = a_0 \wp' + a_1 \wp + a_2, \quad \wp = \wp(x, g_2, g_3), \quad (8) \]

where

\[ a_0 = -60\nu, \quad a_1 = -15b, \quad a_2 = -\frac{b\mu}{4\nu}, \]

\[ b^2 = 16\mu \nu, \quad g_2 = \frac{1}{12} \left(\frac{\mu}{\nu}\right)^2, \quad g_3 = \frac{A\nu + 13\mu^3}{1080\nu^3}, \quad (9) \]

\( A = \text{arbitrary}. \)

**Elliptic Solution II**

This is the solution directly deduced from the duplication formulas.

\[ u = \frac{A(\wp; a_k) + B(\wp; b_l)}{C(\wp; c_m)}, \quad \wp = \wp(x, g_2, g_3), \]
where

\[
\begin{align*}
    a_0 &= \frac{832A\mu^3 \nu + 4733\mu^6 + 32A^2 \nu^2}{1274019840 \nu^5}, \\
    a_1 &= \frac{\mu^2(A\nu + 13\mu^3)}{221184 \nu^4}, \\
    a_2 &= \frac{25\mu^4}{12288 \nu^3}, \\
    a_3 &= \frac{5(A\nu + 13\mu^3)}{288 \nu^2}, \\
    a_4 &= \frac{25\mu^2}{16 \nu}, \\
    a_5 &= 0, \\
    a_6 &= -240 \nu, \\
    b_0 &= \frac{b\mu(8A\nu - 121\mu^3)}{8847360 \nu^4}.
\end{align*}
\]

and

\[
\begin{align*}
    b_1 &= -\frac{b(4A\nu + 49\mu^3)}{9216 \nu^3}, \\
    b_2 &= -\frac{5b}{128} \left( \frac{\mu^2}{\nu} \right), \\
    b_3 &= -\frac{b\mu}{4 \nu}, \\
    b_4 &= -15b, \\
    c_0 &= -\frac{A\nu + 13\mu^3}{276480 \nu^3}, \\
    c_1 &= -\frac{1}{768} \left( \frac{\mu^2}{\nu} \right), \\
    c_2 &= 0, \\
    c_3 &= 1,
\end{align*}
\]  \tag{10}

Elliptic Solution III

The elliptic solution III is reducible to the elliptic solution I.

\[
\begin{align*}
    b^2 &= 16\mu \nu, \\
    g_2 &= \frac{1}{2^4} \frac{1}{12} \left( \frac{\mu}{\nu} \right)^2, \\
    g_3 &= \frac{1}{2^6} \left( \frac{A\nu + 13\mu^3}{1080 \nu^3} \right). \\
\end{align*}
\]  \tag{11}

\[
u = \frac{A(\varphi; a_k) + B(\varphi; b_l)}{C(\varphi; c_m)}, \quad \varphi = \varphi(x, g_2, g_3),
\]
where

\[
\begin{align*}
    a_0 &= \frac{1481683 \mu^3 (A\nu + 13 \mu^3)}{4484937600 \nu^5}, \\
    a_1 &= \frac{\mu^2 (3216679 \mu^3 - 94490 A\nu)}{149497920 \nu^4}, \\
    a_2 &= -\frac{\mu (286 A\nu + 8013 \mu^3)}{75504 \nu^3}, \\
    a_3 &= \frac{86515 A\nu - 1628717 \mu^3}{1557270 \nu^2}, \\
    a_4 &= \frac{12160 \mu^2}{1573 \nu}, \\
    a_5 &= \frac{180 \mu}{11}, \\
    a_6 &= -240 \nu, \\
    b_0 &= -\frac{1481683 b \mu^4}{996652800}.
\end{align*}
\]

and

\[
\begin{align*}
    b^2 &= 16 \mu \nu, \\
    g_2 &= \frac{1}{12} \left( \frac{\mu}{\nu} \right)^2, \\
    g_3 &= \frac{A\nu + 13 \mu^3}{1080 \nu^3}.
\end{align*}
\]

Elliptic Solution IV

The closed form elliptic solution IV contains an arbitrary parameter $c_0$. 

\[
    u = \frac{A(\varphi; a_k) + B(\varphi; b_l)}{C(\varphi; c_m)}, \quad \varphi = \varphi(x, g_2, g_3).
\]
where

\[
\begin{align*}
    a_0 &= \frac{c_0(A\nu + 13\mu^3)}{18\nu^2}, \\
    a_1 &= 5\mu^2\left(\frac{c_0}{\nu} - \frac{A\nu + 13\mu^3}{9504\nu^4}\right), \\
    a_2 &= -\frac{\mu(2A\nu + 51\mu^3)}{528\nu^3}, \\
    a_3 &= -240\nu c_0 + \frac{22A\nu + 151\mu^3}{396\nu^2}, \\
    a_4 &= \frac{80\mu^2}{11\nu}, \\
    a_5 &= \frac{180\mu}{11}, \\
    a_6 &= -240\nu,
\end{align*}
\]

\[
\begin{align*}
    b_0 &= -\frac{b\mu c_0}{4\nu}, \\
    b_1 &= \frac{5b}{2112}\left(\frac{\mu}{\nu}\right)^3 - 15bc_0, \\
    b_2 &= \frac{7b}{44}\left(\frac{\mu}{\nu}\right)^2, \\
    b_3 &= \frac{17b\mu}{22\nu}, \\
    b_4 &= -15b, \\
    c_1 &= -\frac{5}{528}\left(\frac{\mu}{\nu}\right)^2, \\
    c_2 &= -\frac{3\mu}{44\nu}, \\
    c_3 &= 1,
\end{align*}
\]

and

\[
c_0 = \text{arbitrary}, \quad b^2 = 16\mu\nu, \quad g_2 = \frac{1}{12}\left(\frac{\mu}{\nu}\right)^2, \quad g_3 = \frac{A\nu + 13\mu^3}{1080\nu^3}.
\]

Closed form analytic solutions of the Kuramoto-Sivashinsky ordinary differential equation are generated. An algorithm for constructing elliptic solutions is presented. The chaotic behavior of the KS equation was studied in [9]–[10]. We have reviewed the equation numerically and analytically. In fact, we found that for a differential equation which is non-integrable, it may admit some particular elliptic solutions which are analytic. In this paper, we focus on only one equation which is of great interest in Physics. The next step to proceed is to characterize the elliptic particular solutions, particularly, the solution which contains an arbitrary constant.

4. Conclusion

Thanks to the availability of mathematics software (MAPLE) which enables us to perform very complicated and tedious symbolic calculations on a computer. The software helps to recover some explicit and exact solutions in closed form. The ordinary differential equation considered in this paper is non-integrable and chaotic but it still admits some particular solutions which are in general uneasy to look for. We introduce in this paper the method which involves truncating an assumed expression of elliptic function and its derivative. This truncation method allows us to generate four elliptic solutions of the ODE and one of which contains a free constant.
References


