Near Field Asymptotic Formulation for Dendritic Growth Due to Buoyancy

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Abstract

The paper is devoted to the asymptotic analysis for a mathematical model of physical dendritic crystal growth problem. A theoretical framework governing the flow field and the temperature field under the effect of buoyancy-driven convection in the tip region of the dendritic growth is studied. By assuming the asymptotic expansions as the gravity parameter $\text{Gr} \to 0$, the governing equations and the corresponding interface boundary conditions for the leading-order and the first-order expansions have been formulated.

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1. Introduction

This paper is concerned with the single dendritic growth [1]–[7] under the effect of buoyancy-driven convection. We neglect the effects due to the density change during phase transition and the external flow in the far field. The dendrite is subject to the small gravity which means the gravity parameter $\text{Gr}$ is negligible. We also assume that the surface tension is negligible. Therefore we only need to consider the temperature field in liquid melt. The governing equations are:

\begin{equation}
\text{D}_1^2 \Psi = -(\xi^2 + \eta^2) \xi, \tag{1}
\end{equation}

\begin{equation}
\text{Pr} \text{D}_1^2 \xi = \eta_0^2 \left( \frac{\partial \xi}{\partial \xi} - \eta \frac{\partial \xi}{\partial \eta} \right) + \frac{2 \xi}{\eta_0^2 \xi^2 \eta^2} \left( \xi \frac{\partial \Psi}{\partial \xi} - \eta \frac{\partial \Psi}{\partial \eta} \right) \right.
- \frac{1}{\eta_0^2 \xi \eta} \left( \frac{\partial \Psi}{\partial \xi} \frac{\partial \xi}{\partial \eta} - \frac{\partial \Psi}{\partial \eta} \frac{\partial \xi}{\partial \xi} \right) - \frac{\text{Gr} \xi \eta}{\text{T}_\infty} \left( \frac{\partial \text{T}}{\partial \xi} + \xi \frac{\partial \text{T}}{\partial \eta} \right), \tag{2}
\end{equation}
\[ \nabla_i^2 T = \eta_0^2 \left( \xi \frac{\partial T}{\partial \xi} - \eta \frac{\partial T}{\partial \eta} \right) + \frac{1}{\eta_0^2 \xi \eta} \left( \frac{\partial \Psi}{\partial \eta} \frac{\partial T}{\partial \xi} - \frac{\partial \Psi}{\partial \xi} \frac{\partial T}{\partial \eta} \right). \] (3)

We attempt to study the near field asymptotic expansion solutions of the flow field \( \Psi \) and the temperature field \( T \) under the assumption that the gravitational acceleration is negligible. With such consideration, the mathematical formulation follows from assuming the asymptotic expansions in the small gravity parameter \( \text{Gr} \). Note that the entire physical space can be divided into two regions such as the near field and the far field. The inner and the outer asymptotic solutions should be determined independently. We may assume that there exists an intermediate region in which both inner and outer asymptotic solutions are valid. We believe that the solutions can be matched in the intermediate region and the matched solution should be a globally valid asymptotic solution in the whole physical domain.

In Sect. 2, we shall first present the mathematical formulation of the physical problem. Asymptotic analysis has been performed to derive the governing equations and the corresponding boundary conditions. In Sect. 3, we shall derive the zero-th order and the first-order approximations of the PDE systems by assuming the asymptotic expansions of the flow field and the temperature field in the tip region of the dendritic growth.

2. Mathematical formulation

We shall assume that the interface shape \( \eta_S = \mathcal{O}(\text{Gr}^{\frac{1}{2}}) \), so it could be written in the form

\[ \eta_S = \text{Gr}^{\frac{1}{2}} \hat{\eta}_S, \] (4)

where \( \hat{\eta}_S = \mathcal{O}(1) \). The inner solutions are given by

\[ \hat{\Psi}(\xi, \hat{\eta}, \text{Gr}) = \Psi(\xi, \eta, \text{Gr}), \]
\[ \hat{\zeta}(\xi, \hat{\eta}, \text{Gr}) = \zeta(\xi, \eta, \text{Gr}), \]
\[ \hat{T}(\xi, \hat{\eta}, \text{Gr}) = T(\xi, \eta, \text{Gr}), \] (5)

where we have introduced the new inner variables \((\xi, \hat{\eta})\) such that

\[ \hat{\eta} = \frac{\eta}{\text{Gr}^{\frac{1}{2}}} = \mathcal{O}(1). \] (6)

From (1)–(3), the inner solutions are subject to the following governing equations:

\[ \left\{ \frac{\partial^2}{\partial \hat{\eta}^2} - \frac{1}{\hat{\eta}} \frac{\partial}{\partial \hat{\eta}} \right\} \hat{\Psi} = -\text{Gr} \left\{ \frac{\partial^2}{\partial \xi^2} - \frac{1}{\xi} \frac{\partial}{\partial \xi} \right\} \hat{\Psi} - \text{Gr} \xi^2 \hat{\zeta} - \text{Gr}^2 \hat{\eta}^2 \hat{\zeta}, \] (7)
with the interface boundary conditions at \( \hat{\eta} = \hat{\eta}_s \). Then we are able to derive each order of the inner region expansions successively.

Asymptotic analysis in the near field \( \eta = \mathcal{O}\left( \text{Gr}^{-\frac{1}{2}} \right) \)

Now we consider the following inner region asymptotic expansions in the limit \( \text{Gr} \rightarrow 0 \):

\[
\begin{align*}
\dot{\Psi} &= \text{Gr} \left\{ \hat{\Psi}_0(\xi, \hat{\eta}) + \text{Gr} \hat{\Psi}_1(\xi, \hat{\eta}) + \cdots \right\}, \\
\dot{\hat{\eta}} &= \text{Gr}^\alpha \left\{ \hat{\eta}_0(\xi, \hat{\eta}) + \text{Gr} \hat{\eta}_1(\xi, \hat{\eta}) + \cdots \right\}, \\
\dot{T} &= \text{Gr} \left\{ \hat{T}_0(\xi, \hat{\eta}) + \text{Gr} \hat{T}_1(\xi, \hat{\eta}) + \cdots \right\}, \\
\dot{\hat{h}} &= \hat{h}_0(\xi) + \text{Gr} \hat{h}_1(\xi) + \cdots.
\end{align*}
\]

It could be seen that \( \alpha = 0 \) for the less degenerate case. In order to obtain the inner region solution, we substitute the expansions (14) into the governing equations (11)–(13). Then we are able to derive each order of the inner region expansions successively.
The zero-order approximation solution is subject to the following governing equations:

\[
\begin{align*}
\left\{ \frac{\partial^2}{\partial \tilde{\eta}^2} - \frac{1}{\tilde{\eta}} \frac{\partial}{\partial \tilde{\eta}} \right\} \hat{\Psi}_0 + \xi^2 \hat{\zeta}_0 &= 0, \quad (15) \\
\left\{ \frac{\partial^2}{\partial \tilde{\eta}^2} - \frac{1}{\tilde{\eta}} \frac{\partial}{\partial \tilde{\eta}} \right\} \hat{\zeta}_0 &= 0, \quad (16) \\
\left\{ \frac{\partial^2}{\partial \tilde{\eta}^2} + \frac{1}{\tilde{\eta}} \frac{\partial}{\partial \tilde{\eta}} \right\} \hat{T}_0 &= 0, \quad (17)
\end{align*}
\]

with the interface conditions at \( \hat{\eta} = \hat{h}_0 \) (near field),

\[
\begin{align*}
\hat{T}_0 &= 0, \quad (18) \\
\frac{\partial \hat{T}_0}{\partial \hat{\eta}} + \eta_0^2 (\xi \hat{h}_0' + \hat{h}_0) &= 0, \quad (19) \\
\left( \frac{\partial \hat{\Psi}_0}{\partial \xi} + \eta_0^4 \xi \hat{h}_0' \right) + \hat{h}_0' \left( \frac{\partial \hat{\Psi}_0}{\partial \hat{\eta}} + \eta_0^4 \xi^2 \hat{h}_0 \right) &= \eta_0^4 (\xi \hat{h}_0) (\xi \hat{h}_0' + \hat{h}_0), \quad (20) \\
\frac{\partial \hat{\Psi}_0}{\partial \hat{\eta}} &= 0. \quad (21)
\end{align*}
\]

From the condition (19), we may assume that the solution \( \hat{T}_0 \) only depends on the variable \( \hat{\eta} \), namely \( \hat{T}_0 = \hat{T}_0(\hat{\eta}) \), when \( \hat{h}_0 \) is supposed to be a constant. The constant \( \hat{h}_0 \) could be normalized to be 1 by properly choosing the parameter \( \eta_0^2 \), which is related to the undercooling temperature \( T_\infty \).

Now, the free boundary value problem (15)–(21) can be easily solved. The solution is as follows:

\[
\begin{align*}
\hat{\zeta}_0(\xi, \hat{\eta}) &= a_0 \xi^2 \hat{\eta}^2 + b_0 \xi^2 + c_0, \\
\hat{\Psi}_0(\xi, \hat{\eta}) &= -\xi^2 \left\{ \xi^2 \left( \frac{a_0}{8} \hat{\eta}^4 + \frac{b_0}{2} \hat{\eta}^2 \ln \hat{\eta} + d_0 \hat{\eta}^2 + e_0 \right) \\
&\quad + \left( \frac{c_0}{2} \hat{\eta}^2 \ln \hat{\eta} + d_1 \hat{\eta}^2 + e_1 \right) \right\}, \\
\hat{T}_0(\hat{\eta}) &= a_1 \ln \hat{\eta} + b_1,
\end{align*}
\]

where \( a_0, b_0, c_0, d_0, e_0, a_1, b_1, d_1 \) and \( e_1 \) are some undetermined constants. We can express \( b_0, d_0, e_0, a_1, b_1, d_1 \) and \( e_1 \) in terms of the constants \( a_0 \) and \( c_0 \) by applying the boundary conditions (18)-(21) at \( \hat{\eta} = 1 \), we eventually obtain the zero-order inner solution,

\[
\begin{align*}
\hat{\zeta}_0(\xi, \hat{\eta}) &= a_0 \xi^2 \hat{\eta}^2 + c_0, \\
\hat{\Psi}_0(\xi, \hat{\eta}) &= -\xi^2 \left[ \frac{a_0}{8} \xi^2 (\hat{\eta}^4 - 2\hat{\eta}^2 + 1) + 2c_0 (2\hat{\eta}^2 \ln \hat{\eta} - \hat{\eta}^2 + 1) \right], \\
\hat{T}_0(\hat{\eta}) &= -\eta_0^2 \ln \hat{\eta}.
\end{align*}
\]
The zero-order solution contains arbitrary constants $a_0$ and $c_0$. The solution satisfy the near field conditions on the interface, but fail to satisfy the far field boundary conditions at $\hat{n} \to \infty$. The inner asymptotic expansion solution is not valid in the far field. In fact, we should consider a different asymptotic expansion solution which could satisfy all the far field boundary conditions. This solution is called the outer expansion solution. The inner and the outer expansion solutions would be matched in an intermediate region, while $a_0$ and $c_0$ could then be determined by matching. We will show this step by step in the followings. Note also that the leading order inner solution \(23\) do not have singularity at the tip $\xi = 0$. So, at this stage we do not need to consider the asymptotic solution in the near tip region.

The first-order approximation solution is subject to the following governing equations:

\[
\left\{ \frac{\partial^2}{\partial \hat{n}^2} + \frac{1}{\hat{n}} \frac{\partial}{\partial \hat{n}} \right\} \hat{\Psi}_1 + \xi^2 \hat{\xi}_1 = -\left\{ \frac{\partial^2}{\partial \xi^2} - \frac{1}{\xi} \frac{\partial}{\partial \xi} \right\} \hat{\Psi}_0 - \hat{n}^2 \hat{\xi}_0, \tag{24}\]

\[
\text{Pr} \left\{ \frac{\partial^2}{\partial \hat{n}^2} - \frac{1}{\hat{n}} \frac{\partial}{\partial \hat{n}} \right\} \hat{\xi}_1 = - \text{Pr} \left\{ \frac{\partial^2}{\partial \xi^2} - \frac{1}{\xi} \frac{\partial}{\partial \xi} \right\} \hat{\xi}_0 + \eta_0^2 \left( \frac{\partial \hat{\xi}_0}{\partial \xi} - \hat{n} \frac{\partial \hat{\xi}_0}{\partial \hat{n}} \right) + \frac{2 \hat{\xi}_0}{\eta_0^2 \xi^2 \hat{n}^2} \left( \frac{\partial \hat{\Psi}_0}{\partial \xi} - \hat{n} \frac{\partial \hat{\Psi}_0}{\partial \hat{n}} \right) - \frac{1}{\eta_0^2 \xi \hat{n}} \left( \frac{\partial \hat{\Psi}_0}{\partial \xi} \frac{\partial \hat{\xi}_0}{\partial \hat{n}} - \frac{\partial \hat{\Psi}_0}{\partial \hat{n}} \frac{\partial \hat{\xi}_0}{\partial \xi} \right), \tag{25}\]

\[
\left\{ \frac{\partial^2}{\partial \hat{n}^2} + \frac{1}{\hat{n}} \frac{\partial}{\partial \hat{n}} \right\} \hat{T}_1 = - \left\{ \frac{\partial^2}{\partial \xi^2} + \frac{1}{\xi} \frac{\partial}{\partial \xi} \right\} \hat{T}_0 + \eta_0^2 \left( \frac{\partial \hat{T}_0}{\partial \xi} - \hat{n} \frac{\partial \hat{T}_0}{\partial \hat{n}} \right) + \frac{1}{\eta_0^2 \xi \hat{n}} \left( \frac{\partial \hat{\Psi}_0}{\partial \xi} \frac{\partial \hat{T}_0}{\partial \hat{n}} - \frac{\partial \hat{\Psi}_0}{\partial \hat{n}} \frac{\partial \hat{T}_0}{\partial \xi} \right), \tag{26}\]

with the interface conditions at $\hat{n} = \hat{n}_0$,

\[
\frac{\partial \hat{T}_1}{\partial \hat{n}} - \hat{n}_0 \frac{\partial \hat{T}_0}{\partial \xi} + \eta_0^2 (\xi \hat{h}_1' + \hat{h}_1) = 0, \tag{27}\]

\[
\frac{\partial \hat{\Psi}_1}{\partial \xi} + 2 \eta_0^4 \xi \hat{h}_0 \hat{h}_1 + \hat{h}_0' \left( \frac{\partial \hat{\Psi}_1}{\partial \hat{n}} + \eta_0^4 \xi^2 \hat{h}_1 \right) + \hat{h}_1' \left( \frac{\partial \hat{\Psi}_0}{\partial \hat{n}} + \eta_0^4 \xi^2 \hat{h}_0 \right) = \eta_0^4 \xi (\xi \hat{h}_1)' + \hat{h}_0 h_1, \tag{28}\]

\[
\frac{\partial \hat{\Psi}_1}{\partial \hat{n}} + \eta_0^4 \xi^2 \hat{h}_1 - \hat{h}_0' \left( \frac{\partial \hat{\Psi}_0}{\partial \xi} + \eta_0^4 \xi \hat{h}_0 \right) + \eta_0^4 \xi \hat{h}_0' \hat{h}_1 - \eta_0^4 \xi^2 \hat{h}_1 = 0. \tag{29}\]
4. Conclusion

In [8]–[10] we have introduced the method of matched asymptotic expansions and successfully applied it to the physical problem of dendritic growth under the effect of convection motion induced by an oscillating external force. The asymptotic formulation was based on the assumption of the small Reynolds number. In this paper, we undertake the further study and have an investigation on the dendritic growth under the effect of buoyancy-driven convection. We use the asymptotic approach for the mathematical formulation as the gravity parameter $\text{Gr} \to 0$. Our analytical framework will provide a basis to resolve the yet unsolved problems of selection of tip velocity of dendrite and the formation of pattern on the interface under the effect of buoyant flow at the near-tip region of the dendrite.

References


