On Limits of Mean Labeling for Three Star Graphs

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Abstract

In this paper, we prove that if $\ell \leq m < n$, then the three star $K_{1,\ell} \wedge K_{1,m} \wedge K_{1,n}$ is a mean graph if and only if $m + 1 \leq n \leq \ell + m + 4$ when $1 \leq \ell \leq 9$ and $\ell + m - 8 \leq n \leq \ell + m + 4$ when $\ell > 9$.

Key Words: Mean graph, wedge and star.

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1. INTRODUCTION

The condition for a graph to be mean is that $p = q + 1$. We study the path, star and the characterization of the mean labeling of stars. We have proved that the disjoint union of any path $P_n = P_1 \cup P_2 \cup \ldots \cup P_m$ with $m - 1$ edges joining the pendent vertices of distinct paths is a mean graph and $K_{1,m}$ is not a mean graph for $m \geq 4$. Also, we have proved [6] that the two star $K_{1,m} \cup K_{1,n}$ with an edge in common is a mean graph if and only if $|m - n| \leq 4$.

1.1 Definition

The three star is the disjoint union of $K_{1,a}$ , $K_{1,b}$ and $K_{1,c}$ . It is denoted by $K_{1,a} \cup K_{1,b} \cup K_{1,c}$.

1.2 Mean Graph

A graph with $p$ vertices and $q$ edges is said to be a mean graph if there exists a function $f$ from the vertex set of $G$ to $\{0,1,2,\ldots,q\}$ such that the induced map $f^*$ from
the edge set of $G$ to $\{1,2,\ldots,q\}$ defined by

$$f^*(e = uv) = \begin{cases} \frac{f(u) + f(v)}{2} & \text{if } f(u) + f(v) \text{ is even} \\ \frac{f(u) + f(v) + 1}{2} & \text{if } f(u) + f(v) \text{ is odd} \end{cases}$$

then the resulting edges get distinct labels from the set $\{1,2,\ldots,q\}$.

1.3 Wedge
A Wedge is defined as a bridge connecting two components of a graph, denoted as $\wedge$. $\omega(G \wedge) < \omega(G)$.

$K_{1,m} \cup K_{1,n}$ is a two star and is a two component or a disconnected graph, whereas $K_{1,m} \wedge K_{1,n}$ is a two star but a connected graph. Which means adding a wedge to a disconnected graph with two components becomes a connected or a single component graph. And a disconnected graph with three components and two wedges becomes a connected or a single component graph.

1.4 Theorem
If $\ell \leq m < n$, then the three star $K_{1,\ell} \wedge K_{1,m} \wedge K_{1,n}$ is a mean graph if and only if

1) $\ell + m - 8 \leq n \leq \ell + m + 4$ when $\ell > 9$.
2) $m + 1 \leq n \leq \ell + m + 4$ when $1 \leq \ell \leq 9$.

Proof:
Consider, $G = K_{1,\ell} \wedge K_{1,m} \wedge K_{1,n}$, then $G$ is a connected graph whose vertex and edge sets are given as follows,

$$V(G) = \{u, v, w\} \cup \{u_i : 1 \leq i \leq \ell\} \cup \{v_j : 1 \leq j \leq m\} \cup \{w_k : 1 \leq k \leq n\}.$$

$$E(G) = \{uu_i : 1 \leq i \leq \ell\} \cup \{vv_j : 1 \leq j \leq m\} \cup \{ww_k : 1 \leq k \leq n\} \cup \{u_i v_j : \text{for an } i \text{ and a } j\} \cup \{v_j w_k : \text{for a } j \text{ and a } k\}.$$ 

Therefore $G$ has $\ell + m + n + 3$ vertices and $\ell + m + n + 2$ edges.

Vertex labeling and edge labeling of $G$ is given by the functions $f$ and $f^*$ respectively, where
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\[ f : V(G) \rightarrow \{0, 1, 2, \ldots, q = \ell + m + n + 2\} \] and
\[ f : E(G) \rightarrow \{1, 2, 3, \ldots, q = \ell + m + n + 2\}. \]

Let \( \{t_{1,i} : 1 \leq i \leq \ell\} \) be the label given to the vertices of \( K_{1,\ell}\),
\( \{t_{2,j} : 1 \leq i \leq m\} \) be the label given to the vertices of \( K_{1,m}\),
\( \{t_{3,k} : 1 \leq i \leq n\} \) be the label given to the vertices of \( K_{1,n}\) and
\( \{x_{1,i} : 1 \leq i \leq \ell\} \) be the label given to the edges of \( K_{1,\ell}\),
\( \{x_{2,j} : 1 \leq i \leq m\} \) be the label given to the edges of \( K_{1,m}\),
\( \{x_{3,k} : 1 \leq i \leq n\} \) be the label given to the edges of \( K_{1,n}\).

**PART I \( \ell > 9 \)**

We have to prove that \( G = K_{1,\ell} \land K_{1,m} \land K_{1,n} \) is a mean graph if and only if \( n \) lies between \( \ell + m - 8 \) and \( \ell + m + 4 \).

First, we prove that \( G = K_{1,\ell} \land K_{1,m} \land K_{1,n} \) is a mean graph for each and every \( n \) between \( \ell + m - 8 \) and \( \ell + m + 4 \). Then there occurs thirteen cases. The vertex and edge labeling of each case is presented below.

**Case: 1 \( n = \ell + m + 4 \).**

The vertex labeling of \( G \): \( t_{1,0} = 0, t_{2,0} = 2 \) and \( t_{3,0} = q - 1 = \ell + m + n + 1 \).

\[ t_{1,i} = 2i + 2 \quad \text{for } 1 \leq i \leq \ell. \]
\[ t_{2,j} = 2\ell + 2j + 2 \quad \text{for } 1 \leq j \leq m. \]
\[ t_{3,k} = 2k - 1 \quad \text{for } 1 \leq k \leq n - 2, \]
\[ t_{3,n-1} = q - 2 = \ell + m + n \quad \text{and} \quad t_{3,n} = q = \ell + m + n + 2. \]

The edge labeling of \( G \):
\( x_{1,i} \) is \( i + 1 \) for \( 1 \leq i \leq \ell \); \( x_{2,j} \) is \( \ell + j + 2 \) for \( 1 \leq j \leq m \); \( x_{3,k} \) is \( \ell + m + k + 2 \) for \( 1 \leq k \leq n - 2 \); \( x_{3,n-1} \) is \( q - 1 = \ell + m + n + 1 \); \( x_{3,n} \) is \( q = \ell + m + n + 2 \). Also, \( t_{1,0}t_{2,0} \) and \( t_{1,\ell}t_{3,1} \) are the wedges labeled \( 1 \) and \( \ell + 2 \) respectively.

Hence, the vertex and edges of \( G \) has been assigned distinct labels. Therefore, \( G \) is a mean graph when \( n = \ell + m + n + 4 \).

**Case: 2 \( n = \ell + m + 3 \).**

The vertex labeling of \( G \): \( t_{1,0} = 0, t_{2,0} = 2 \) and \( t_{3,0} = q - 1 = \ell + m + n + 1 \).
The edge labeling of G:

\[ x_{i,j} = \begin{cases} 
    i+1 & \text{for } 1 \leq i \leq \ell; \\
    \ell + j + 2 & \text{for } 1 \leq j \leq m; \\
    \ell + m + k + 2 & \text{for } 1 \leq k \leq n.
\end{cases} \]

Also, \( t_{1,0}, t_{2,0}, t_{3,0} \) and \( t_{2,1}, t_{3,1} \) are the wedges labeled 1 and \( \ell + 2 \) respectively.

Hence, the vertex and edges of G has been assigned distinct labels. Therefore, G is a mean graph when \( n = \ell + m + n + 3 \).

**Case: 3** \( n = \ell + m + 2 \).

The vertex labeling of G: \( t_{1,0} = 0, t_{2,0} = 1 \) and \( t_{3,0} = q - 1 = \ell + m + n + 1 \).

\[ \begin{align*}
    t_{i,i} &= 2i + 1 & \text{for } 1 \leq i \leq \ell. \\
    t_{2,j} &= 2\ell + 2j + 1 & \text{for } 1 \leq j \leq m. \\
    t_{3,k} &= 2k & \text{for } 1 \leq k \leq n.
\end{align*} \]

The edge labeling of G:

\[ x_{i,j} = \begin{cases} 
    i+1 & \text{for } 1 \leq i \leq \ell; \\
    \ell + j + 2 & \text{for } 1 \leq j \leq m; \\
    \ell + m + k + 2 & \text{for } 1 \leq k \leq n.
\end{cases} \]

Also, \( t_{1,0}, t_{2,0}, t_{3,0} \) and \( t_{2,1}, t_{3,1} \) are the wedges labeled 1 and \( \ell + m + 2 \) respectively.

Hence, the vertex and edges of G has been assigned distinct labels. Therefore, G is a mean graph when \( n = \ell + m + n + 2 \).

**Case: 4** \( n = \ell + m + 1 \).

The vertex labeling of G: \( t_{1,0} = 0, t_{2,0} = 1 \) and \( t_{3,0} = q - 1 = \ell + m + n + 1 \).

\[ \begin{align*}
    t_{i,i} &= 2i & \text{for } 1 \leq i \leq \ell. \\
    t_{2,j} &= 2\ell + 2j & \text{for } 1 \leq j \leq m. \\
    t_{3,k} &= 2k + 1 & \text{for } 1 \leq k \leq n.
\end{align*} \]

The edge labeling of G:

\[ x_{i,j} = \begin{cases} 
    i & \text{for } 1 \leq i \leq \ell; \\
    \ell + j + 1 & \text{for } 1 \leq j \leq m; \\
    \ell + m + k + 2 & \text{for } 1 \leq k \leq n.
\end{cases} \]

Also, \( t_{1,0}, t_{2,1}, t_{3,1} \) are the wedges labeled \( \ell + 1 \) and \( \ell + m + 2 \) respectively.

Hence, the vertex and edges of G has been assigned distinct labels. Therefore, G is a mean graph when \( n = \ell + m + n + 1 \).
Case: 5 $n = \ell + m$.
The vertex labeling of $G$: $t_{1,0} = 0, t_{2,0} = 2$ and $t_{3,0} = q - 1 = \ell + m + n + 1$.
$$
t_{1,i} = 2i - 1 \quad \text{for } 1 \leq i \leq \ell.
t_{2,j} = 2\ell + 2j - 1 \quad \text{for } 1 \leq j \leq m.
t_{3,k} = 2k + 2 \quad \text{for } 1 \leq k \leq n.
$$
The edge labeling of $G$:
$$
x_{1,i} = i \quad \text{for } 1 \leq i \leq \ell; \quad x_{2,j} = \ell + j + 1 \quad \text{for } 1 \leq j \leq m; \quad x_{3,k} = \ell + m + k + 2 \quad \text{for } 1 \leq k \leq n.
$$
Also, $t_{1,0}t_{2,1}$ and $t_{2,0}t_{3,1}$ are the wedges labeled $\ell + 1$ and $\ell + m + 2$ respectively.
Hence, the vertex and edges of $G$ has been assigned distinct labels. Therefore, $G$ is a mean graph when $n = \ell + m + n$.

Case: 6 $n = \ell + m - 1$.
The vertex labeling of $G$: $t_{1,0} = 1, t_{2,0} = 3$ and $t_{3,0} = q - 1 = \ell + m + n + 1$.
$$
t_{1,i} = 2i - 2 \quad \text{for } 1 \leq i \leq \ell.
t_{2,j} = 2\ell + 2j - 2 \quad \text{for } 1 \leq j \leq m.
t_{3,k} = 2k + 3 \quad \text{for } 1 \leq k \leq n.
$$
The edge labeling of $G$:
$$
x_{1,i} = i \quad \text{for } 1 \leq i \leq \ell; \quad x_{2,j} = \ell + j + 1 \quad \text{for } 1 \leq j \leq m; \quad x_{3,k} = \ell + m + k + 2 \quad \text{for } 1 \leq k \leq n.
$$
Also, $t_{1,0}t_{2,1}$ and $t_{2,0}t_{3,1}$ are the wedges labeled $\ell + 1$ and $\ell + m + 2$ respectively.
Hence, the vertex and edges of $G$ has been assigned distinct labels. Therefore, $G$ is a mean graph when $n = \ell + m + n - 1$.

Case: 7 $n = \ell + m - 2$.
The vertex labeling of $G$: $t_{1,0} = 2, t_{2,0} = 4$ and $t_{3,0} = q - 1 = \ell + m + n + 1$.
$$
t_{1,i} = 0;
t_{1,i} = 2i - 3 \quad \text{for } 2 \leq i \leq \ell.
t_{2,j} = 2\ell + 2j - 3 \quad \text{for } 1 \leq j \leq m.
t_{3,k} = 2k + 4 \quad \text{for } 1 \leq k \leq n.
$$
The edge labeling of $G$:
$$
x_{1,i} = i \quad \text{for } 2 \leq i \leq \ell; \quad x_{2,j} = \ell + j + 1 \quad \text{for } 1 \leq j \leq m; \quad x_{3,k} = \ell + m + k + 2 \quad \text{for } 1 \leq k \leq n.
$$
Also, $t_{1,0}t_{2,1}$ and $t_{2,0}t_{3,1}$ are the wedges labeled $\ell + 1$ and $\ell + m + 2$ respectively.
Hence, the vertex and edges of $G$ has been assigned distinct labels. Therefore, $G$ is a mean graph when $n = \ell + m + n - 2$. 

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Case: 8 $n = \ell + m - 3$.
The vertex labeling of $G$: $t_{1,0} = 0, t_{2,0} = q - 1 = \ell + m + n + 1$ and $t_{3,0} = 2\ell + 3$.

\begin{align*}
  t_{1,i} &= 2i \quad \text{for } 1 \leq i \leq \ell. \\
  t_{2,j} &= 2\ell + 2j \quad \text{for } 1 \leq j \leq m - 2, \\
  t_{2,m-1} &= q - 2 = \ell + m + n, \\
  t_{2,m} &= q = \ell + m + n + 2, \\
  t_{3,k} &= 2k - 1 \quad \text{for } 1 \leq k \leq \ell + 1; \\
  t_{3,h} &= 2h + 1 \quad \text{for } \ell + 2 \leq h \leq n.
\end{align*}

The edge labeling of $G$:

\begin{align*}
x_{i,j} &= i \text{ for } 1 \leq i \leq \ell; \quad x_{2,j} = 2\ell + m + j - 1 \text{ for } 1 \leq j \leq m - 2; \quad x_{2,m-1} = q = \ell + m + n + 1 \text{ for } 1 \leq j \leq m - 2; \quad x_{2,m} = q - 1 = \ell + m + n + 1 \text{ for } 1 \leq j \leq m - 2; \quad x_{3,k} = \ell + k + 1 \text{ for } 1 \leq k \leq \ell + 1; \quad x_{3,h} = \ell + h + 2 \text{ for } 1 \leq k \leq \ell + 1; \\
  x_{3,h} &= \ell + h + 2 \text{ for } 1 \leq k \leq \ell + 1. \\
\end{align*}

Also, $t_{1,1}, t_{3,1}$ and $t_{2,2} t_{1,1}$ are the wedges labeled $\ell + 1$ and $\ell + m + 3$ respectively.
Hence, the vertex and edges of $G$ has been assigned distinct labels. Therefore, $G$ is a mean graph when $n = \ell + m + n - 3$.

Case: 9 $n = \ell + m - 4$.
The vertex labeling of $G$: $t_{1,0} = 1, t_{2,0} = q - 1 = \ell + m + n + 1$ and $t_{3,0} = 2\ell - 1$.

\begin{align*}
  t_{1,i} &= 2i - 2 \quad \text{for } 1 \leq i \leq \ell. \\
  t_{2,j} &= 2\ell + 2j - 2 \quad \text{for } 1 \leq j \leq m. \\
  t_{3,k} &= 2k + 1 \quad \text{for } 1 \leq k \leq \ell - 2; \\
  t_{3,h} &= 2h + 3 \quad \text{for } \ell - 1 \leq h \leq n.
\end{align*}

The edge labeling of $G$:

\begin{align*}
x_{i,j} &= i \text{ for } 1 \leq i \leq \ell; \quad x_{2,j} = 2\ell + m + j - 2 \text{ for } 1 \leq j \leq m; \quad x_{3,k} = \ell + k \text{ for } 1 \leq k \leq \ell - 2; \quad x_{3,h} = \ell + h + 1 \text{ for } 1 \leq k \leq \ell - 2; \\
x_{3,h} &= \ell + h + 1 \text{ for } 1 \leq k \leq \ell - 2. \\
\end{align*}

Also, $t_{1,1}, t_{2,2} t_{1,2}$ and $t_{2,2} t_{3,1}$ are the wedges labeled $2\ell - 1$ and $\ell + m + 2$ respectively.
Hence, the vertex and edges of $G$ has been assigned distinct labels. Therefore, $G$ is a mean graph when $n = \ell + m + n - 4$.

Case: 10 $n = \ell + m - 5$.
The vertex labeling of $G$: $t_{1,0} = 1, t_{2,0} = q - 1 = \ell + m + n + 1$ and $t_{3,0} = 2\ell + 1$.
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The edge labeling of $G$:

$v_{1,i}$ is $i$ for $1 \leq i \leq \ell$; $v_{2,j}$ is $2\ell + m + j - 3$ for $1 \leq j \leq m - 2$; $v_{2,m-1}$ is $q - 1 = \ell + m + n + 1$; $v_{2,m}$ is $q = \ell + m + n + 2$; $v_{3,k}$ is $\ell + k + 1$ for $1 \leq k \leq \ell - 1$; $v_{3,h}$ is $\ell + h + 2$ for $\ell - 1 \leq h \leq n$. Also, $t_{1,i}t_{3,i}$ and $t_{2,m}t_{3,2}$ are the wedges labeled $\ell + 1$ and $2\ell + 1$ respectively.

Hence, the vertex and edges of $G$ has been assigned distinct labels. Therefore, $G$ is a mean graph when $n = \ell + m + n - 5$.

**Case: 11** $n = \ell + m - 6$.

The vertex labeling of $G$: $t_{1,0} = 1, t_{2,0} = q - 1 = \ell + m + n + 1$ and $t_{3,0} = 2\ell$.

The edge labeling of $G$:

$v_{1,i}$ is $i$ for $1 \leq i \leq \ell$; $v_{2,j}$ is $2\ell + m + j - 4$ for $1 \leq j \leq m - 2$; $v_{2,m-1}$ is $q - 1 = \ell + m + n + 1$; $v_{2,m}$ is $q = \ell + m + n + 2$; $v_{3,k}$ is $\ell + k + 1$ for $1 \leq k \leq \ell - 2$; $v_{3,h}$ is $\ell + h + 2$ for $\ell - 1 \leq h \leq n$. Also, $t_{1,i}t_{3,i}$ and $t_{2,m}t_{3,2}$ are the wedges labeled $\ell + 1$ and $\ell + m$ respectively.

Hence, the vertex and edges of $G$ has been assigned distinct labels. Therefore, $G$ is a mean graph when $n = \ell + m + n - 6$.

**Case: 12** $n = \ell + m - 7$.

The vertex labeling of $G$: $t_{1,0} = 2, t_{2,0} = q - 1 = \ell + m + n + 1$ and $t_{3,0} = 2\ell - 1$. 
The edge labeling of $G$:

\[ t_{1,i} = 0; \]
\[ t_{1,i} = 2i - 3 \quad \text{for } 2 \leq i \leq \ell. \]
\[ t_{2,j} = 2\ell + 2j - 4 \quad \text{for } 1 \leq j \leq m - 2, \]
\[ t_{2,m-1} = q - 2 = \ell + m + n, \]
\[ t_{2,m} = q = \ell + m + n + 2. \]
\[ t_{3,k} = 2k + 2 \quad \text{for } 1 \leq k \leq \ell - 3; \]
\[ t_{3,h} = 2h + 5 \quad \text{for } \ell - 2 \leq h \leq n. \]

Hence, the vertex and edges of $G$ has been assigned distinct labels. Therefore, $G$ is a mean graph when $n = \ell + m + n - 7$.

**Case: 13** $n = \ell + m - 8$.

The vertex labeling of $G$: $t_{1,0} = 2$, $t_{2,0} = q - 1 = \ell + m + n + 1$ and $t_{3,0} = 2\ell - 2$.

\[ t_{1,i} = 0; \]
\[ t_{1,i} = 2i - 3 \quad \text{for } 2 \leq i \leq \ell. \]
\[ t_{2,i} = 2\ell + 4; \]
\[ t_{2,j} = 2\ell + 2j - 5 \quad \text{for } 2 \leq j \leq m - 2, \]
\[ t_{2,m-1} = q - 2 = \ell + m + n, \]
\[ t_{2,m} = q = \ell + m + n + 2. \]
\[ t_{3,k} = 2k + 2 \quad \text{for } 1 \leq k \leq \ell - 4; \]
\[ t_{3,h} = 2h + 6 \quad \text{for } \ell - 3 \leq h \leq n. \]

Hence, the vertex and edges of $G$ has been assigned distinct labels. Therefore, $G$ is a mean graph when $n = \ell + m + n - 8$. 
Thus \( G = K_{1,\ell} \land K_{1,m} \land K_{1,n} \) is a mean graph for all \( n \) in \( \ell + m - 8, \ell + m + 4 \).

Now, we have to prove that \( G = K_{1,\ell} \land K_{1,m} \land K_{1,n} \) is not a mean graph for \( \ell + m - 9 \geq n \geq \ell + m + 5 \). Then there arises two cases,

- \( \ell + m - 9 \geq n \).
- \( n \geq \ell + m + 5 \).

At first we shall consider the case when \( n \leq \ell + m - 9 \). Primarily, consider \( n = \ell + m - 9 \).

None of the labeling defined above suits \( G \) when \( n = \ell + m - 9 \). For instance comparing \( G \) when \( n = \ell + m - 9 \) with the labeling defined in the case when \( n = \ell + m - 8 \), we see that \( t_{2,m-2} \) and \( t_{2,m-1} \) receives the same label. Similarly we can see that \( G \) does not match with any labeling in the previous cases when \( n = \ell + m - 9 \).

We shall now try to give a refined proof with the help of simulation analysis, consider the graph \( G = K_{1,10} \land K_{1,10} \land K_{1,11} \), then \( G \) has 34 vertices and 33 edges, so the vertices of \( G \) should be labeled from \( \{0, 1, \ldots, q = 33\} \) and the edges should be labeled from \( \{1, 2, \ldots, q = 33\} \). We are supposed to align distinct labels to the vertices so that they generate distinct labels to the edges. Let us first label the vertices of \( K_{1,10} \), such that its edges are labeled \( \{0, 1, \ldots, 10\} \) (we assign consecutive integers to edges so that it lies in a smaller range and finding a vertex label in common to the smallest and the greatest edge label in that range seem make mean labeling possible). The possibilities to obtain the edge label 1 are 0 and 1 or 0 and 2. Therefore, 0 should be the label of an end of the edge 1, let it be \( t_{1,0} \), i.e., \( t_{1,0} = 0 \). Then the possibilities of \( t_{1,i} \) where \( 1 \leq i \leq \ell \), for obtaining the edge labels \( \{1, 2, \ldots, 10\} \) are \( \{1 \text{ or } 2, \text{ or } 3 \text{ or } 4, \text{ or } 5 \text{ or } 6, \text{ or } 7 \text{ or } 8, \text{ or } 9 \text{ or } 10, \text{ or } 11 \text{ or } 12, \text{ or } 13 \text{ or } 14, \text{ or } 15 \text{ or } 16, \text{ or } 17 \text{ or } 18, \text{ or } 19 \text{ or } 20 \} \). Also, label the vertices of another \( K_{1,10} \), such that its edges are labeled \( \{24, 25, \ldots, 33\} \). The only possibility to get the edge label 33 is 33 and 32 or vice versa, let \( t_{2,0} = 33 \) and \( t_{2,32} = 32 \). Then the possibilities of \( t_{2,j} \) for \( 1 \leq j \leq m \), for obtaining the edge labels \( \{24, 25, \ldots, 33\} \) are \( \{14 \text{ or } 15, \text{ or } 16 \text{ or } 17, \text{ or } 18 \text{ or } 19, \text{ or } 20 \text{ or } 21, \text{ or } 22 \text{ or } 23, \text{ or } 24 \text{ or } 25, \text{ or } 26 \text{ or } 27, \text{ or } 28 \text{ or } 29, \text{ or } 30 \text{ or } 31 \} \).

And the edges of \( K_{1,11} \) and the two wedges should be labeled from the remaining edge labels \( \{11, 12, \ldots, 23\} \). First, consider \( \{11, 12, \ldots, 21\} \) are the edge labels of \( K_{1,11} \) and 22 and 23 be the wedge labels. Now let us see the possibilities of getting a common vertex label \( (t_{3,0}) \) to 11 and 21. Possibilities of vertex pair (say \( t_{3,0} \) and \( t_{3,1} \)) to induce the edge label 11 are, respectively,(we don’t include the possibilities involving the vertex label 0 because it has been already used to generate the edge label 1)
There is no other possibility of obtaining the edge label 11. Possibilities of vertex pair (say \(t_{3,0}\) and \(t_{3,n}\)) to induce the edge label 21 are, respectively, (we don't include the possibilities involving the vertex label 33 and 32 because it has been already used to generate the edge label 33)

\(20\) and \(22\) or \(21\), \(21\) and \(20\), \(22\) and \(20\) or \(19\), \(23\) and \(19\) or \(18\), \(24\) and \(18\) or \(17\), \(25\) and \(17\) or \(16\), \(26\) and \(16\) or \(15\), \(27\) and \(15\) or \(14\), \(28\) and \(14\) or \(13\), \(29\) and \(13\) or \(12\), \(30\) and \(12\) or \(11\), \(31\) and \(11\) or \(10\).

There is no other possibility of obtaining the edge label 21. Therefore, the vertex label in common (\(t_{3,0}\)) that generates the edges 11 as well as 21 are 20 and 21.

Suppose that all the \(t_{1,i}\)'s where \(1 \leq i \leq 10\) are labeled even i.e., \{2, 4, \ldots, 20\} and all the \(t_{2,j}\)'s where \(1 \leq j \leq 9\) are labeled odd i.e., \{15, 17, \ldots, 31\}. Then the remaining vertex labels to label the vertices \(t_{3,k}, 0 \leq k \leq 11\) are \{1, 3, 5, 7, 9, 11, 13, 22, 24, 26, 28, 30\}, we see that both the common vertex labels 20 and 21 are not possible to label \(t_{3,0}\). Therefore, \(G\) is not a mean graph. We can also see that 20 and 21 will not be possible on considering the other choices of \(t_{1,i}\)'s as well as \(t_{2,j}\)'s.

Hence, \(G = K_{1,\ell} \land K_{1,m} \land K_{1,n}\) is not a mean graph when \(n = \ell + m - 9\) and it follows for the smaller values of \(n\) too. Therefore \(G = K_{1,\ell} \land K_{1,m} \land K_{1,n}\) is not a mean graph when \(n < \ell + m - 9\).

**PART II 1 \leq \ell \leq 9**

We have to prove that \(G = K_{1,\ell} \land K_{1,m} \land K_{1,n}\) is a mean graph if and only if \(n\) lies between \(m + 1\) and \(\ell + m + 4\).

First, we prove that \(G = K_{1,\ell} \land K_{1,m} \land K_{1,n}\) is a mean graph for each and every \(n\) between \(m + 1\) and \(\ell + m + 4\). For each case of \(n\), we can see that \(G = K_{1,\ell} \land K_{1,m} \land K_{1,n}\) obeys any one of the labeling defined in PART I. Therefore \(G\) is a mean graph for all \(n\) in \([m + 1, \ell + m + 4]\).

Now, we prove that \(G\) is not a mean graph when \(n < m\) and \(n > \ell + m + 4\).
On Limits Of Mean Labeling For Three Star Graphs

\[ G = K_{1, \ell} \land K_{1, m} \land K_{1, n} \] is not a mean graph when \( \ell < m \) simply because it barrs the basic condition of the theorem \( \ell \leq m < n \).

Next, \( G = K_{1, \ell} \land K_{1, m} \land K_{1, n} \) is not a mean graph when \( n > \ell + m + 4 \).

To prove this statement let us consider the graph \( G = K_{1, 2} \land K_{1, 2} \land K_{1, 3} \) where

\[ V(G) = \{v_{i, j} : 1 \leq i \leq 2, 0 \leq j \leq 2\} \cup \{v_{3, j} : 0 \leq j \leq 9\} \]

\[ E(G) = \{v_{i, 0}v_{i, j} : 1 \leq i \leq 2; 1 \leq j \leq 2\} \cup \{v_{3, 0}v_{3, j} : 1 \leq j \leq 9\} \]

Assume the contradiction, that \( G \) is a mean graph.

Then there exists a function \( f \) from the vertex set of \( G \) to \( \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\} \) such that the induced map \( f^* \) from the edge set of \( G \) to \( \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\} \) defined by

\[ f^*(e = uv) = \begin{cases} \frac{f(u) + f(v)}{2} & \text{if } f(u) + f(v) \text{ is even} \\ \frac{f(u) + f(v) + 1}{2} & \text{if } f(u) + f(v) \text{ is odd} \end{cases} \]

then the resulting edges get distinct labels.

Let \( t_{i,j} \) be the label given to the vertices \( v_{i,j} \) and \( x_{i,j} \) be the label given to the edges \( v_{i,0}v_{i,j} \).

Let us first consider the case that \( t_{3,0} = 15 \).

If \( t_{3,j} = 2n \) and \( t_{3,k} = 2n + 1 \) for some \( n \) and for some \( j \) and \( k \); then

\[ f^*(v_{3,0}v_{3,j}) = \left\lfloor \frac{15 + 2n}{2} \right\rfloor = 8 + n = \left\lfloor \frac{15 + 2n + 1}{2} \right\rfloor = f^*(v_{3,0}v_{3,k}) \]

This is not possible as \( f^* \) is a bijection.

Therefore the nine numbers \( t_{3,j} \) for \( 1 \leq j \leq 9 \), must be among \( 0, (0 \text{ or } 1), (2 \text{ or } 3), (4 \text{ or } 5), (6 \text{ or } 7), (8 \text{ or } 9), (10 \text{ or } 11), (12 \text{ or } 13), \text{ and } 14. \)

i.e., there are only eight possibilities to label nine vertices which is not sufficient, which means labeling is not possible.

Therefore, \( G \) is not a mean graph when \( t_{3,0} = 15 \).

Now let us consider the case that \( t_{3,0} = 14 \).

If \( t_{3,j} = 2n - 1 \) and \( t_{3,k} = 2n \) for some \( n \) and for some \( j \) and \( k \); then

\[ f^*(v_{3,0}v_{3,j}) = \left\lfloor \frac{14 + 2n - 1}{2} \right\rfloor = 7 + n = \left\lfloor \frac{14 + 2n}{2} \right\rfloor = f^*(v_{3,0}v_{3,k}) \]

This is not possible as \( f^* \) is a bijection.

Therefore the nine numbers \( t_{3,j} \) for \( 1 \leq j \leq 9 \), must be among \( 0, (1 \text{ or } 2), (3 \text{ or } 4), (5 \text{ or } 6), (7 \text{ or } 8), (9 \text{ or } 10), (11 \text{ or } 12), (13 \text{ or } 14), \text{ and } 15. \)

This is not possible as \( f^* \) is a bijection.

Therefore the nine numbers \( t_{3,j} \) for \( 1 \leq j \leq 9 \), must be among \( 0, (0 \text{ or } 1), (2 \text{ or } 3), (4 \text{ or } 5), (6 \text{ or } 7), (8 \text{ or } 9), (10 \text{ or } 11), (12 \text{ or } 13), \text{ and } 14. \)

Therefore, \( G \) is not a mean graph when \( t_{3,0} = 14 \).
Then $x_{3,j}$ takes all the values of \{7, 8, \ldots, 15\}. We have the following fixed values $t_{3,0} = 14$, $t_{3,1} = 0, t_{3,8} = 13$ and $t_{3,9} = 15$.

Next, $t_{3,7}$ should be either 11 or 12.

Suppose that $t_{3,7} = 11$ then, 12 should be the label of pendent or non-pendent vertices of $K_{1,2}$ components of $G$.

If 12 is a label of any vertex of $K_{1,2}$ components i.e., $v_{i,j}$ for $1 \leq i \leq 2, 0 \leq j \leq 2$, then $x_{i,j} \geq 7$, which is not possible.

Therefore, let $t_{3,7} = 12$.

Now, 11 will be the label of pendent or non-pendent vertices of $K_{1,2}$ components of $G$. Suppose 11 is a label of a non-pendent vertex of $K_{1,2}$. Without loss of generality, let $t_{1,0} = 11$, then $t_{1,1} = 1$ (if $t_{1,j} \geq 2$ then $x_{1,j} \geq 7$ which is not possible), then $t_{1,2}$ has no possibilities.

Therefore, 11 is not a label of a non-pendent vertex and it should be the label of a pendent vertex.

(i.e.) let $t_{1,1} = 11$, then $t_{1,0} = 1$ (if $t_{1,0} \geq 2$ then $x_{1,1} \geq 7$ which is not possible).

Now we have

$t_{3,1} = 0, t_{3,0} = 14, t_{3,9} = 15, t_{3,8} = 13, t_{3,7} = 12, t_{1,1} = 11, t_{1,0} = 1, x_{1,1} = 6$.

Next $t_{3,6}$ should be either 9 or 10.

If $t_{3,6} = 9$, then let $t_{1,2} = 10$.

Which implies $x_{1,2} = 6$ but we already have $x_{1,1} = 6$.

Therefore 10 should be the label of $t_{2,j}, 0 \leq j \leq 2$.

Then $t_{2,j} \geq 6$ which is not possible.

Therefore $t_{3,6} = 10$.

Also, $t_{1,2} = 9$, then $x_{1,2} = 5$.

Now we have

$t_{3,1} = 0, t_{3,0} = 14, t_{3,9} = 15, t_{3,8} = 13, t_{3,7} = 12, t_{3,6} = 10, t_{1,1} = 11, t_{1,0} = 1, t_{1,2} = 9$

$x_{1,1} = 6$ and $x_{1,2} = 5$.

Next let us consider $t_{3,5}$.

$t_{3,5}$ is either 7 or 8. The remaining possible edge labels are $x_{2,j} \leq 4$ which is not possible when $t_{2,j} = 7$ or 8.

Therefore, $G$ is not a mean graph when $t_{3,0} = 14$. 
Similarly we can prove that \( G \) is not a mean graph for all other possible values of \( t_{3,0} \). Hence, \( G = K_{1,\ell} \wedge K_{1,m} \wedge K_{1,n} \) is not a mean graph when \( n = \ell + m + 5 \). It follows for the greater values of \( n \) too. Therefore, \( G = K_{1,\ell} \wedge K_{1,m} \wedge K_{1,n} \) is not a mean graph when \( n = \ell + m + 5 \).

**APPLICATION OF MEAN LABELING**

The communications network addressing: A communication network is composed of nodes, each of which has computing power and can transmit and receive messages over communication links, wireless or cabled. The basic network topologies include fully connected, mesh, star, ring, tree, bus. A single network may consist of several interconnected subnets of different topologies. Networks are further classified as Local Area Networks (LAN), e.g. inside one building, or Wide Area Networks (WAN), e.g. between buildings. It might be useful to assign each user terminal a node label, subject to the constraint that all connecting edges (communication links) receive distinct labels. In this way, the numbers of any two communicating terminals automatically specify (by simple subtraction) the link label of the connecting path; and conversely, the path label uniquely specifies the pair of user terminals which it interconnects. Researches may get some information related to graph labeling and its applications in communication field and can get some ideas related to their field of research.

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