Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 21, Number 1 (2025), pp. 15-34

http://ripublication.com/gjpam23/gjpamv21n1_02.pdf

The Prime Number Theorem, Riemann Zeta Function Zeros, and Dynamical Zeta Functions

Darrell Cox[1]

Grayson County College[1]*, United States*

Abstract

Chebyshev's second function, the Möbius function, and Mersenne primes are used to derive a function that estimates the number of primes less than a given amount. A relationship between this function and Riemann zeta function zeros is investigated. Dynamical zeta functions are derived using this function.

Kkeywords : Chebyshev's second function, Möbius function, Mersenne primes, dynamical zeta functions

1. INTRODUCTION

Chebyshev's second function is the summatory Mangoldt function, that is,

$$\psi(x) = \sum_{n \leq x} \Lambda(n), x > 0. \tag{1}$$

$\Lambda(n)$ equals $\log(p)$ if $n = p^m$ for some prime p and some $m \geq 1$ or 0 otherwise. The prime number theorem is equivalent to the asymptotic formula

$$\sum_{n \leq x} \Lambda(n) \sim x, x \to \infty \tag{2}$$

This asymptotic formula states that

$$\lim_{x \to \infty} \frac{\psi(x)}{x} = 1. \tag{3}$$

Let v_n, $n = 1, 2, 3, \ldots, 10000$, denote $\sum_{i|n}(\psi_{i+1} - \psi_i)\mu(i)$ where $\mu(i)$ denotes the Möbius function. The Möbius function is defined as follows. $\mu(1)$ is set to 1. For $n > 1$, write $n = p_1^{a_1} \cdots p_k^{a_k}$. Then $\mu(n) = (-1)^k$ if $a_1 = a_2 = \ldots = a_k = 1$ or 0 otherwise. ψ_1 is set to 0. For prime n other than Mersenne primes (3, 7, 31, 127, 8191,...), v_n then equals $\log(2)$. If n is a Mersenne prime, then $v_n = 0$ since $n + 1$

is a prime power and the ψ values increase by $\log(2)$ at this point (cancelling out the difference between ψ_2 and ψ_1). In general, if n is odd, v_n equals an integer multiple of $\log(2)$ (including a multiple of 0). Note that the Möbius function zeros out any $\psi_{i+1}-\psi_i$ value in the sum where i is not square-free. A plot of the v_n values versus the n values is

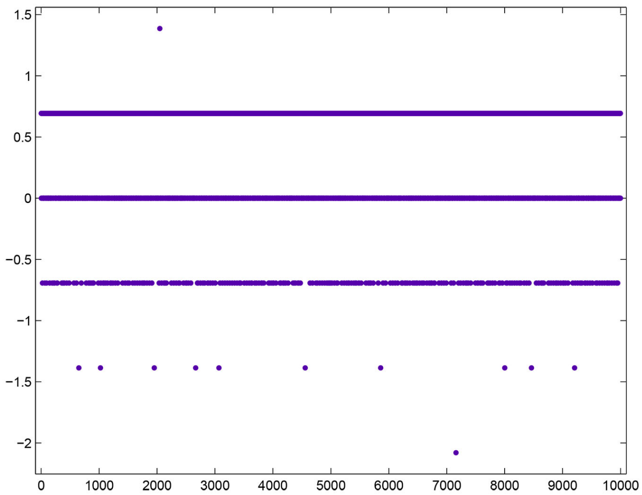

Figure 1

The respective counts of the number of elements in each line are 1, 3000, 1739, 235, 23, and 2 (some points are visually indistinguishable from other points).

The sum of the counts times the respective multiples of $\log(2)$ is used to estimate the number of primes. Let v_n' denote $v_n/\log(v_n)$ and $\pi(n)$ denote the number of primes less than or equal to n. A plot of v_n' for $n = 512, 514, 516,...,710$ is

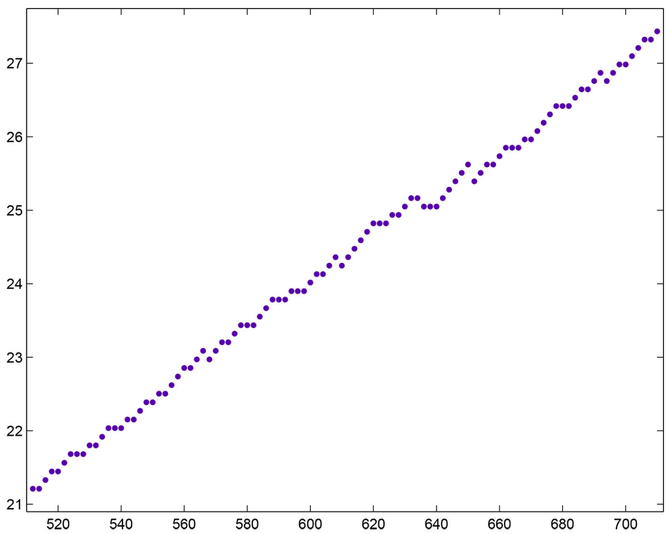

Figure 2

The n values up to 512 are not considered because the v_n values are negative. Note that there are some equal v'_n values. A plot of the corresponding $\pi(n)$ values is

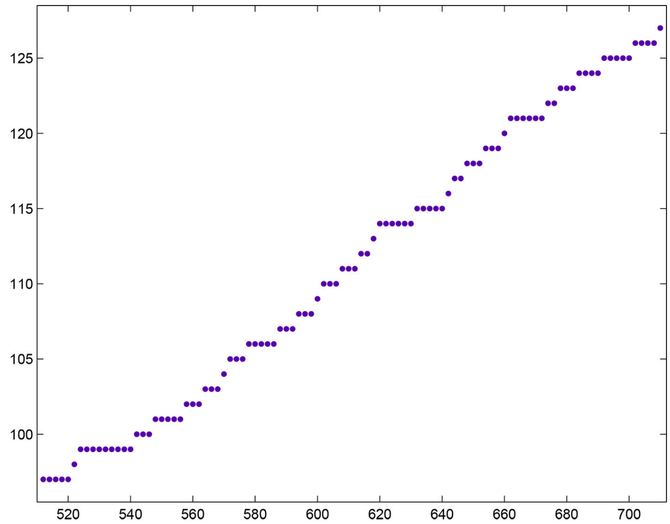

Figure 3

A plot of v'_n versus $\pi(n)$ is

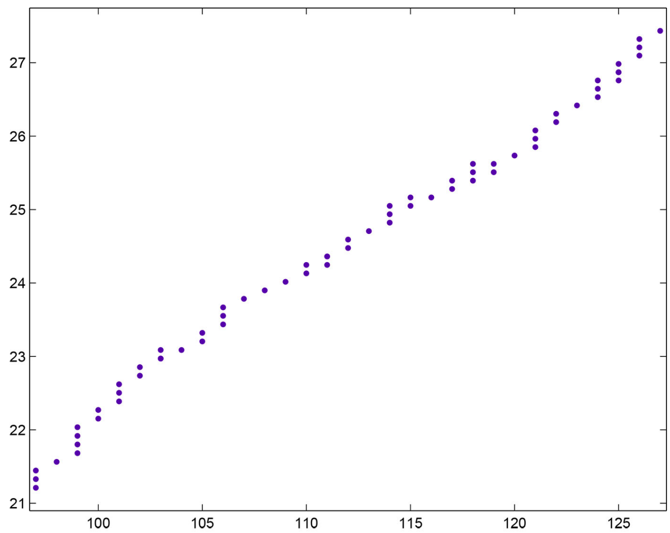

Figure 4

Other than the v'_n values that are equal, the v'_n values are equally distributed along the y-axis for each $\pi(n)$ value.

A plot of $\pi(n)$ for n=49802, 49804, 49806,. . .,50000 is

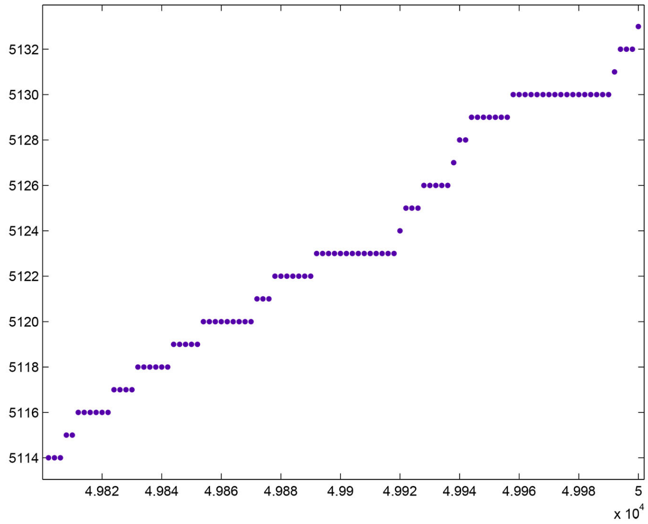

Figure 5

A plot of v'_n versus $\pi(n)$ for n=49802, 49804, 49806,. . .,50000 is

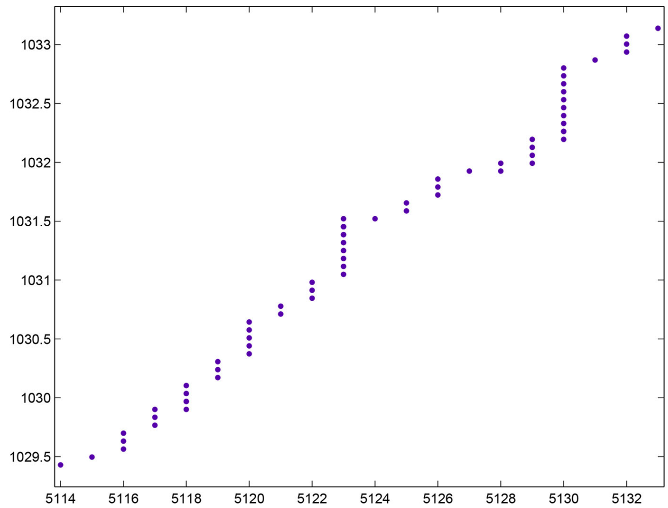

Figure 6

In the following table, the first column is the $\pi(n)$ values. The second column is the number of points these values are mapped to. The third column is the numbers of v'_n values that are equal. The sum of the values in parentheses for a particular row is the $\pi(n)$ value and the number of values in parentheses is the number of points in the mapping. For example, in the third row, there are three v'_n values that are not equal.

3 $\rightarrow$ 1 (3)
2 $\rightarrow$ 1 (2)
6 $\rightarrow$ 3 (1,1,4)
4 $\rightarrow$ 3 (1,2,1)

$6 \to 4$ (2,1,2,1)
$5 \to 3$ (1,2,2)
$9 \to 5$ (1,2,3,1,2)
$3 \to 2$ (1,2)
$7 \to 3$ (3,1,3)
$14 \to 8$ (2,1,2,2,1,3,2,1)
$1 \to 1$ (1)
$3 \to 2$ (2,1)
$5 \to 3$ (3,1,1)
$1 \to 1$ (1)
$2 \to 2$ (1,1)
$7 \to 4$ (3,1,2,1)
$17 \to 10$ (2,1,1,1,1,3,2,1,2,3)
$1 \to 1$ (1)
$3 \to 3$ (1,1,1)
$1 \to 1$ (1)

A plot of v'_n versus $\pi(n)$ for n up to 20000 is

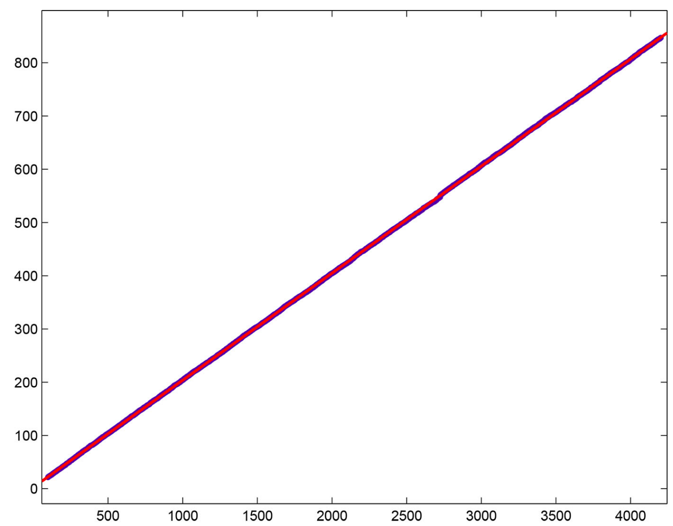

Figure 7

For a linear least-squares fit of the curve, $p_1 = 0.2011$ with a 95% confidence interval of (0.201, 0.2011), $p_2 = 2.729$ with a 95% confidence interval of (2.7, 2.757), SSE=$1.722 \cdot 10^4$, R-squared=1, and RMSE=0.9339. For $n <= 80000$, $p_1 = 0.2007$ with a 95% confidence ineval of (0.2007, 0.2007), $p_2 = 3.283$ with a 95% confidence interval of (3.253, 3.312), SSE=$7.62 \cdot 10^4$, R-squared=1, and RMSE=1.385.

2. RIEMANN ZETA FUNCTION ZEROS

The Riemann zeta function $\zeta(s)$ for $0 < Re(s) < 1$ can be computed from the η function;

$$\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s} = (1 - 2^{1-s})\zeta(s) \tag{4}$$

A plot of the real components of $\zeta(s)$ for the first non-trivial zeta functon zero $(s = (0.5, 14.1347251417))$ and $n \leq 200$ is

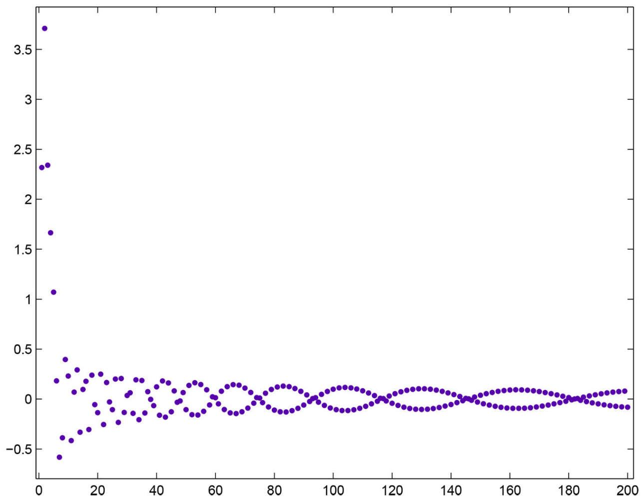

Figure 8

A plot of the values from $n = 147$ to 184 where successive values are connected is

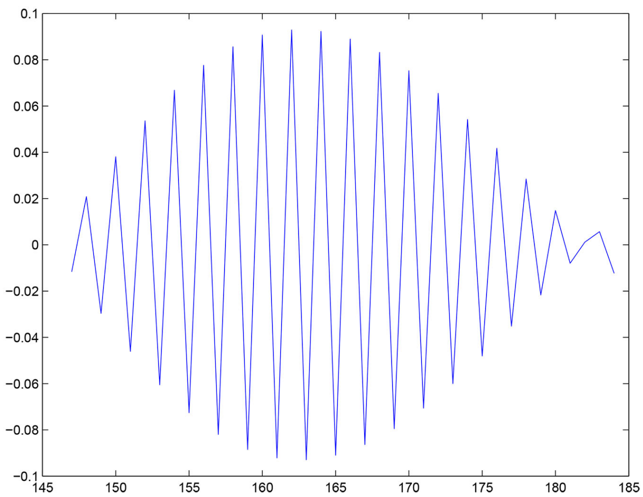

Figure 9

A plot of the n values of the inflection points (where the curve crosses the x-axis from above) for $n \leq 200$ is

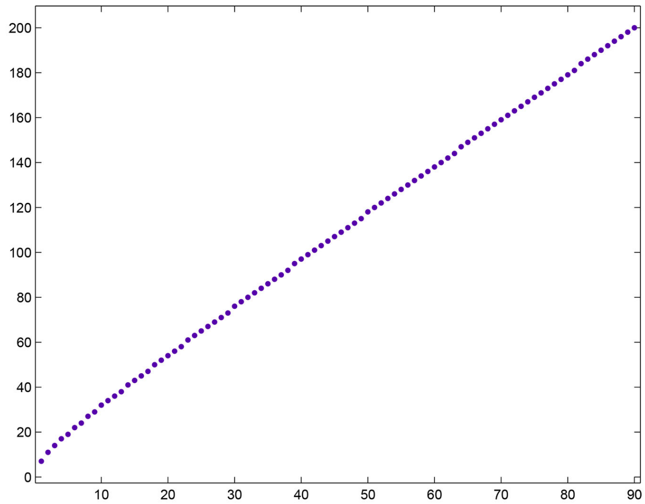

Figure 10

The n values of the inflection points for $n \leq 95$ are 7, 11, 14, 17, 19, 22, 24, 27, 29, 32, 34, 36, 38, 41, 43, 45, 47, 50, 52, 54, 56, 58, 61, 63, 65, 67, 69, 71, 73, 76, 78, 80, 82, 84, 86, 88, 90, 92, and 95. The n values that are at least three greater than the previous n values are 11, 14, 17, 22, 27, 32, 41, 50, 61, 76, and 95. Except for 11, the n values are three greater than the previous n value. The number of n values between 76 and 95 for example is $\frac{95-76-3}{2}$. A plot of the logarithms of the n values that are at least three greater than the previous n values for $n \leq 100000$ is

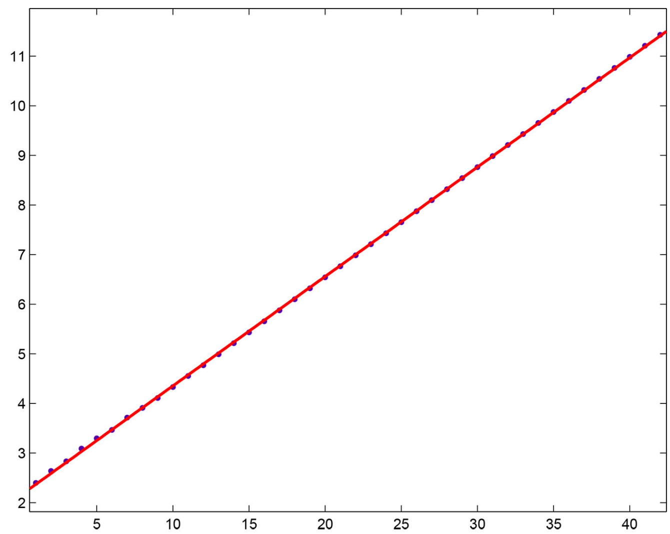

Figure 11

For a linear least-squares fit of the curve, $p_1 = 0.2205$ with a 95% confidence interval of (0.22, 0.2211), $p_2 = 2.15$ with a 95% confidence interval of (2.136, 2.163), SSE=0.01785, R-squared=0.9999, and RMSE=0.02113.

Let $\zeta'(s)$ denote the function where v'_n values are substituted for the imaginary part of the zeta function zeros in the above formula. A plot of $\zeta'(s)$ for $n \leq 512 \leq 20000$ is

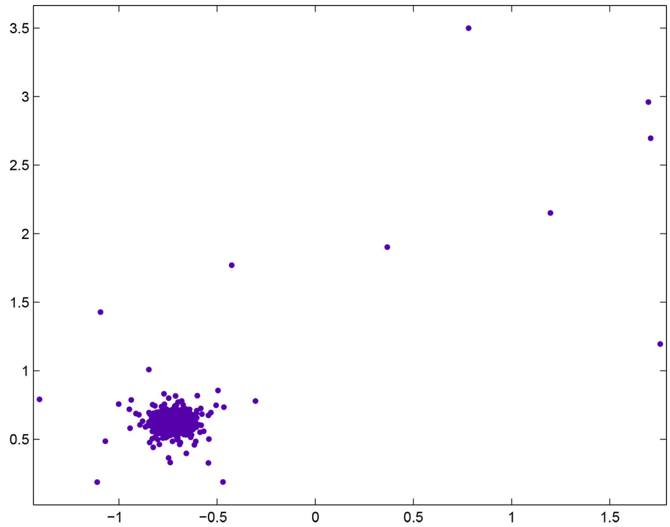

Figure 12

A plot of these $\zeta'(s)$ values where the first 50 points are omitted is

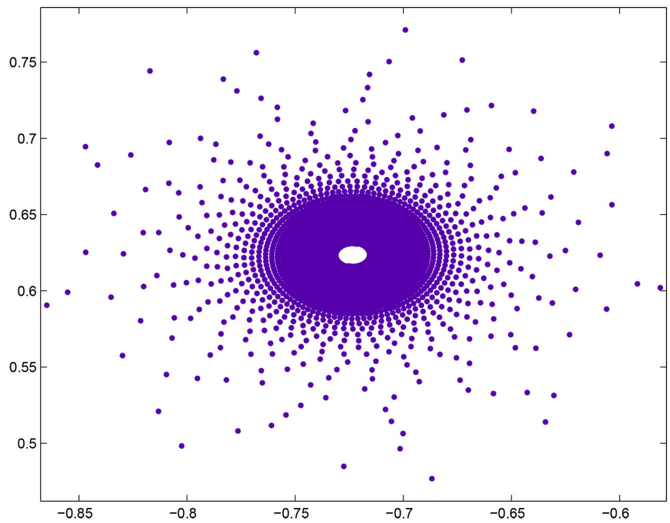

Figure 13

This converging logarithmic spiral is centered on about (0.619, -0.72) (instead of (0, 0) for zeta function zeros). The curve is translated to the x-axis by subtracting 0.619

from the real part. The n values of the inflection points that are at least three greater than the previous n value can then be determined (for $512 \leq n \leq 70000$). A plot of the logarithms of the n values of the current inflection points and the logarithms of the differences between the n values of the current and previous inflection points is

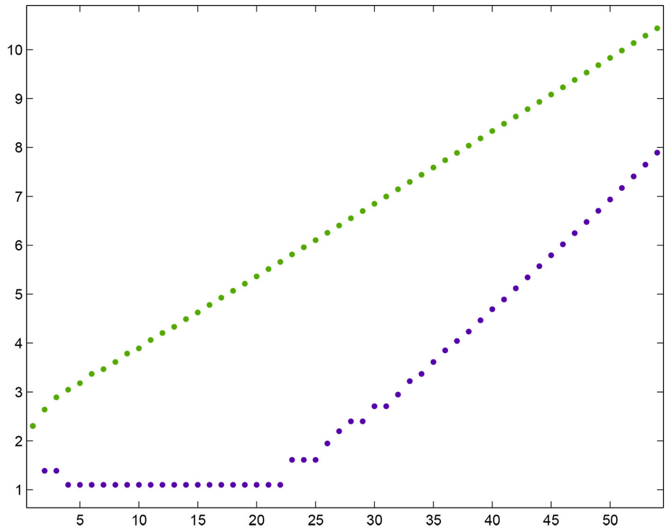

Figure 14

For a linear least-squares fit of the logarithms of the n values of the current inflection points (neglecting the first six points), $p_1 = 0.1483$ with a 95% confidence interval of (0.1481, 0.1486), $p_2 = 2.408$ with a 95% confidence interval of (2.399, 2.416), SSE=0.006285, R-squared=1, and RMSE=0.01169.

3. A DYNAMICAL ZETA FUNCTION

Dynamical zeta functions are generating functions for the lengths of closed orbits of a map f that sends a set M to itself. An example is the map $x \rightarrow 1 - \mu x^2$ of the interval $[-1, 1]$ to itself. For a special value of $\mu \approx 1.40155...$ (the Feiningenbaum value), this map has one periodic order of 2^n for every integer $n \geq 0$. This ζ then satisfies the functional equation $\zeta(z^2) = (1 - z)\zeta(z)$. See Kargin [1] for more details on dynamical zeta functions.

4. EXPONENTIALLY WEIGHTED DIRICHLET SERIES

A Dirichlet series with exponential terms is

$$D(s) = \sum_{k=1}^{\infty} e^{-ks} \tag{5}$$

where $s = (a, b)$. For $\Re(s) > 0$, the series converges to $e^{-s}/(1 - e^{-s})$. The real part can be expressed as $\sum_{k=1}^{\infty} e^{-ka} \cos(kb)$ and the imaginary part can be expressed as $\sum_{k=1}^{\infty} e^{-ka} \sin(kb)$. A partial sum of this function will be denoted by $D(n, a, b)$. See the Methods section for the C code for generating a variant dynamical zeta function from the v'_n values using this transformation.

A plot of the imaginary parts of $D(n, a, b)$ versus the real parts for $a = \frac{1}{2}$ and b the first thousand v'_n values is

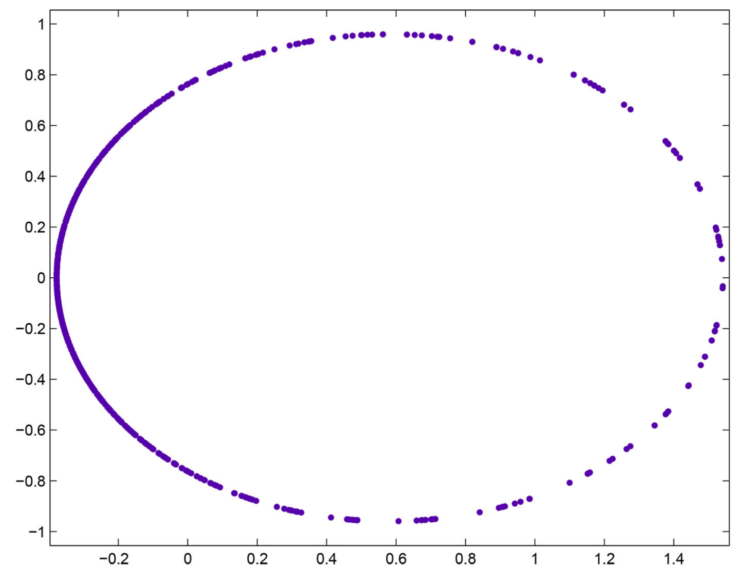

Figure 15

5. VARIANT DYNAMICAL ZETA FUNCTIONS

The variant dynamical zeta functions corresponding to $\Re s = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ satisfy functional relationships similar to $\zeta(z^2) = (1 - z)\zeta(z)$. The real parts of the z values are set to the reciprocals of $\Re s = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ and the imaginary parts are set to the imaginary parts of the corresponding $D(n, a, b)$ values.

For the linear least-squares fits of the real parts of $\zeta(z^2)$ versus the real parts of $(1 - z)\zeta(z)$, the slopes are 1. The y-intercepts are $\Re z \cdot (\Re z * 2 - 1)$. For the linear least-squares fits of the imaginary parts of $\zeta(z^2)$ versus the imaginary parts of $(1 - z)\zeta(z)$, the slopes are $2 \cdot \Re z$. The imaginary parts are almost 0. For a quadratic least-squares fit of the real parts of $\zeta(z^2)$ versus the imaginary parts, the curve is an upside-down parabola with parameters of -1, 0 (almost), and $-\Re z \cdot (\Re z - 1)$. For a quadratic least-squares fit of the real parts of $(1 - z) \cdot \zeta(z)$ versus the imaginary parts, the curve is an upside-down parabola with parameters of $-(\frac{1}{2 \cdot \Re z})^2$, 0 (almost), and $\Re z^2$.

These values have normal probability distributions. The means and standard deviations of the normal probability fits determine the above relationships. For example, for $\Re(s) = \frac{1}{2}$ and the first thousand v_n' values, the mean of the normal probability fit of the real parts of $(1-z)\zeta(z)$ is -2.2885 with a 95% confidence interval of (-2.3072, -2.2697) and a standard deviation of 0.3019 with a 95% confidence interval of (0.2892, 0.3158). For a normal probablity fit of the real parts of $\zeta(z^2)$, the mean is 3.7155 with a 95% confidence interval of (3.6928, 3.7303) and a standard deviation of 0.3019 with a 95% confidence interval of (0.2892, 0.3158). The slope of 1 in the above is due to the standard deviations being equal. The y-intercept (6) equals the difference in means (3.7115+2.2885).

For a normal probability fit of the imaginary parts of $(1-z)\zeta(s)$, the mean is -0.01225 with a 95% confidence interval of (-0.0455, 0.0211) and a standard deviation of 0.5372 with a 95% confidence interval of (0.5147, 0.5619). For a normal probability fit of the imaginary parts of $\zeta(z^2)$, the mean is -0.0488 with a 95% confidence interval of (-0.1882, 0.0845) and a standard deviation of 2.1489 with a 95% confidence interval of (2.05863, 2.2472). The slope of 4 is due to the ratio of standard deviations (2.1489/0.5372) being almost 4.

6. ANALOGUE OF EULER'S PRODUCT FORMULA FOR THE RIEMANN ZETA FUNCTION

If f is a flow on M, that is, a map $MxR^{+} \to M$, then the zeta function of this flow is defined as

$$\zeta(s) = \prod_{\omega}(1 - e^{-sl(\omega)})^{-1} \tag{6}$$

where ω denotes a periodic orbit of f and $l(\omega)$ is its length. If it is assumed that prime numbers correspond to periodic orbits of a flow and the lengths of the orbit indexed by p is given by $\log p$, then the zeta function of the flow will be similar to Riemann's zeta function. See the Methods section for C code that computes the above equation with the v_n' values as input. A plot of the real and imaginary parts of the resulting values versus the imaginary parts of $D(n, a, b)$ for the first 1000 v'_n values where $a = 1/2$ is

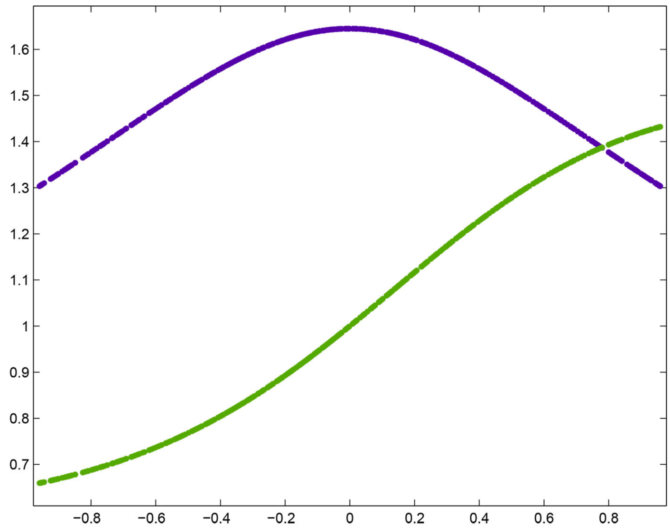

Figure 16

The curves have good quartic least-squares fits. These are small portions of sinusoidal curves.

A plot of the real and imaginary parts of the resulting values versus the imaginary parts of $D(n, a, b)$ for the first 1000 v'_n values where $a = 1/10$ is

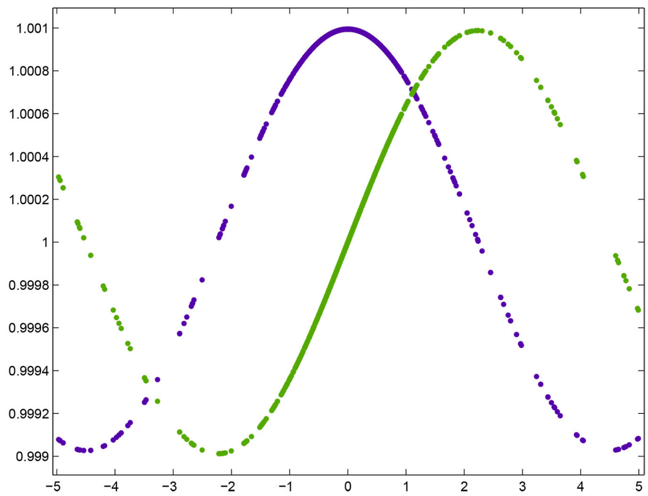

Figure 17

The curves are now good approximations of the sine and cosine functions.

The Euler product formula for the Riemann zeta function is

$$\zeta(s) = \prod_p \frac{1}{1-p^{-s}}, (\Re s > 1) \tag{7}$$

7. THE RIEMANN HYPOTHESIS AND LOGARITHMICALLY WEIGHTED DIRICHLET SERIES

In this section, the real part of $(1-z)\zeta(z)$ is input to the imaginary part of a logarithmically-weighted Dirichlet series and the real part of $\zeta(z^2)$ is input into the imaginary part of a logarithmically-weighted Dirichlet series (see the C code in the Methods section). The real parts of the Dirichlet series are set to $R(z)$. This gives another dynamical zeta function denoted by ζ_1. A plot of the real part of $\zeta_1(z^2)$ versus the real part of $(1-z)\zeta_1(z)$, the imaginary part of $\zeta_1(z^2)$ versus the imaginary part of $(1-z)\zeta_1(z)$, the real part of $(1-z)\zeta_1(z)$ versus the imaginary part, and the real part of $\zeta_1(z^2)$ versus the imaginary part where $\Re(z) = 1.5$ and $n \leq 2000$ is

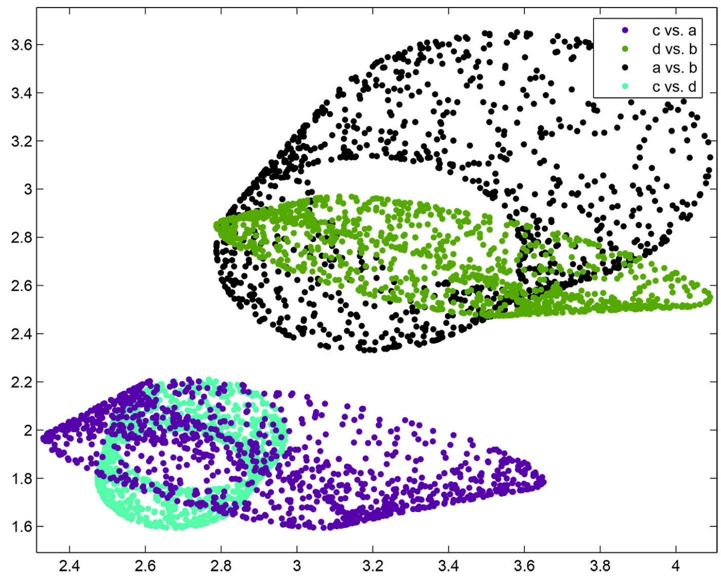

Figure 18

The real and imaginary parts of $\zeta_1(z^2)$ are normally distributed and the real and imaginary parts of $(1-z)\zeta_1(z)$ are normally distributed. In this plot, $\Re((1-z)\zeta_1(z))$, $\Im((1-z)\zeta_1(z))$, $\Re(\zeta_1(z^2))$, and $\Im(\zeta_1(z^2))$ are denoted by a, b, c, and d respectively.

A plot of these values where $\Re(z) = 1.999$ and $n \leq 2000$ is

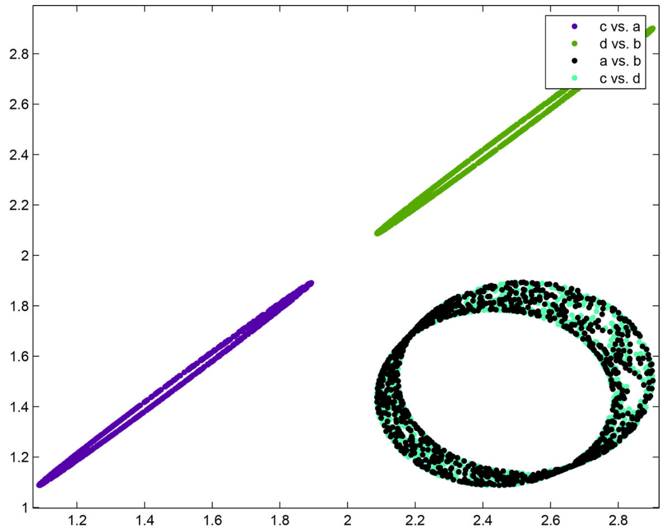

Figure 19

A plot of these values where $\Re(z) = 2.0$ and $n =\leq 2000$ is

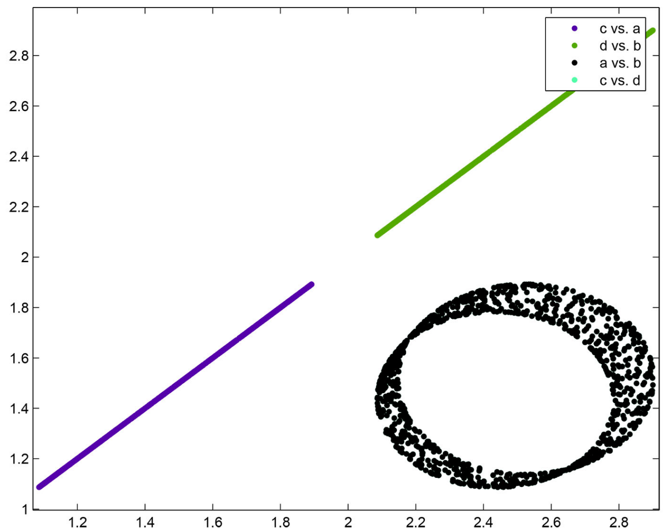

Figure 20

Taking into account the precision of the floating point arithmetic and truncation of infinite sums, the linear least-squares fits of the real part of $\zeta_1(z^2)$ versus the real part of $(1-z)\zeta_1(z)$ and the imaginary part of $\zeta_1(z^2)$ versus the imaginary part of $(1-z)\zeta_1(z)$ are apparently perfect and the slopes are 1. In this case, $\zeta_1(z^2) = (1-z)\zeta_1(z)$. Also, the curve of the real part of $\zeta_1(z^2)$ versus the imaginary part and the curve of the real part of $(1 - z)\zeta_1(z)$ versus the imaginary part overlap.

A plot of these values where $\Re(z) = 2.5$ and $n \le 2000$ is

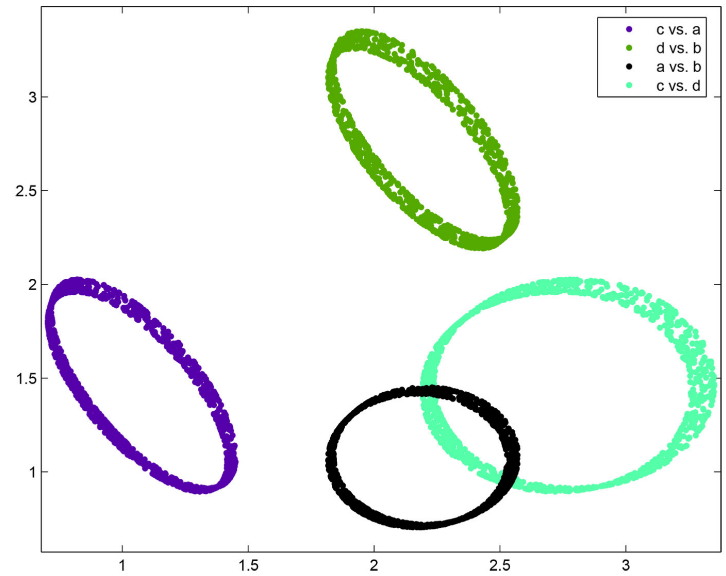

Figure 21

8. METHODS

variant dynamical zeta function

```
#include <math.h>
#include <stdio.h>
#include "v2.h" // v'(n) values
double a=1.5; // a>1 if out=1
unsigned int size=10000;
unsigned int max=50001;
unsigned int part=1;
unsigned int select=1;
unsigned int out=1;
void main() {
unsigned int x,i;
double esumr,esumi,tempr,ap,b,y,p,q,c,d,sum,out1,out2,out3,out4,bp;
FILE *Outfp;
Outfp = fopen("weight1a.dat","w");
ap=1.0/a;
y=1.0-ap;
```

```
for (i=1; <=size; i++) {
    b=zero[i-1];
    esumr=0.0;
    esumi=0.0;
    for (x=1; x<=(max-1); x++) {
        tempr=1.0/exp((double)x*a);
        esumr=esumr+tempr*cos((double)x*b);
        esumi=esumi+tempr*sin((double)x*b);
        }
    if (part==1)
        sum=esumi;
    else
        sum=esumr;
    c=ap*y-sum*sum;
    d=ap*sum+sum*y;
    p=ap*ap-sum*sum;
    q=2*ap*sum;
    if (out==0) {
        fprintf(Outfp," %.16llf, %.16llf, %.16llf, %.16llf \n",c,d,p,q);
        printf(" %d %.10llf, %.10llf, %.10llf, %.10llf \n",i,c,d,p,q);
        continue;
        }
    if (select==1)
        bp=c;
    else
        bp=d;
    esumr=0.0;
    esumi=0.0;
    for (x=1; x<=(max-1); x++) {
        tempr=1.0/log((double)x*a);
        esumr=esumr+tempr*cos((double)x*bp);
        esumi=esumi+tempr*sin((double)x*bp);
        }
    out1=esumr;
    out2=esumi;
    if (select==1)
        bp=p;
    else
```

```
        bp=q;
    esumr=0.0;
    esumi=0.0;
    for (x=1; x<=(max-1); x++) {
        tempr=1.0/log((double)x*a);
        esumr=esumr+tempr*cos((double)x*bp);
        esumi=esumi+tempr*sin((double)x*bp);
        }
    out3=esumr;
    out4=esumi;
    fprintf(Outfp,” %.16llf, %.16llf, %.16llf, %.16llf, \n”,out1,out2,out3,out4);
    if (i==(i/100)*100)
        printf(” %d, %.10llf, %.10llf, %.10llf, %.10llf \n”,i,out1,out2,out3,out4);
    }
fclose(Outfp);
return;
}
```

analogue of Euler's product formula

```
#include <math.h>
#include <stdio.h>
#include ”table5.h”
#include ”v2.h”
double a=1.0/2.0;
unsigned int size=10000;
unsigned int part=1;
unsigned int max=50001;
void main() {
unsigned int x,i,p;
double esumr,esumi,tempr,b,prodr,prodi,ap;
FILE *Outfp;
Outfp = fopen(”weight2.dat”,”w”);
ap=1.0/a;
for (i=1; i<=size; i++) {
    b=zero[i-1];
    esumr=0.0;
    esumi=0.0;
```

```
    for (x=1; x<=(max-1); x++) {
        tempr=1.0/exp((double)x*a);
        esumr=esumr+tempr*cos((double)x*b);
        esumi=esumi+tempr*sin((double)x*b);
        }
    if (part==1)
        b=esumi;
    else
        b=esumr;
    prodr=1.0;
    prodi=1.0;
    for (x=1; x<=2000; x++) {
        p=table[x-1];
        tempr=1.0/(1.0-exp(ap*log((double)p)));
        prodr=prodr*(1.0-tempr*cos(log((double)p)*b));
        prodi=prodi*(1.0-tempr*sin(log((double)p)*b));
        }
    fprintf(Outfp," %.16llf, %.16llf, %.16llf, \n",b,prodr,prodi);
    printf(" %d %.10llf, %.10llf, %.10llf, \n",i,b,prodr,prodi);
    }
fclose(Outfp);
return;
}
```

second dynamical zeta function

```
#include <math.h>
#include <stdio.h>
#include "v2.h"
double a=2.0; // a>1
unsigned int size=2000;
unsigned int max=50001;
unsigned int part=1; // usually set to 1
unsigned int select=1; // usually set to 1
unsigned int out=1;
void main() {
unsigned int x,i;
double esumr,esumi,tempr,ap,b,y,p,q,c,d,sum,out1,out2,out3,out4,bp;
```

```
FILE *Outfp;
Outfp = fopen(”weight1a.dat”,”w”);
ap=1.0/a;
y=1.0-ap;
for (i=1; i<=size; i++) {
    b=zero[i-1];
    esumr=0.0;
    esumi=0.0;
    for (x=1; x<=(max-1); x++) {
        tempr=1.0/exp((double)x*a);
        esumr=esumr+tempr*cos((double)x*b);
            esumi=esumi+tempr*sin((double)x*b);
            }
    if (part==1)
        sum=esumi;
    else
        sum=esumr;
    c=ap*y-sum*sum;
    d=ap*sum+sum*y;
    p=ap*ap-sum*sum;
    q=2*ap*sum;
    if (out==0) {
        fprintf(Outfp,” %.16llf, %.16llf, %.16llf, %.16llf \n”,c,d,p,q);
        if (i==(i/100)*100)
            printf(” %d %.12llf, %.12llf, %.12llf, %.12llf\n”,i,c,d,p,q);
        continue;
        }
    if (select==1)
        bp=c;
    else
        bp=d;
    esumr=0.0;
    esumi=0.0;
    for (x=1; x<=(max-1); x++) {
        tempr=1.0/log((double)x*a);
        esumr=esumr+tempr*cos((double)x*bp);
        esumi=esumi+tempr*sin((double)x*bp);
        }
```

```
    out1=esumr;
    out2=esumi;
    if (select==1)
        bp=p;
    else
        bp=q;
    esumr=0.0;
    esumi=0.0;
    for (x=1; x<=(max-1); x++) {
        tempr=1.0/log((double)x*a);
        esumr=esumr+tempr*cos((double)x*bp);
        esumi=esumi+tempr*sin((double)x*bp);
        }
    out3=esumr;
    out4=esumi;
    fprintf(Outfp," %.16llf, %.16llf, %.16llf, %.16llf, \n",out1,out2,out3,out4);
    if (i==(i/100)*100)
        printf(" %d, %.14llf, %.14llf, %.14llf, %.14llf \n",i,out1,out2,out3,out4);
    }
fclose(Outfp);
return;
}
```

REFERENCES

[1] Vladislav Kargin, Statistical properties of zeta functions' zeros, Probability Surveys, Vol. 11 (2014) 121-160, ISSN: 1549-5787