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Vector Basis {(1,1,1,1),(1,1,1,0),(1,1,0,0),(1,0,0,0)}-Cordial Labeling of Generalized Friendship Graph, Tadpole Graph and Gear Graph

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Abstract

Let G be a (p, q) graph. Let V be an inner product space with basis S. We denote the inner product of the vectors ω_1 and ω_2 by $< \omega_1, \omega_2 >$. Let $\chi : V(G) \to S$ be a function. For edge uv assign the label $< \chi(u), \chi(v) >$. Then χ is called a vector basis S-cordial labeling of G (VB S-cordial labeling) if $|\chi_{\omega_1} - \chi_{\omega_2}| \leq 1$ and $|\delta_i - \delta_j| \leq 1$ where χ_{ω_i} denotes the number of vertices labeled with the vector ω_i and δ_i denotes the number of edges labeled with the scalar i. A graph which admits a vector basis S-cordial labeling is called a vector basis S-cordial graph. In this paper, we investigate the vector basis {(1,1,1,1),(1,1,1,0),(1,1,0,0),(1,0,0,0)}-cordial labeling of certain graphs such as the generalized friendship graph, tadpole graph, gear graph, $C_{m,n}$, alternate triangular snake and alternate quadrilateral snake.
Keywords. friendship graph, tadpole graph, gear graph, alternate triangular snake and alternate quadrilateral snake.
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1. INTRODUCTION

We consider only finite, simple and undirected graphs. In 1967, Rosa introduced the graph labeling of a graph. For all standard terminologies and notations, we refer to Harary [6] and Herstein [7]. We provide brief summary and basic definitions that are relevant to the current investigations

Definition 1.1. *[13] The generalized friendship graph $F_{n,m}$ is a graph of m cycles (all of order n) meeting at a common vertex.*

Definition 1.2. *[15] The tadpole graph $T_{m,n}$ is a graph in which the path P_n is attached to any one vertex of the cycle C_m.*

Definition 1.3. *[4] The gear graph G_n is obtained from the wheel W_n by adding a vertex between every pair of adjacent vertices of the rim of the wheel graph W_n.*

Definition 1.4. *[3] Let C_m and C_n be two even cycles where m and n are even integers. Then the graph $C_{m,n}$ is a graph obtained by sharing a common edge of C_m and C_n.*

Definition 1.5. *[11] The triangular snake T_n is obtained from a path $P_n : u_1u_2 \dots u_n$ by joining u_i and u_{i+1} to a new vertex v_i for $1 \leq i \leq n-1$. That is every edge of a path is replaced by a triangle.*

Definition 1.6. *[11] An alternate triangular snake AT_n is obtained from a path $P_n : u_1u_2 \dots u_n$ by joining u_i and u_{i+1} alternatively ($i = 1, 3, 5, \dots$) to a new vertex v_i. That is every alternate edge of a path is replaced by a triangle.*

Definition 1.7. *[12] The quadrilateral snake Q_n is obtained from a path $P_n : u_1u_2 \dots u_n$ by joining u_i and u_{i+1} to two new vertex v_i and w_i for $1 \leq i \leq n-1$ respectively and then joining v_i and w_i. That is every edge of a path is replaced by C_4.*

Definition 1.8. *[12] The alternate quadrilateral snake AQ_n is obtained from a path $P_n : u_1u_2 \dots u_n$ by joining u_i and u_{i+1} alternatively ($i = 1, 3, 5, \dots$) to two new vertex v_i and w_i for $1 \leq i \leq n-1$ respectively and then joining v_i and w_i. That is every alternate edge of a path is replaced by C_4.*

The notion of cordial labeling of a graph was introduced in [2]. Ansari Saima [1] has investigated the mean cordial labeling patterns in shadow graphs of paths. Pair mean cordial labeling of some snake related graphs was discussed in [10]. Group mean cordial labeling of some triangular snake and quadrilateral snake related graphs have investigated by Rajalekshmi and Kala [11, 12].
Sugumaran and Mohan have investigated the difference cordial labeling of some special graphs and related to fan graphs in [14]. The 3-product edge cordial labeling of tadpole, book and flower graphs was examined in [15]. HMC labeling of some triangular snake graphs was discussed by Gowri and Jayapriya [5].
Tribonacci cordial labeling of graphs was introduced by Sarbari Mitra and Soumya Bhoumik in [13]. Jeba Jesintha et al. [8] have proved that the tadpole graph attached to k-polygonal snakes, double polygonal snakes and alternate k-polygonal snakes by an

edge are cordial. Product cordial labeling of graphs related to helm, closed helm and gear graph were explored in [4]. For a dynamic survey of various graphs labeling along with bibliographic references we refer to Gallian [3].
Ponraj and Jeya have been introduced the new graph labeling method called vector basis S-cordial labeling and investigated the vector basis $\{$(1,1,1,1), (1,1,1,0), (1,1,0,0),(1,0,0,0)$\}$ -cordial labeling of certain thorn graphs in [9]. In this paper, we examines the vector basis $\{$(1,1,1,1),(1,1,1,0),(1,1,0,0),(1,0,0,0)$\}$-cordial labeling of certain graphs such as the generalized friendship graph, tadpole graph, gear graph, $C_{m,n}$, alternate triangular snake and alternate quadrilateral snake.

2. VECTOR BASIS S-CORDIAL LABELING

Let G be a (p, q) graph. Let V be an inner product space with basis S. We denote the inner product of the vectors ω_1 and ω_2 by $< \omega_1, \omega_2 >$. Let $\chi : V(G) \to S$ be a function. For edge uv assign the label $< \chi(u), \chi(v) >$. Then χ is called a vector basis S-cordial labeling of G (VB S-cordial labeling) if $|\chi_{\omega_1} - \chi_{\omega_2}| \leq 1$ and $|\delta_i - \delta_j| \leq 1$ where χ_{ω_i} denotes the number of vertices labeled with the vector ω_i and δ_i denotes the number of edges labeled with the scalar i. A graph which admits a vector basis S-cordial labeling is called a vector basis S-cordial graph. The figure (1) shows a simple example of a vector basis $\{$(1,1,1,1),(1,1,1,0),(1,1,0,0),(1,0,0,0)$\}$-cordial graph.

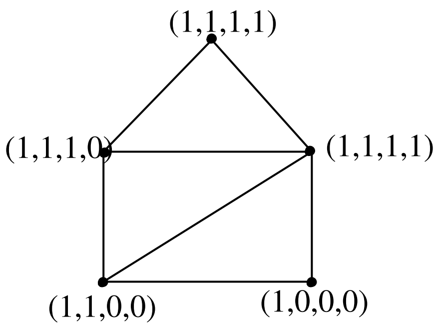

Figure 1. An example of vector basis $\{$(1,1,1,1),(1,1,1,0),(1,1,0,0),(1,0,0,0)$\}$-cordial graph

3. MAIN RESULTS

In this paper, we investigate the vector basis $\{$(1,1,1,1),(1,1,1,0),(1,1,0,0),(1,0,0,0)$\}$-cordial labeling of some special graphs like the generalized friendship graph, tadpole graph, gear graph, $C_{m,n}$, alternate triangular snake and alternate quadrilateral snake.

Theorem 3.1. *The generalized friendship graph $F_{n,m}$ is a vector basis $\{$(1,1,1,1), (1,1,1,0),(1,1,0,0),(1,0,0,0)$\}$-cordial for all n and $m \equiv 0 \pmod 4$.*

Proof. Let $V(F_{n,m}) = \{u, u(i,j) \mid 1 \leq i \leq m \,\text{and}\, 1 \leq j \leq n-1\}$ and $E(F_{n,m}) = uu_{i,1}, uu_{i,n-1}, u_{i,j}u_{i,j+1} \mid 1 \leq i \leq m \,\text{and}\, 1 \leq i \leq n-2$ respectively be the vertex and edge sets of $F_{n,m}$.Then $|V(F_{n,m})| = p = m(n-1)+1$ and $|E(F_{n,m})| = q = mn$. Assign the vectors to the vertices $u, u_{1,1}, u_{1,2}, \ldots, u_{1,n-1}, u_{2,1}, u_{2,2}, \ldots, u_{2,n-1}, \ldots, u_{m,1}, u_{m,2}, \ldots, u_{m,n-1}$ in some order. Let $m \equiv 0 \pmod 4$. Then $m = 4t$. Assign the vector (1,1,1,1) to the vertex u. Assign the vector (1,1,1,1) to the vertices of the first t cycles. Also assign the vector (1,1,1,0) to the vertices of the next t cycles. Then assign the vector (1,1,0,0) to the vertices of the next t cycles. Finally assign the vector (1,0,0,0) to the vertices of remaining t cycles. Clearly the above labeling pattern provides a vector basis {(1,1,1,1),(1,1,1,0),(1,1,0,0),(1,0,0,0)}-cordial labeling for the generalized friendship graph $F_{n,m}$ if $m \equiv 0 \pmod 4$. □

Example 3.1. *The figure (2) shows a vector basis {(1,1,1,1),(1,1,1,0),(1,1,0, 0),(1,0,0,0)}-cordial labeling of the generalized friendship graph* $F_{5,4}$.

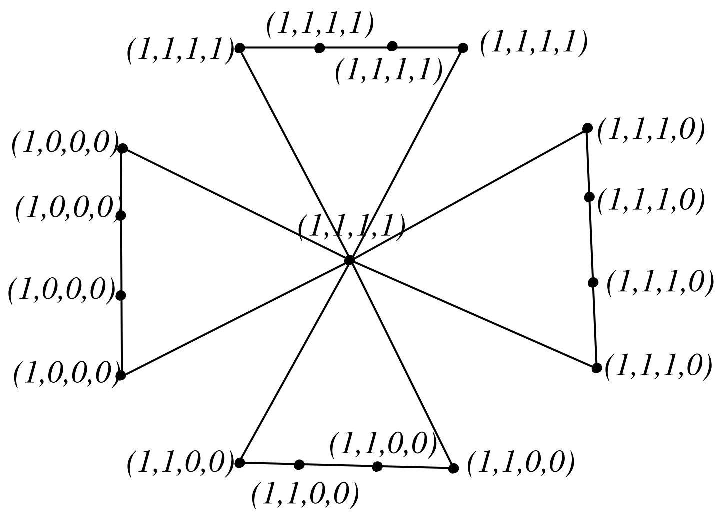

Figure 2. *A vector basis {(1,1,1,1),(1,1,1,0),(1,1,0,0),(1,0,0,0)}-cordial labeling of* $F_{5,4}$.

Theorem 3.2. *The generalized friendship graph* $F_{n,m}$ *is a vector basis {(1,1,1,1), (1,1,1,0),(1,1,0,0),(1,0,0,0)}-cordial for all* $m \equiv 1 \pmod 4$ *and* $n \equiv 1, 2, 3 (mod 4)$.

Proof. Let us assign labels to the following vertices $u, u_{1,1}, u_{1,2}, \ldots, u_{1,n-1}, u_{2,1}, u_{2,2}, \ldots, u_{2,n-1}, \ldots, u_{m,1}, u_{m,2}, \ldots, u_{m,n-1}$ in that order. Let $m \equiv 1 \pmod 4$. Then $m = 4t + 1$. Assign the label as in Theorem 3.1. for first $4t$ cycles. Next assign the label to the last cycle. There are three cases arises.

Case (i):$n \equiv 1 \pmod 4$

Then $n = 4k + 1$. Assign vector (1,1,1,1) to the k vertices (except u). So assign the vector (1,1,1,0) to the next k vertices. Then assign the vector (1,1,0,0) to the next k vertices. Further assign the vector (1,0,0,0) to the next k vertices.
Case (ii):$n \equiv 2 \pmod 4$
Take $n = 4k + 2$. Assign vector (1,1,1,1) to the first k vertices (except u) and assign the vector (1,1,1,0) to the next $k + 1$ vertices. More over assign the vector (1,1,0,0) to the next k vertices and assign the vector (1,0,0,0) to the next k vertices.
Case (iii):$n \equiv 3 \pmod 4$
Let $n = 4k + 3$. Assign vector (1,1,1,1) to the first k vertices (except u) and assign the vector (1,1,1,0) to the next $k + 1$ vertices. More over assign the vector (1,1,0,0) to the next $k+1$ vertices and assign the vector (1,0,0,0) to the next k vertices. Hence the above labeling pattern provides a vector basis {(1,1,1,1),(1,1,1,0),(1,1,0,0),(1,0,0,0)}-cordial labeling for the generalized friendship graph $F_{n,m}$ if $m \equiv 1 \pmod 4$ and $n \equiv 1, 2, 3 \pmod 4$. □

Theorem 3.3. *The generalized friendship graph $F_{n,m}$ is a vector basis {(1,1,1,1), (1,1,1,0),(1,1,0,0),(1,0,0,0)}-cordial for all $m \equiv 2 \pmod 4$ and n is odd.*

Proof. Now assign the vector to the following vertices $u, u_{1,1}, u_{1,2}, \ldots, u_{1,n-1}, u_{2,1}, u_{2,2}, \ldots, u_{2,n-1}, \ldots, u_{m,1}, u_{m,2}, \ldots, u_{m,n-1}$ in that order. Let $m \equiv 2 \pmod 4$. Then $m = 4t+2$. Assign the label as in case (i) of Theorem 3.1 for first $4t$ cycles. Next assign the label to the remaining two cycles. Let n is odd. There are two cases arises.
Case (i):$n \equiv 1 \pmod 4$
Then $n = 4k + 1$. Note that $p = 8k$ and $q = 8k + 2$. Let us assign the vector (1,1,1,1) to the first $2k$ vertices (except u) and assign the vector (1,1,1,0) to the next $2k$ vertices. Thereafter assign the vector (1,1,0,0) to the next $2k$ vertices and assign the vector (1,0,0,0) to the next $2k$ vertices.
Case (ii):$n \equiv 3 \pmod 4$
Let $n = 4k + 3$. Note that $p = 8k + 4$ and $q = 8k + 6$. Assign vector (1,1,1,1) to the first $2k+1$ vertices (except u) and assign the vector (1,1,1,0) to the next $2k+1$ vertices. More over assign the vector (1,1,0,0) to the next $2k + 1$ vertices and assign the vector (1,0,0,0) to the next $2k + 1$ vertices.
Hence the above labeling pattern provides a vector basis {(1,1,1,1),(1,1,1,0),(1,1, 0,0), (1,0,0,0)}-cordial labeling for the generalized friendship graph $F_{n,m}$ if $m \equiv 2 \pmod 4$ and n is odd. □

Theorem 3.4. *The generalized friendship graph $F_{n,m}$ is a vector basis {(1,1,1,1), (1,1,1,0),(1,1,0,0),(1,0,0,0)}-cordial for all $m \equiv 3 \pmod 4$ and $n \equiv 1 \pmod 4$.*

Proof. Let us now assign the vector to the following vertices $u, u_{1,1}, u_{1,2}, \ldots, u_{1,n-1}$, $u_{2,1}, u_{2,2}, \ldots, u_{2,n-1}, \ldots, u_{m,1}, u_{m,2}, \ldots, u_{m,n-1}$ in that order. Let $m \equiv 3 \pmod 4$. Then $m = 4t+3$. Assign the label as in case (i) of Theorem 3.1 for first $4t$ cycles. Next assign the label to the remaining three cycles. Take $n \equiv 1 \pmod 4$. Then $n = 4k+1$. Note that $p = 12k$ and $q = 12k+3$. Let us assign the vector (1,1,1,1) to the first $3k$ vertices (excluding u) and assign the vector (1,1,1,0) to the next $3k$ vertices. Thereafter assign the vector (1,1,0,0) to the next $3k$ vertices and assign the vector (1,0,0,0) to the next $3k$ vertices.
Thus the above labeling pattern provides a vector basis {(1,1,1,1),(1,1,1,0),(1,1,0 ,0), (1,0,0,0)}-cordial labeling for the generalized friendship graph $F_{n,m}$ if $m \equiv 3 \pmod 4$ and $n \equiv 1 \pmod 4$. □

Theorem 3.5. *The tadpole graph $T_{m,n}$, $m > 3$ is a vector basis {(1,1,1,1),(1,1,1, 0),(1,1,0,0),(1,0,0,0)}-cordial if $m \equiv 0 \pmod 4$ and $n \equiv 0, 2, 3 \pmod 4$.*

Proof. Let $V(T_{m,n}) = \{u, u_i, v_j \mid 1 \leq i \leq m \,\text{and}\, 2 \leq j \leq n\}$ and $E(T_{m,n}) = \{u_i u_(i+1), u_1 u_m, u_1 v_2, v_j v_(j+1) \mid 1 \leq i \leq m \,\text{and}\, 2 \leq j \leq n\}$ respectively be the vertex and edge sets of the tadpole graph $T_{m,n}$. Then $|V(T_{m,n})| = p = m+n-1$ and $|E(T_{m,n})| = q = m+n-1$. Assign the vectors in the following order $u_1, u_2, \ldots, u_m, v_2, v_3, \ldots, v_n$. Let $m \equiv 0 \pmod 4$. Then $m = 4t_1$. For the vertices in the cycles $m = 4t_1$. There are m vertices in the cycle. Assign the vector (1,1,1,1) to the first t_1 vertices and assign the vector (1,1,1,0) to the next t_1 vertices. Thereafter assign the vector (1,1,0,0) to the next t_1 vertices and assign the vector (1,0,0,0) to the next t_1 vertices.
Case (i): $n \equiv 0 \pmod 4$
For the vertices in the path $n = 4t_2$. There are $n-1$ vertices in the path. Assign the vector (1,1,1,1) to the first t_2 vertices and assign the vector (1,1,1,0) to the next t_2 vertices. Thereafter assign the vector (1,1,0,0) to the next t_2 vertices and assign the vector (1,0,0,0) to the next $t_2 - 1$ vertices.
Case (ii):$n \equiv 2 \pmod 4$
Then $n = 4t_2+2$. Let us assign the vector (1,1,1,1) to the first t_2+1 vertices and assign the vector (1,1,1,0) to the next t_2 vertices. Further assign the vector (1,1,0,0) to the next t_2 vertices and assign the vector (1,0,0,0) to the next t_2 vertices.
Case (iii):$n \equiv 3 \pmod 4$
Then $n = 4t_2+3$. Now assign the vector (1,1,1,1) to the first t_2+1 vertices and assign the vector (1,1,1,0) to the next t_2+1 vertices. More over assign the vector (1,1,0,0) to the next t_2 vertices and assign the vector (1,0,0,0) to the next t_2 vertices.
Thus the above labeling pattern provides a vector basis {(1,1,1,1),(1,1,1,0),(1,1,0 ,0), (1,0,0,0)}-cordial labeling for the tadpole graph $T_{m,n}$ if $m \equiv 0 \pmod 4$ and $n \equiv 0, 2, 3 \pmod 4$. □

Theorem 3.6. *The tadpole graph $T_{m,n}$, $m > 3$ is a vector basis {(1,1,1,1),(1,1,1, 0),(1,1, 0,0),(1,0,0,0)}-cordial if $m \equiv 1 \pmod 4$ and $n \equiv 1, 2, 3 \pmod 4$.*

Proof. Let us assign the vectors in the following order $u_1, u_2, \ldots, u_m, v_2, v_3, \ldots, v_n$. Let $m \equiv 1 \pmod 4$. Then $m = 4t_1 + 1$. Assign the vector (1,1,1,1) to the first $t_1 + 1$ vertices and assign the vector (1,1,1,0) to the next t_1 vertices. Thereafter assign the vector (1,1,0,0) to the next t_1 vertices and assign the vector (1,0,0,0) to the next t_1 vertices.

Case (i): $n \equiv 1 \pmod 4$

Then $n = 4t_2 + 1$. Assign the vector (1,1,1,1) to the first t_2 vertices and assign the vector (1,1,1,0) to the next t_2 vertices. Also assign the vector (1,1,0,0) to the next t_2 vertices and assign the vector (1,0,0,0) to the next t_2 vertices.

Case (ii):$n \equiv 2 \pmod 4$

Then $n = 4t_2 + 2$. Now assign the vector (1,1,1,1) to the first t_2 vertices and assign the vector (1,1,1,0) to the next $t_2 + 1$ vertices. Further assign the vector (1,1,0,0) to the next t_2 vertices and assign the vector (1,0,0,0) to the next t_2 vertices.

Case (iii):$n \equiv 3 \pmod 4$

Then $n = 4t_2 + 3$. So assign the vector (1,1,1,1) to the first t_2 vertices and assign the vector (1,1,1,0) to the next $t_2 + 1$ vertices. More over assign the vector (1,1,0,0) to the next $t_2 + 1$ vertices and assign the vector (1,0,0,0) to the next t_2 vertices.

Hence the above labeling method provides a vector basis {(1,1,1,1),(1,1,1,0), (1,1,0,0), (1,0,0,0)}-cordial labeling for the tadpole graph $T_{m,n}$ if $m \equiv 1 \pmod 4$ and $n \equiv 1, 2, 3 \pmod 4$. □

Theorem 3.7. *The tadpole graph $T_{m,n}$, $m > 3$ is a vector basis {(1,1,1,1),(1,1,1, 0),(1,1,0,0),(1,0,0,0)}-cordial if $m \equiv 2 \pmod 4$ and $n \equiv 0, 1, 2 \pmod 4$.*

Proof. Assign the vectors in the following order $u_1, u_2, \ldots, u_m, v_2, v_3, \ldots, v_n$. Let $m \equiv 2 \pmod 4$. Then $m = 4t_1 + 2$. Assign the vector (1,1,1,1) to the first $t_1 + 1$ vertices and assign the vector (1,1,1,0) to the next $t_1 + 1$ vertices. Thereafter assign the vector (1,1,0,0) to the next t_1 vertices and assign the vector (1,0,0,0) to the next t_1 vertices.

Case (i): $n \equiv 0 \pmod 4$

Then $n = 4t_2$. Assign the vector (1,1,1,1) to the first t_2 vertices and assign the vector (1,1,1,0) to the next $t_2 - 1$ vertices. Also assign the vector (1,1,0,0) to the next t_2 vertices and assign the vector (1,0,0,0) to the next t_2 vertices.

Case (ii):$n \equiv 1 \pmod 4$

Then $n = 4t_2 + 1$. Also assign the vector (1,1,1,1) to the first t_2 vertices and assign the vector (1,1,1,0) to the next t_2 vertices. Further assign the vector (1,1,0,0) to the next t_2 vertices and assign the vector (1,0,0,0) to the next t_2 vertices.

Case (iii):$n \equiv 2 \pmod 4$
Then $n = 4t_2 + 2$. So assign the vector (1,1,1,1) to the first t_2 vertices and assign the vector (1,1,1,0) to the next t_2 vertices. More over assign the vector (1,1,0,0) to the next $t_2 + 1$ vertices and assign the vector (1,0,0,0) to the next t_2 vertices.
Clearly the above labeling method provides a vector basis {(1,1,1,1),(1,1,1,0),(1,1,0 ,0),(1,0,0,0)}-cordial labeling for the tadpole graph $T_{m,n}$ if $m \equiv 2 \pmod 4$ and $n \equiv 0, 1, 2 \pmod 4$. □

Theorem 3.8. *The tadpole graph $T_{m,n}$, $m > 3$ is a vector basis {(1,1,1,1),(1,1,1,0), (1,1, 0,0),(1,0,0,0)}-cordial if $m \equiv 3 \pmod 4$ and $n \equiv 0, 1, 3 \pmod 4$.*

Proof. Assign the vectors in the following order $u_1, u_2, \ldots, u_m, v_2, v_3, \ldots, v_n$. Let $m \equiv 3 \pmod 4$. Then $m = 4t_1 + 3$. Assign the vector (1,1,1,1) to the first $t_1 + 1$ vertices and assign the vector (1,1,1,0) to the next $t_1 + 1$ vertices. Thereafter assign the vector (1,1,0,0) to the next $t_1 + 1$ vertices and assign the vector (1,0,0,0) to the next t_1 vertices.
Case (i): $n \equiv 0 \pmod 4$
Then $n = 4t_2$. Assign the vector (1,1,1,1) to the first t_2 vertices and assign the vector (1,1,1,0) to the next t_2 vertices. Also assign the vector (1,1,0,0) to the next $t_2 - 1$ vertices and assign the vector (1,0,0,0) to the next t_2 vertices.
Case (ii):$n \equiv 1 \pmod 4$
Then $n = 4t_2 + 1$. So assign the vector (1,1,1,1) to the first t_2 vertices and assign the vector (1,1,1,0) to the next t_2 vertices. Further assign the vector (1,1,0,0) to the next t_2 vertices and assign the vector (1,0,0,0) to the next t_2 vertices.
Case (iii):$n \equiv 2 \pmod 4$
Then $n = 4t_2 + 2$. So assign the vector (1,1,1,1) to the first $t_2 + 1$ vertices and assign the vector (1,1,1,0) to the next t_2 vertices. More over assign the vector (1,1,0,0) to the next t_2 vertices and assign the vector (1,0,0,0) to the next $t_2 + 1$ vertices.
Clearly the above labeling method provides a vector basis {(1,1,1,1),(1,1,1,0),(1, 1,0,0), (1,0,0,0)}-cordial labeling for the tadpole graph $T_{m,n}$ if $m \equiv 3 \pmod 4$ and $n \equiv 0, 1, 3 \pmod 4$. □

Example 3.2. *The figure (3) shows a vector basis {(1,1,1,1),(1,1,1,0),(1,1,0,0), (1,0,0,0)}-cordial labeling of the tadpole graph $T_{9,9}$.*

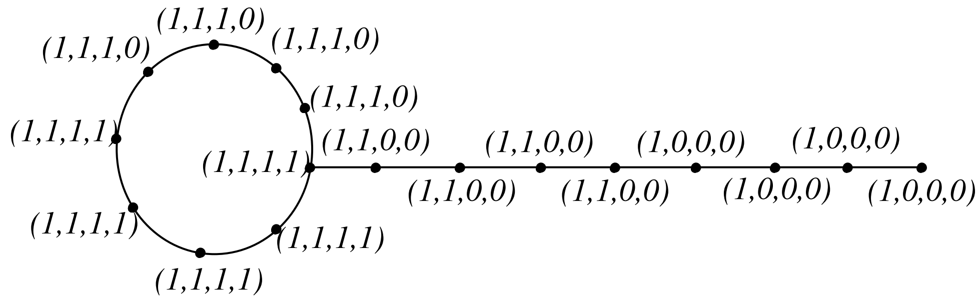

Figure 3. *A vector basis {(1,1,1,1),(1,1,1,0),(1,1,0,0), (1,0,0,0)}-cordial labeling of $T_{9,9}$.*

Theorem 3.9. *The gear graph G_n is a vector basis {(1,1,1,1),(1,1,1, 0),(1,1,0,0), (1,0,0,0)}-cordial if and only if $n \equiv 2 \pmod 4$.*

Proof. Let $V(G_n) = \{u, u_i, v_i \mid 1 \leq i \leq n\}$ and $E(G_n) = \{uu_i, u_iv_i \mid 1 \leq i \leq n\} \cup \{v_iu_{i+1}, v_nu_1 \mid 1 \leq i \leq n-1\}$ respectively be the vertex and edge sets of the gear graph G_n. Then $|V(G_n)| = p = 2n+1$ and $|E(G_n)| = q = 3n$. Assign the vectors to the vertices in the following order $u, u_1, v_1, u_2, v_2, \ldots, u_n, v_n$.

Case (i):$n \equiv 0 \pmod 4$

Let $n = 4t$. Then $p = 8t+1$ and $q = 12t$. From $2t+1$ vertices with vertex label (1,1,1,1), we get only $2t+1$ edges with edge label 4, this is a contradiction.

Case (ii):$n \equiv 1 \pmod 4$

Let n=4t+1. Then p=8t+3 and q=12t+3. From 2t+1 vertices with vertex label (1,1,1,1), we get only 2t+1 edges with edge label 4, this is a contradiction.

Case (ii):$n \equiv 2 \pmod 4$

Let n=4t+2. Then p=8t+5 and q=12t+6. Assign the vector (1,1,1,1) to the vertex u. Then assign the vector (1,1,1,1) to the first 2t+1 vertices and assign the vector (1,1,1,0) to the next 2t+1 vertices. Finally assign the vector (1,1,0,0) to the next 2t+1 vertices and assign the vector (1,0,0,0) to the next 2t+1 vertices.

Case (iv): $n \equiv 3 \pmod 4$

Let $n = 4t+3$. Then $p = 8t+7$ and $q = 12t+9$. From $2t+2$ vertices with vertex label (1,1,1,1), we get only $3t+1$ edges with edge label 4, we get a contradiction.

Hence the above labeling method provides a vector basis {(1,1,1,1),(1,1,1,0),(1,1, 0,0), (1,0,0,0)}-cordial labeling for the gear graph G_n if $n \equiv 2 \pmod 4$. $\square$

Example 3.3. *The figure (4) shows a vector basis $\{$(1,1,1,1),(1,1,1,0),(1,1,0,0), (1,0,0,0)$\}$-cordial labeling of the gear graph G_6.*

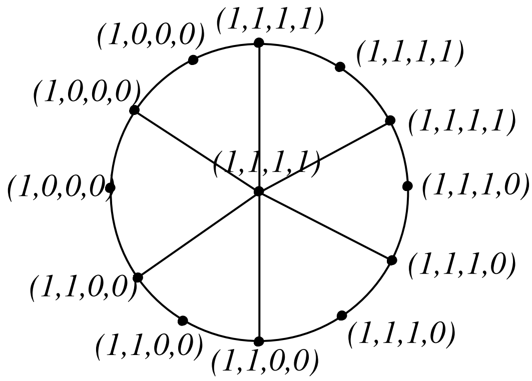

Figure 4. *A vector basis $\{$(1,1,1,1),(1,1,1,0),(1,1,0,0),(1,0,0,0)$\}$-cordial labeling of G_6.*

Theorem 3.10. *The graph $C_{m,n}$ is a vector basis $\{$(1,1,1,1),(1,1,1,0),(1,1,0, 0),(1,0,0,0)$\}$-cordial if (1) $m \equiv 0 \pmod 4$ and $n \equiv 0 \pmod 4$, (2)$m \equiv 2 \pmod 4$ and $n \equiv 2 \pmod 4$.*

Proof. Let $V(C_{m,n}) = \{u_i, v_j \mid 1 \leq i \leq n \,\text{and}\, 2 \leq j \leq n-1\}$ and $E(C_{m,n}) = \{u_iu_{i+1}, u_mu_1, v_jv_{j+1}, u_1v_2, v_{n-1}u_m \mid 1 \leq i \leq m-1 \,\text{and}\, 2 \leq j \leq n-2\}$ respectively be the vertex and edge sets of the graph $C_{m,n}$. Then $|V(C_{m,n}))| = p = m+n-2$ and $|E(C_{m,n})| = q = m+n-1$. Assign the vectors in the following order $u_1, u_2, \ldots, u_m, v_2, v_3, \ldots, v_{n-1}$.

(1) $m \equiv 0 \pmod 4$ and $n \equiv 0 \pmod 4$

Case (i): when $m = n$

Let $m = n = 4t$. Then $p = 8t-2$. Assign the vector (1,1,1,1) to the vertex u_m. Then assign the vector (1,1,1,1) to the first $2t$ vertices and assign the vector (1,1,1,0) to the next $2t$ vertices (except u_m). Finally assign the vector (1,1,0,0) to the next $2t-2$ vertices and assign the vector (1,0,0,0) to the next $2t-1$ vertices.

Case (ii): when $m < n$

Let $m = 4t_1$ and $n = 4t_2$. Then $p = 4(t_1+t_2)-2$ and $q = 4(t_1+t_2)-1$. Assign the vector (1,1,1,1) to the first t_1+t_2 vertices and assign the vector (1,1,1,0) to the next t_1+t_2 vertices . Finally assign the vector (1,1,0,0) to the next t_1+t_2-1 vertices and assign the vector (1,0,0,0) to the next t_1+t_2-1 vertices.

(2) $m \equiv 0 \pmod 4$ and $n \equiv 0 \pmod 4$

Case (i): when $m = n$

Let $m = n = 4t+2$. Then $p = 8t+2$. Assign the vector (1,1,1,1) to the vertex u_m. Then assign the vector (1,1,1,1) to the first $2t+1$ vertices and assign the vector (1,1,1,0)

to the next $2t+1$ vertices (except u_m). Finally assign the vector (1,1,0,0) to the next $2t-1$ vertices and assign the vector (1,0,0,0) to the next $2t$ vertices.
Case (ii):when $m < n$
Let $m = 4t_1 + 2$ and $n = 4t_2 + 2$. Then $p = 4(t_1 + t_2) + 2$ and $q = 4(t_1 + t_2) + 3$. Assign the vector (1,1,1,1) to the first $t_1 + t_2 + 1$ vertices and assign the vector (1,1,1,0) to the next $t_1 + t_2 + 1$ vertices . Finally assign the vector (1,1,0,0) to the next $t_1 + t_2$ vertices and assign the vector (1,0,0,0) to the next $t_1 + t_2$ vertices.
Hence the above labeling method provides a vector basis {(1,1,1,1),(1,1,1,0),(1,1, 0,0),(1,0,0,0)}-cordial labeling for the graph $C_{m,n}$ if (1) $m \equiv 0 \pmod 4$ and $n \equiv 0 \pmod 4$, (2) $m \equiv 2 \pmod 4$ and $n \equiv 2 \pmod 4$. □

Theorem 3.11. *The alternate triangular snake AT_n is a vector basis {(1,1,1,1), (1,1,1,0),(1,1,0,0),(1,0,0,0)}-cordial for all even values of n.*

Proof. Denoting by $V(AT_n) = \{u_i, v_j \mid 1 \le i \le n \text{ and } 1 \le j \le \frac{n}{2}\}$ and $E(AT_n) = \{u_iu_{i+1}, u_{2j-1}v_j, v_ju_{2j} \mid 1 \le i \le n-1 \text{ and } 1 \le j \le \frac{n}{2}\}$ respectively the vertex set and edge set of the alternate triangular snake AT_n. Note that $p = |V(AT_n)| = \frac{3n}{2}$ and $q = |E(AT_n)| = 2n - 1$. Assign the vector to the vertices in the following order $u_1, v_1, u_2, u_3, v_2,$
$u_4, u_5, v_3, u_6, \ldots, u_{n-1}, v_{\frac{n}{2}}, u_n$. Let $m = \frac{n}{2}$. Then $p = 3m$
Case (i): $m \equiv 0 \pmod 4$
Let $m = 4t$. Then $p = 12t$ and $q = 16t - 1$. Assign the vector (1,1,1,1) to the first $3t$ vertices and assign the vector (1,1,1,0) to the next $3t$ vertices. Moreover assign the vector (1,1,0,0) to the next $3t$ vertices and assign the vector (1,0,0,0) to the next $3t$ vertices.
Case (ii): $m \equiv 1 \pmod 4$
Let $m = 4t + 1$. Then $p = 12t + 3$ and $q = 16t + 3$. Assign the vector (1,1,1,1) to the first $3t + 1$ vertices and assign the vector (1,1,1,0) to the next $3t + 1$ vertices. Also assign the vector (1,1,0,0) to the next $3t + 1$ vertices and assign the vector (1,0,0,0) to the next $3t$ vertices. Finally interchange the labels of vertices u_{6t+1} and v_{3t+1}.
Case (iii): $m \equiv 2 \pmod 4$
Let $m = 4t + 2$. Then $p = 12t + 6$ and $q = 16t + 7$. Assign the vector (1,1,1,1) to the first $3t + 2$ vertices and assign the vector (1,1,1,0) to the next $3t + 1$ vertices. Also assign the vector (1,1,0,0) to the next $3t + 2$ vertices and assign the vector (1,0,0,0) to the next $3t + 1$ vertices.
Case (iv): $m \equiv 3 \pmod 4$
Let $m = 4t + 3$. Then $p = 12t + 9$ and $q = 16t + 11$. Assign the vector (1,1,1,1) to the first $3t + 3$ vertices and assign the vector (1,1,1,0) to the next $3t + 2$ vertices. Also assign the vector (1,1,0,0) to the next $3t + 2$ vertices and assign the vector (1,0,0,0) to

the next $3t+2$ vertices.
Clearly this labeling pattern is a vector basis {(1,1,1,1),(1,1,1,0),(1,1, 0,0), (1, 0,0,0)}-cordial labeling of the alternate triangular snake AT_n for all n. □

Theorem 3.12. *The alternate quadrilateral snake AQ_n is a vector basis {(1,1, 1,1),(1,1,1,0),(1,1,0,0),(1,0,0 cordial if and only if $\frac{n}{2} \equiv 0 \pmod 4$.*

Proof. Denoting by $V(AQ_n) = \{u_i, v_j, w_j \mid 1 \leq i \leq n \text{ and } 1 \leq j \leq \frac{n}{2}\}$ and $E(AT_n) = \{u_iu_{i+1}, u_{2j-1}v_j, v_jw_j, w_ju_{2j} \mid 1 \leq i \leq n-1 \text{ and } 1 \leq j \leq \frac{n}{2}\}$ respectively the vertex set and edge set of the alternate quadrilateral snake AQ_n. Note that $p = |V(AQ_n)| = 2n$ and $q = |E(AQ_n)| = \frac{5n-2}{2}$. Assign the vector to the vertices in the following order $u_1, v_1, w_1, u_2, v_2, w_2, \ldots, u_{n-1}, v_{\frac{n}{2}}, w_{\frac{n}{2}}, u_n$. Let $m = \frac{n}{2}$. Then $p = 4m$ and $q = 5m - 1$

Case (i): $m \equiv 0 \pmod 4$

Let $m = 4t$. Then $p = 16t$ and $q = 20t - 1$. Assign the vector (1,1,1,1) to the first $4t$ vertices and assign the vector (1,1,1,0) to the next $4t$ vertices. Moreover assign the vector (1,1,0,0) to the next $4t$ vertices and assign the vector (1,0,0,0) to the next $4t$ vertices.

Case (ii): $m \equiv 1 \pmod 4$

Let $m = 4t+1$. Then $p = 16t+4$ and $q = 20t+4$. From $4t+1$ vertices, we cannot get $5t+1$ edges with edge label 4. This is a contradiction.

Case (iii): $m \equiv 2 \pmod 4$

Let $m = 4t+2$. Then $p = 16t+8$ and $q = 20t+9$. From $4t+2$ vertices, we cannot get $5t+2$ edges with edge label 4. We get a contradiction.

Case (iv): $m \equiv 3 \pmod 4$

Let $m = 4t+3$. Then $p = 16t+12$ and $q = 20t+14$. From $4t+3$ vertices, we cannot get $5t+3$ edges with edge label 4. This is a contradiction.

Clearly this labeling pattern is a vector basis {(1,1,1,1),(1,1,1,0),(1,1,0,0), (1,0,0,0)}-cordial labeling of the alternate quadrilateral snake AQ_n if $\frac{n}{2} \equiv 0 \pmod 4$. □

Example 3.4. *The figure (5) shows a vector basis {(1,1,1,1),(1,1,1,0),(1,1,0,0), (1,0,0,0)}-cordial labeling of the alternate quadrilateral snake AQ_8.*

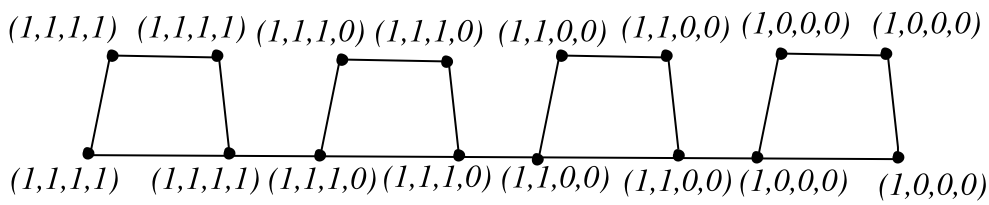

Figure 5. *A vector basis {(1,1,1,1),(1,1,1,0),(1,1,0,0),(1,0,0,0)}-cordial labeling of AQ_8.*

4. CONCLUSION

In this paper, we have investigated the vector basis $\{$(1,1,1,1),(1,1,1,0),(1,1,0,0), (1,0,0,0)$\}$-cordial labeling of certain graphs such as the generalized friendship graph, tadpole graph, gear graph, $C_{m,n}$, alternate triangular snake and alternate quadrilateral snake. The vector basis $\{$(1,1,1,1),(1,1,1,0),(1, 1,0,0),(1,0,0,0)$\}$-cordial labeling of some standard graphs such as step ladder graph, generalized Petersen graph, generalized Jahangir graph, generalized prism graph, generalized web graph and king graph are the open problems for the future research work.

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