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The Prime Number Theorem, Mersenne Primes, the Log Integral Function, and Riemann's R(x) Function

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Abstract

Chebyshev's second function, the Möbius function, and Mersenne primes are used to derive a function that estimates the number of primes less than a given amount. This function is compared to the log integral function and Riemann's $R(x)$ function.

Keywords : Chebyshev's second function, Möbius function, Mersenne primes, log integral function, Riemann's R(x) function

1. INTRODUCTION

Chebyshev's second function is the summatory Mangoldt function, that is,

$$\psi(x) = \sum_{n \leq x} \Lambda(n), x > 0. \tag{1}$$

$\Lambda(n)$ equals $\log(p)$ if $n = p^m$ for some prime p and some $m \geq 1$ or 0 otherwise. The prime number theorem is equivalent to the asymptotic formula

$$\sum_{n \leq x} \Lambda(n) \sim x, x \to \infty \tag{2}$$

This asymptotic formula states that

$$\lim_{x \to \infty} \frac{\psi(x)}{x} = 1. \tag{3}$$

The log integral function is

$$li(x) = \lim_{\delta \to +0} (\int_0^{1-\delta} + \int_{1+\delta}^{x}) \frac{dt}{\log t}, (x > 1). \tag{4}$$

Let $\pi(x)$ denote the number of primes less than or equal to x and $\pi_0(x)$ denote $\pi(x) - \frac{1}{2}$ if x is a prime or $\pi(x)$ otherwise. In 1859 Riemann [1] published and in 1895 von Mangoldt [2] proved, the following formula:

$$\pi_0(x) = \sum_{n=1}^{\infty} \mu(n) f(x^{1/n})/n, \tag{5}$$

where $\mu(n)$ is the Möbius function, and

$$f(x) = \sum_{n=1}^{\infty} \pi_0(x^{1/n})/n = li(x) - \sum_{\rho} li(x^{\rho}) + \int_x^{\infty} \frac{dt}{(t^2-1)t\log t} - \log 2, \tag{6}$$

where the sum means $\lim_{T\to\infty} \sum_{|\rho|\leq T} li(x^{\rho})$, and the ρ's are the non-trivial zeros of the Riemann zeta function:

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \prod_{p} (1-p^{-s})^{-1}. \tag{7}$$

In the sum over the ρ's, each ρ-term appears a number of times equal to the multiplicity of the zero ρ. Since $f(x) = 0$ for $1 < x < 2$, the sum in (5) is actually finite and equals

$$\pi_0(x) = \sum_{n=1}^{N} \mu(n) f(x^{1/n})/n \tag{8}$$

for all $x < 2^{N+1}$, because then $x^{1/n} < 2$ for all $n \geq N+1$ and so $f(x^{1/n}) = 0$.

Taking only the first term $li(x)$ of (6) and introducing it into the "inversion formula" (5), Riemann got his approximation to $\pi_0(x)$:

$$\pi_0(x) \approx R(x) = \sum_{n=1}^{\infty} \mu(n) li(x^{1/n})/n \tag{9}$$

The right-hand side of (9) can be transformed into Gram's series (see Lehmer's [3] article)

$$R(x) = 1 + \sum_{n=1}^{\infty} \frac{(\log x)^n}{n! n \zeta(n+1)}. \tag{10}$$

Since

$$\frac{d^k R(e^t)}{dt^k} = \sum_{n=k}^{\infty} \frac{t^{n-k}}{(n-k)! n \zeta(n+1)} > 0 \tag{11}$$

for $t > 0$ it is obvious that $R(x)$ cannot describe the more detailed behavior of $\pi_0(x)$, which is certainly not a function with all derivatives > 0. $R(x)$ smoothes the values of $\pi_0(x)$ and gives a kind of mean-value correct smooth approximation to $\pi_0(x)$.

Equation (13) in Riesel and Göhl's [4] article is:

$$-\{li(x^{\rho}) + li(x^{\bar{\rho}})\} = -2\Re li(x^{\rho}) \sim -2\Re \frac{\sqrt{x}e^{i\alpha \log x}}{(\frac{1}{2} + i\alpha)\log x} \tag{12}$$

$$= \frac{-2\sqrt{x}}{|\rho| \log x} \cos(\alpha \log x - \arg \rho) \tag{13}$$

where $\rho = \frac{1}{2} + i\alpha$ (a root on the critical line). Thus for large x, the contribution to $\pi_0(x)$ from two complex conjugate zeros $\frac{1}{2} \pm i\alpha$ of $\zeta(s)$ is an oscillating function with an amplitude varying with x as $2\sqrt{x}/(|\rho| \log x)$ and with consecutive zeros x_{k+1} and x_k connected by the relationship $x_{k+1} = x_k \cdot e^{\pi/\alpha}$. The larger $|\rho|$ becomes, the smaller is the amplitude and the faster are the oscillations. See their article for graphs of this function for the first five non-trivial zeta function zeros. A plot of the function for the first non-trivial zeta function zero ((0.5, 14.134725)) is

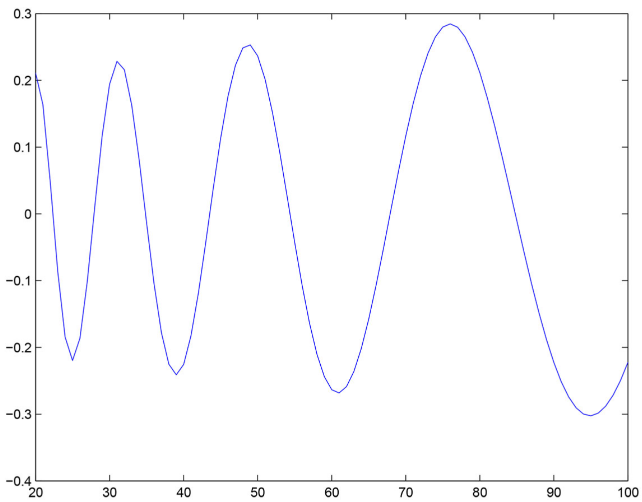

Figure 1

Most of the other material above was also taken from this article.

2. A FUNCTION FOR ESTIMATING $\pi(n)$

Let v_j denote $\sum_{i|j}(\psi_{i+1} - \psi_i)\mu(i)$ where $\mu(i)$ denotes the Möbius function and j is odd. The Möbius function is defined as follows. $\mu(1)$ is set to 1. For $n > 1$, write $n = p_1^{a_1} \cdots p_k^{a_k}$. Then $\mu(n) = (-1)^k$ if $a_1 = a_2 = \ldots = a_k = 1$ or 0 otherwise. ψ_1 is set to 0. For prime j other than Mersenne primes (3, 7, 31, 127, 8191,...), v_j then equals $\log(2)$. If j is a Mersenne prime, then $v_j = 0$ since $j + 1$ is a prime power and the ψ values increase by $\log(2)$ at this point (cancelling out the difference between ψ_2 and ψ_1). In general, v_j equals an integer multiple of $\log(2)$ (including a multiple of

0). For $j < 200000$, there are 30 v_j values equal to $2\log 2$, 60146 v_j values equal to $\log 2$, 34639 v_j values equal to 0, 4949 v_j values equal to $-\log 2$, 221 v_j values equal to $-2\log 2$, and 15 v_j values equal to $-3\log 2$.

Case 1: $v_j = \log 2$
$v_j = \log(2)$ for any j value that is a product of non-Mersenne primes. Note that the Möbius function zeros out any $\psi_{i+1} - \psi_i$ value in the sum where i is not square-free. $v_j = \log(2)$ for any j value that is a product of powers of non-Mersenne primes. This does not account for all v_j values that equal $\log(2)$ though. Note that $2^4 - 1 = 3 \cdot 5$ and $2^9 - 1 = 7 \cdot 73$, products of a Mersenne prime and a non-Mersenne prime. These products behave like non-Mersenne primes. $v_j = \log(2)$ for any j value that is a power of 3 or 5 times 15. Similarly, $v_j = \log(2)$ for any j value that is a power of 7 or 73 times 511. There are likely to be other such products of Mersenne and non-Mersenne primes. The first 100 j values are 1, 5, 11, 13, 15, 17, 19, 23, 25, 29, 37, 41, 43, 45, 47, 53, 55, 59, 61, 65, 67, 71, 73, 75, 79, 83, 85, 89, 95, and 97. A plot of the six hundred and two j values less than 1000 is

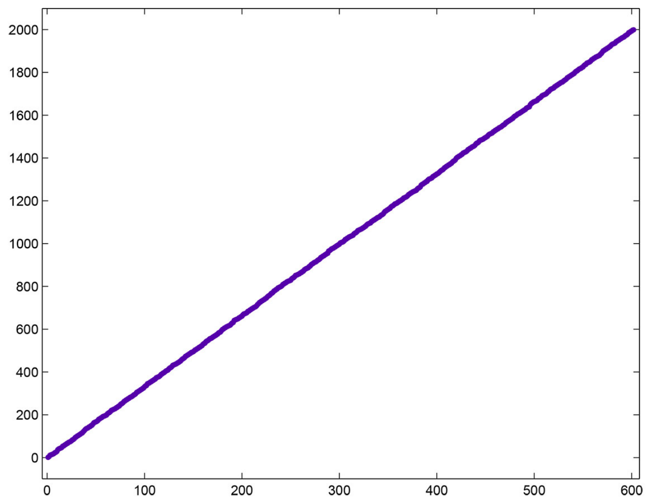

Figure 2

For a linear least-squares fit of the curve, $p_1 = 3.325$ with a 95% confidence interval of (3.324, 3.326), $p_2 = -1.771$ with a 95% confidence interval of (-2.179, -1.362), SSE=3893, R-squared=1, and RMSE=2.547. A formulation of the Riemann hypothesis (that the real part of the non-trivial zeta function zeros equals $\frac{1}{2}$) is that $\psi(x)$ is essentially square root close to the function $f(x) = x$. The above linear curve may be relevant to this. See Mazur and Stein [5] for other formulations of the Riemann hypothesis.

Case 2: $v_j = 0$
The j values less than 200 where $v_j = 0$ are 3, 7, 9, 27, 31, 33, 35, 39, 49, 51, 57, 69, 77, 81, 87, 91, 99, 105, 111, 117, 119, 123, 127, 129, 133, 141, 153, 155, 159, 161, 171, 175, 177, and 183. In general, these values can be factored into powers of Mersenne primes or products of powers of Mersenne primes and powers of non-Mersenne primes. There is at most one distinct Mersenne prime factor in each j value. Again, 3*5 and 7*73 behave like non-Mersenne primes. The j value of 155 factors into $(3 \cdot 5) \cdot 7$ so that there is only one distinct "Mersenne prime" factor. For j values that are not a power of a Mersenne prime, there is at least one non-Mersenne prime factor. A plot of the one thousand, seven hundred, and thirty one j values less than 5000 is

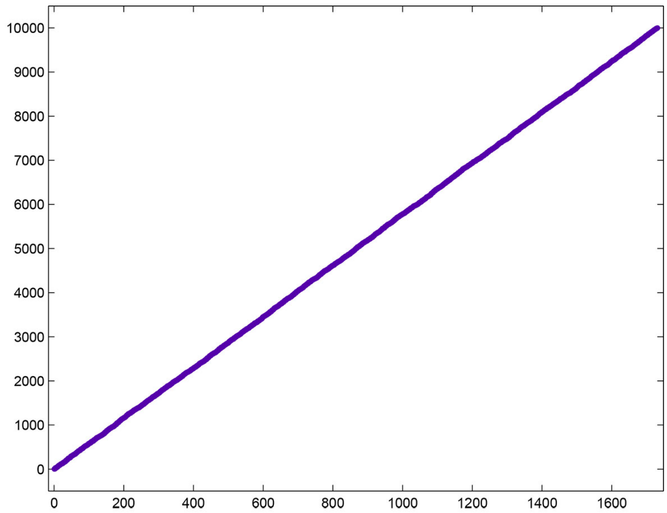

Figure 3

For a linear least-squares fit of the curve, $p_1 = 5.781$ with a 95% confidence interval of (5.78, 5.782), $p_2 == 10.91$ with a 95% confidence interval of $(-11.88, -9.941)$, SSE=$1.833 \cdot 10^8$, R-squared=1, and RMSE=10.3.

Case 3: $v_j = -\log 2$
The j values less than 1000 where $v_j = -\log 2$ are 21, 63, 93, a47, 189, 217, 231 273, 279, 357, 381, 399, 441, 483 567, 609, 693, 777, 819, 837, 861, 889, 903, and 987. In general, these values can be factored into powers of two Mersenne primes or powers of two Merseene primes and powers of non-Mersenne primes. There are most two distinct "Mersenne prime" factors of each j value. A plot of the seven hundred and forty j values less than 15000 is

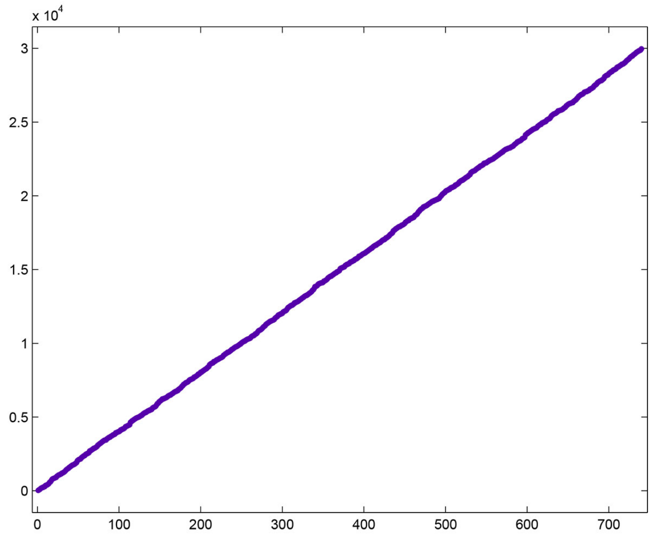

Figure 4

For a linear least-squares fit of the curve, $p_1 = 40.37$ with a 95% confidence interval of (40.35, 40.39), $p_2 = -17.8$ with a 95% confidence interval of $(-27.35, -8.251)$, SSE$=3.224 \cdot 10^6$, R-squared=0.9999, and RMSE=66.09.

Similar factorizations occur in the other cases.

Let w_n denote $\frac{\sum_{i=1}^{n} v_{2i-1}}{\log \sum_{i=1}^{n} v_{2i-1}}$. C code for computing w_n is given in the Methods section. Let $\pi_1(n)$ denote $\pi(2n)$ (this avoids duplicate $\pi(n)$ values for consecutive n values). A plot of w_n versus $\pi_1(n)$ for $n = 1000$ is

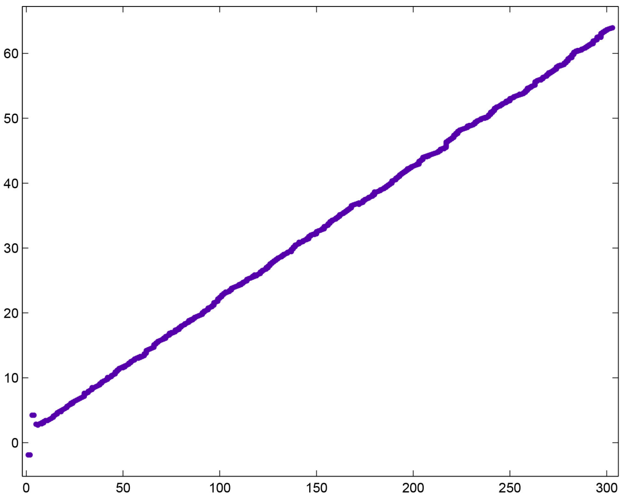

Figure 5

For a linear least-squares fit of the curve, $p_1 = 0.2063$ with a 95% confidence interval of (0.2061, 0.2065), $p_2 = 1.354$ with a 95% confidence interval of (1.313, 1.394), SSE=88.36, R-squared=0.9997, and RMSE=0.2975. (SSE denotes the sum of squared errors and RMSE denotes square root of mean squared errors.)

A plot of the slopes of the linear least-squares fits of the curves for n equal to 1000,2000,3000,. . .,200000 is

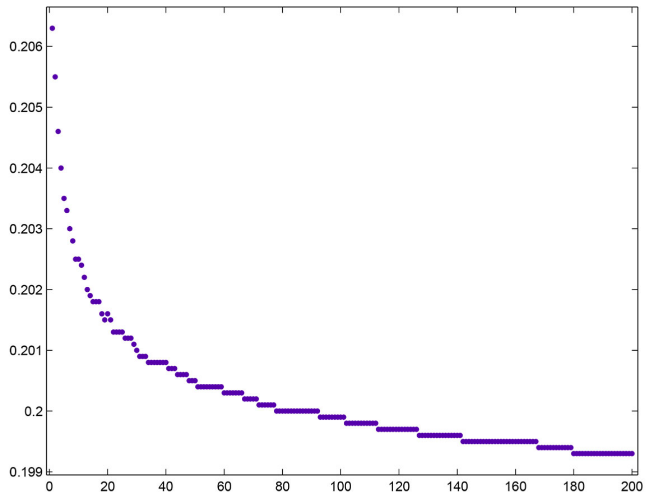

Figure 6

A plot of the y-intercepts of the linear least-squares fits is

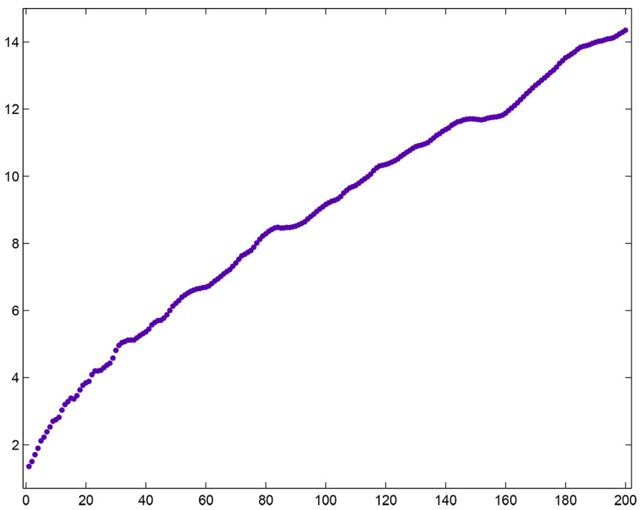

Figure 7

A plot of the SSE values of the linear least-squares fits is

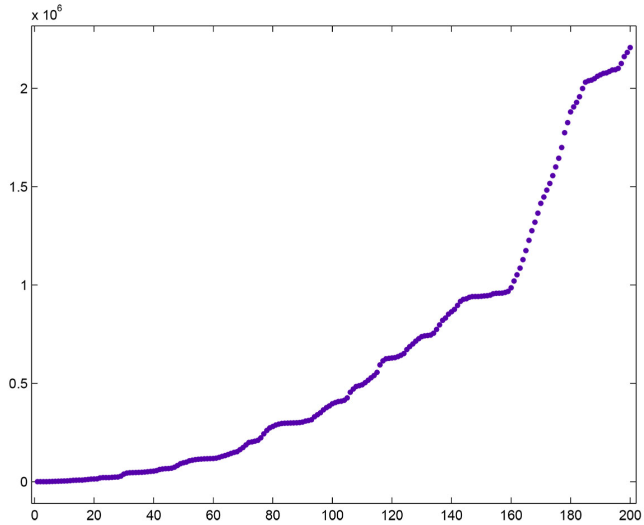

Figure 8

A plot of the RMSE values of the linear least-squares fits is

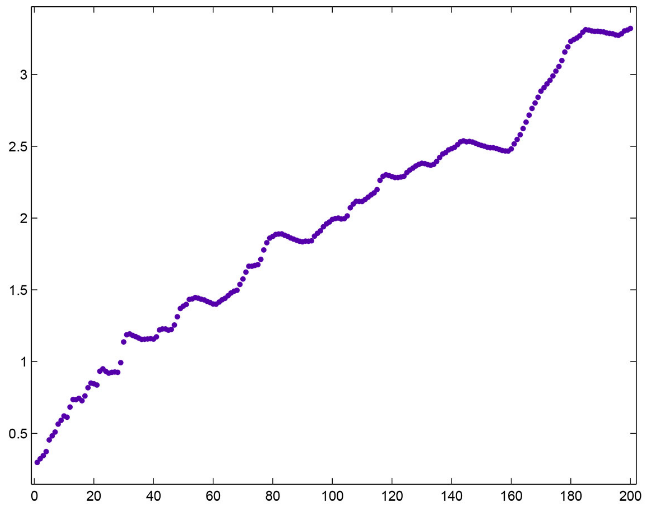

Figure 9

Presumably, the oscillations in the curve are due to the zeta function zeros.

3. w_n VERSUS LOG INTEGRAL FUNCTION

A C program for computing $li(x)$ is given in the Methods section. A plot of w_n for $n = 1000$ versus $li(n)$ for $n = 1, 2, 3, \ldots 1000$ is

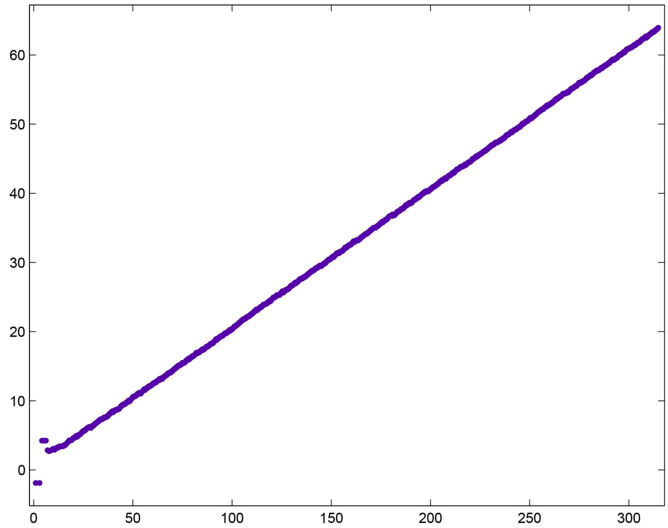

Figure 10

For a linear least-squares fit of the curve, $p_1 = 0.2016$ with a 95% confidence interval of (0.2014, 0.2017), $p_2 = 0.3853$ with a 95% confidence interval of (0.3537, 0.4169), SSE=51.39, R-squared=0.9998, and RMSE=0.2269.

A plot of the slopes of the linear least-squares fits for maximum n values of 1000, 2000, 3000. . ., 200000 is

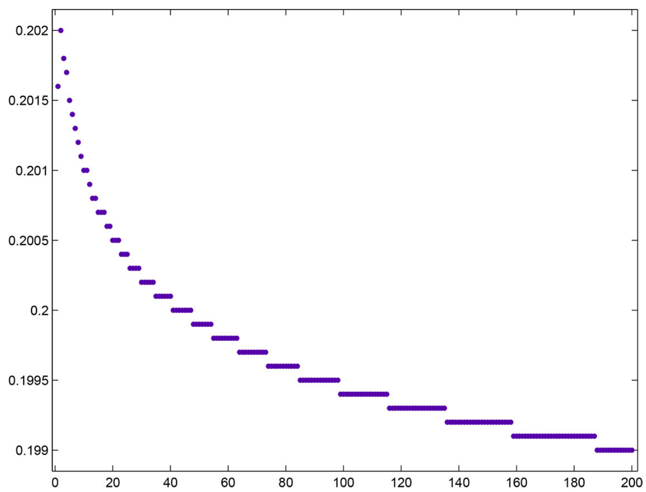

Figure 11

A plot of the y-intercepts of the linear least-squares fits is

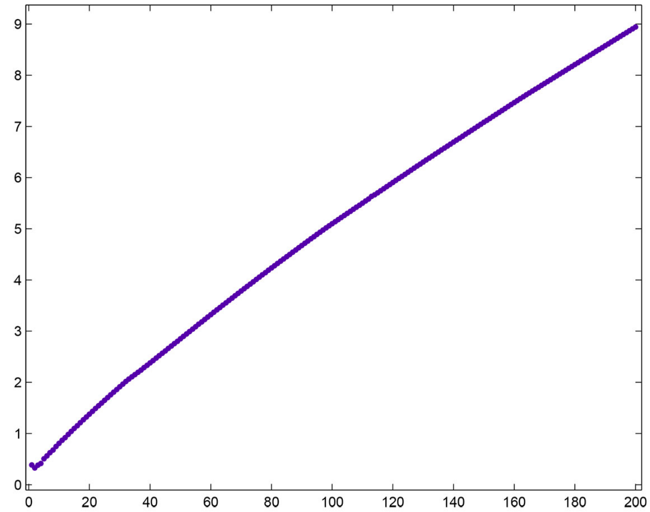

Figure 12

A plot of the SSE values of the linear least-squares fits is

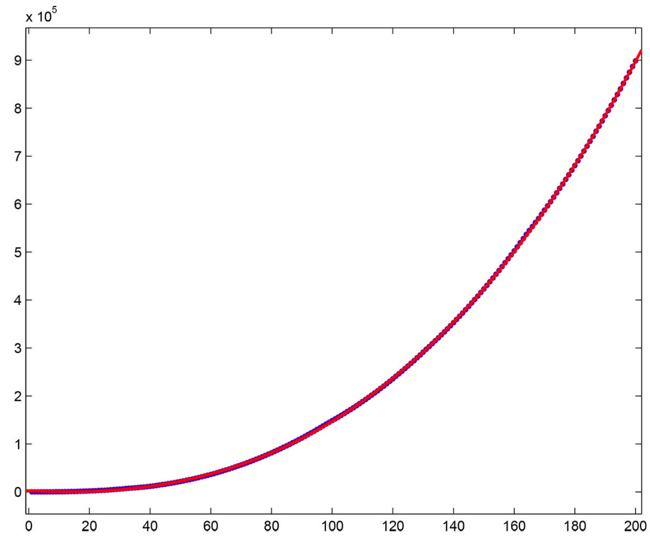

Figure 13

For a cubic least-squares fit of the curve, $p_1 = 0.06231$ with a 95% confidence interval of (0.06144, 0.06319), $p_2 = 11.63$ with a 95% confidence interval of (11.36, 11.89), $p_3 = -342.4$ with a 95% confidence interval of $(-365.5, -319.3)$, $p_4 = 2689$ with a 95% confidence interval of (2152, 3227), SSE=$1.753 \cdot 10^8$, R-squared=1, and RMSE=945.8.

A plot of the SSE values versus the y-intercepts is

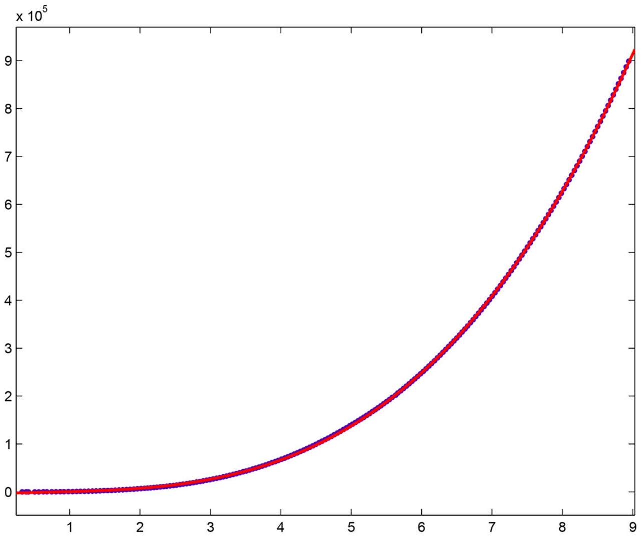

Figure 14

For a cubic least-squares fit of the curve, $p_1 = 1552$ with a 95% confidence interval of (1541, 1563), $p_2 = -3307$ with a 95% confidence interval of $(-3464,\ -3150)$, $p_3 = 5999$ with a 95% confidence interval of (5347, 6650), $p_4 = -3516$ with a 95% confidence interval of $(-4267,\ -2766)$, SSE=$1.856 \cdot 10^8$, R-squared=1, and RMSE=973.

A plot of the RMSE values of the linear least-squares fits is

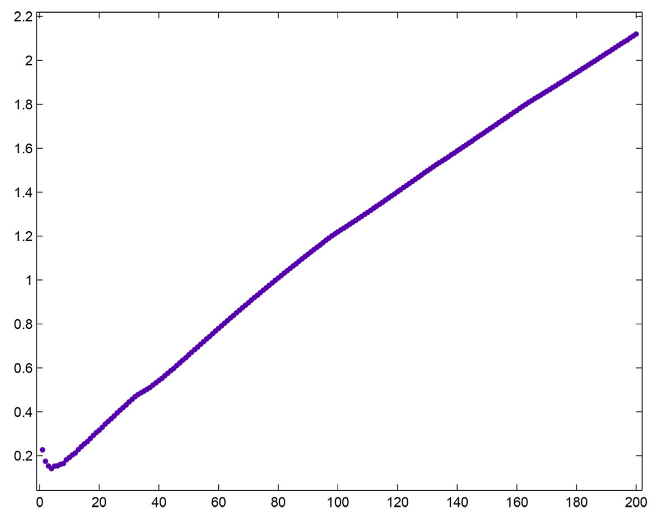

Figure 15

In 1899, de la Vallée [6] proved that

$$\pi(x) = Li(x) + O(xe^{-a\sqrt{\log x}}) \tag{14}$$

as $x \to \infty$ where $Li(x) = li(x) - li(2)$ and a is some constant. In 1901, von Koch [7] proved that if the Riemann hypothesis is true, the above error term can be improved to

$$\pi(x) = Li(x) + O(\sqrt{x} \log x) \tag{15}$$

In 1976, Schoenfeld [8] showed, by assuming the Riemann hypothesis, that

$$|\pi(x) - li(x)| < \frac{\sqrt{x} \log x}{8\pi} \tag{16}$$

for $x \geq 2657$.
A plot of $li(n) - w_n$ versus $\frac{\sqrt{n}\log(n)}{8\pi}$ for $n = 1, 2, 3, \ldots, 1000000$ is

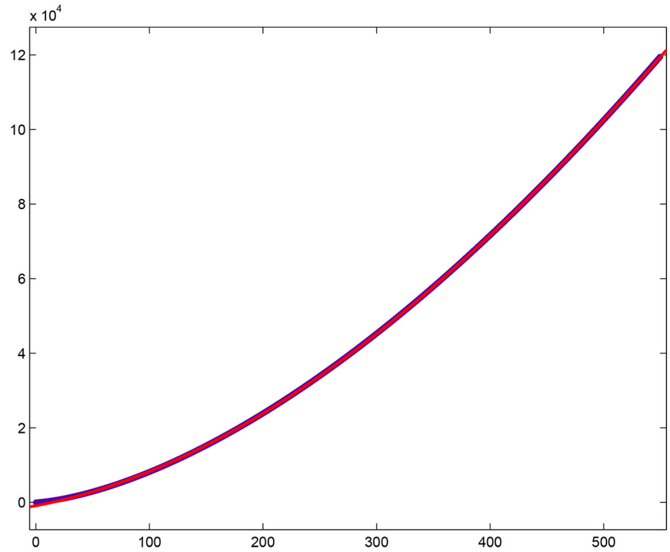

Figure 16

For a cubic least-squares fit of the curve, $p_1 = -0.0001077$ with a 95% confidence interval of (-0.0001078, -0.0001077), $p_2 = 0.3525$ with a 95% confidence interval of (0.3525, 0.3525), $p_3 = 57.6$ with a 95% confidence interval of (57.59, 57.62), $p_4 = -895.1$ with a 95% confidence interval of (-896, -894.1), SSE=$4.139 \cdot 10^9$, R-squared=1, and RMSE=64.33. For a cubic least-squares fit of the curve for $n = 1, 2, 3, \ldots, 500000$, $p_1 = -0.0001886$ with a 95% confidence interval of (-0.0001887, -0.0001885), $p_2 = 0.4044$ with a 95% confidence interval of (0.4043, 0.4045), $p_3 = 48.17$ with a 95% confidence interval of (48.16, 48.19), $p_4 = -468.5$ with a 95% confidence interval of (-469.2, -467.8), SSE=$5.995 \cdot 10^8$, R-squared=1, and RMSE=34.63.

A plot of $\pi_1(n) - w_n$ versus $\frac{\sqrt{n}\log(n)}{8\pi}$ for $n = 1, 2, 3, \ldots, 1000000$ is

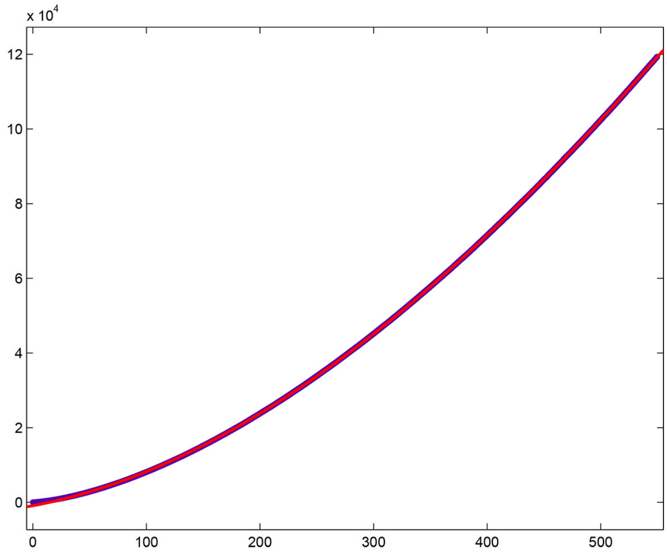

Figure 17

For a cubic least-squares fit of the curve, $p_1 = -0.0001081$ with a 95% confidence interval of (-0.0001081, -0.0001081), $p_2 = 0.3529$ with a 95% confidence interval of (0.3528, 0.3529), $p_3 = 57.27$ with a 95% confidence interval of (57.26, 57.29), $p_4 = -908.3$ with a 95% confidence interval of (-909.3, -907.2), SSE=$4.446 \cdot 10^9$, R-squared=1, and RMSE=66.68. For a cubic least-squares fit of the curve for $n = 1, 2, 3, \ldots, 500000$, $p_1 = -0.0001889$ with a 95% confidence interval of (-0.000189, -0.0001887), $p_2 = 0.4049$ with a 95% confidence interval of (0.4048, 0.4049), $p_3 = 47.81$ with a 95% confidence interval of (47.8, 47.83), $p_4 = -479.6$ with a 95% confidence interval of (-480.4, -478.3), SSE=$6.645 \cdot 10^8$, R-squared=1, and RMSE=36.45.

A plot of $R(n) - w_n$ versus $\frac{\sqrt{n}\log(n)}{8\pi}$ for $n = 1, 2, 3, \ldots, 1000000$ is

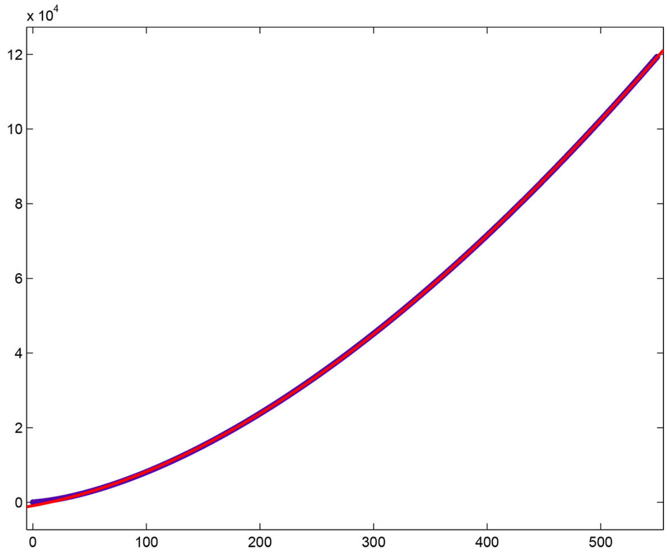

Figure 18

For a cubic least-squares fit of the curve, $p_1 = -0.000108$ with a 95% confidence interval of (-0.000108, -0.0001079), $p_2 = 0.3528$ with a 95% confidence interval of (0.3528, 0.3529), $p_3 = 57.28$ with a 95% confidence interval of (57.26, 57.29), $p_4 = -908$ with a 95% confidence interval of (-909, -907), SSE=$4.182 \cdot 10^9$, R-squared=1, and RMSE=64.67. For a cubic least-squares fit of the curve for $n = 1, 2, 3, \ldots, 500000$, $p_1 = -0.0001893$ with a 95% confidence interval of (-0.0001894, -0001891), $p_2 = 0.405$ with a 95% confidence interval of (0.4049, 0.405), $p_3 = 47.8$ with a 95% confidence interval of (47.78, 46.81), $p_4 = -479.4$ with a 95% confidence interval of (-480.1, -478.6), SSE=$6.092 \cdot 10^8$, R-squared=1, and RMSE=34.91.

Other than the first and third parameters, the parameters appear to decrease as n increases. Plots of the parameters of the cubic least squares fits of $li(n) - w_n$ versus $\frac{\sqrt{n}\log(n)}{8\pi}$ for n upper bounds of 25000, 50000, 75000,. . .,2000000 are as follows. A plot of the p_1 parameters is

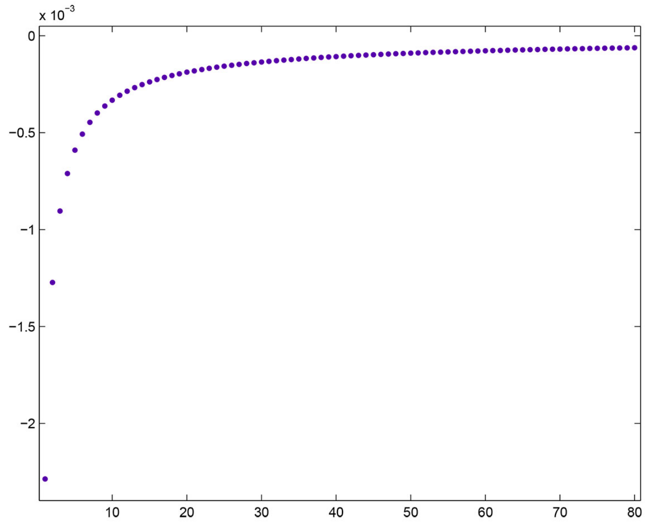

Figure 19

The curve resembles the logarithm function. A plot of the p_2 parameters is

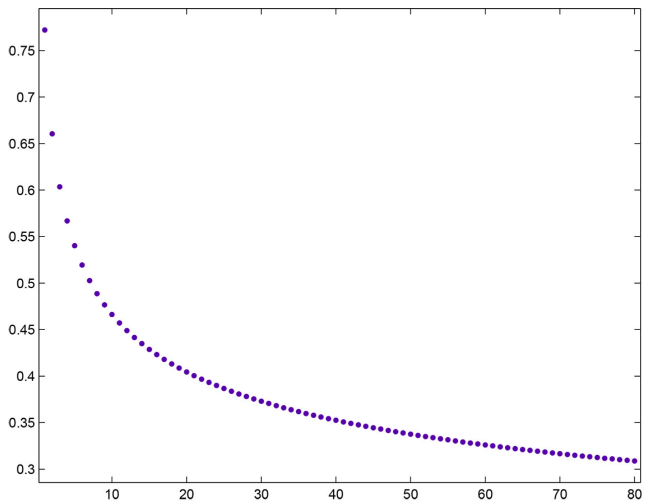

Figure 20

A plot of the p_3 parameters is

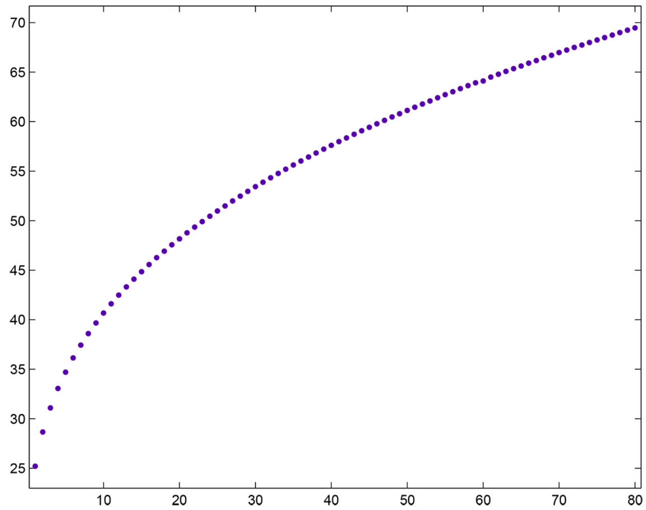

Figure 21

A plot of the p_4 parameters is

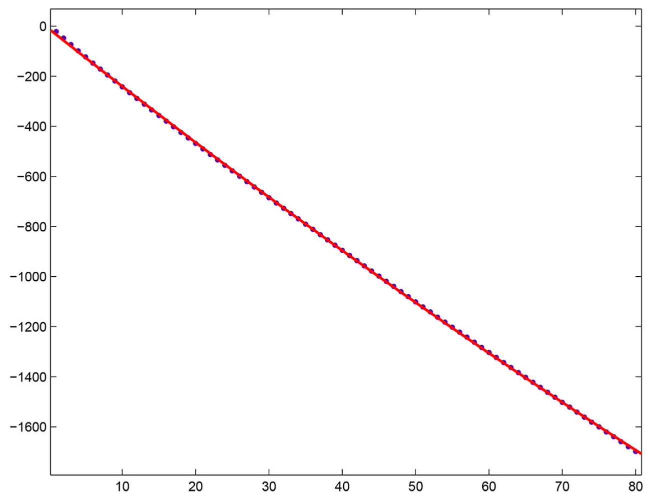

Figure 22

The curve is quadratic. A plot of the SSE values is

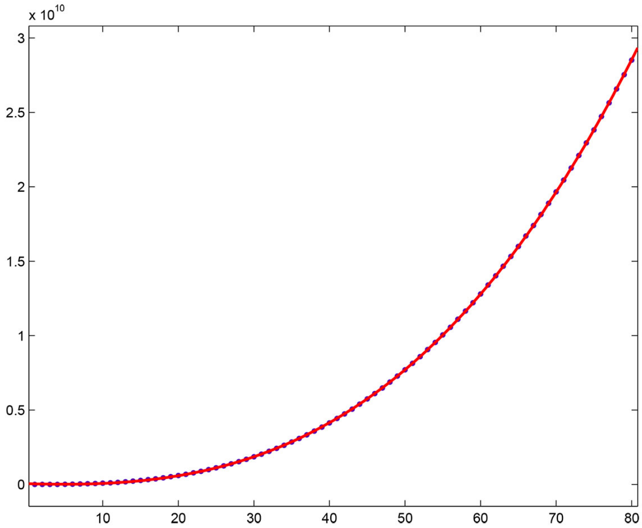

Figure 23

The curve is cubic. A plot of the RMSE values is

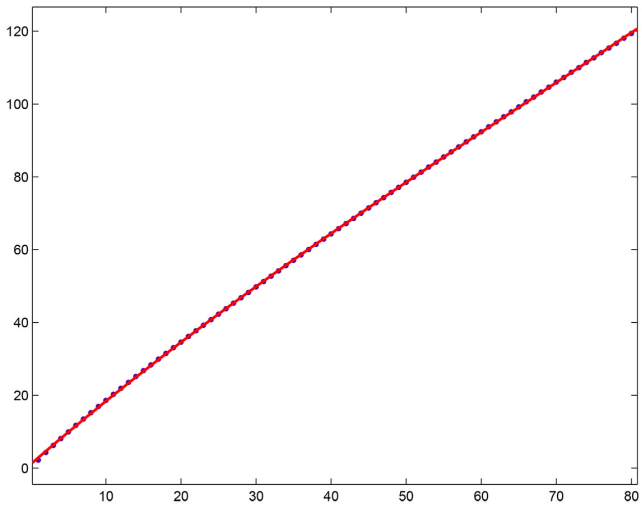

Figure 24

The curve is cubic.

A plot of the p_2 parameters versus the p_3 parameters is

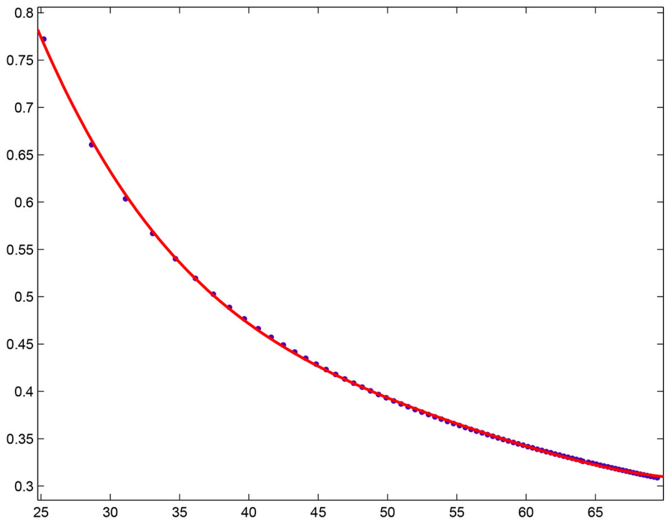

Figure 25

The curve is quartic.

Similar cubic curves are obtained for $\pi_1(n) - w_n$. A plot of the p_1 parameters for n upper bounds of 25000, 50000, 75000,...,2000000 is

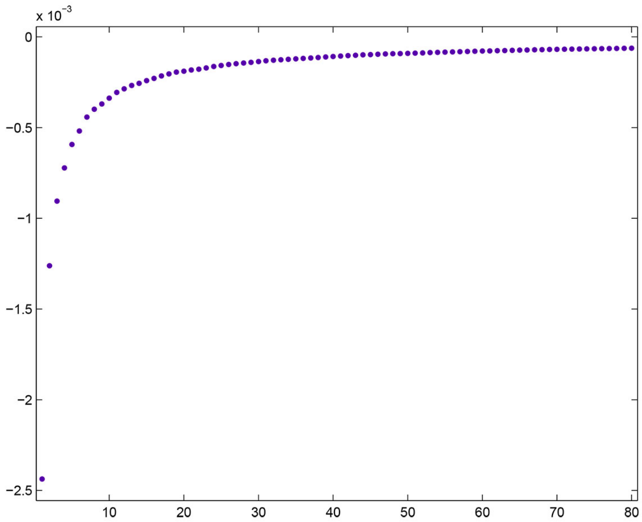

Figure 26

A plot of the p_2 parameters is

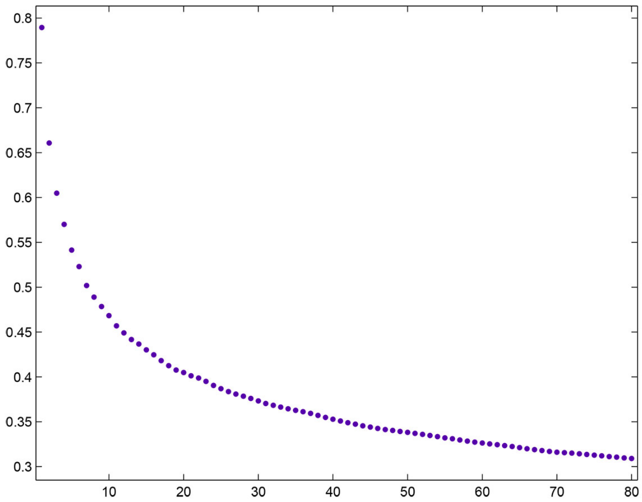

Figure 27

A plot of the p_3 parameters is

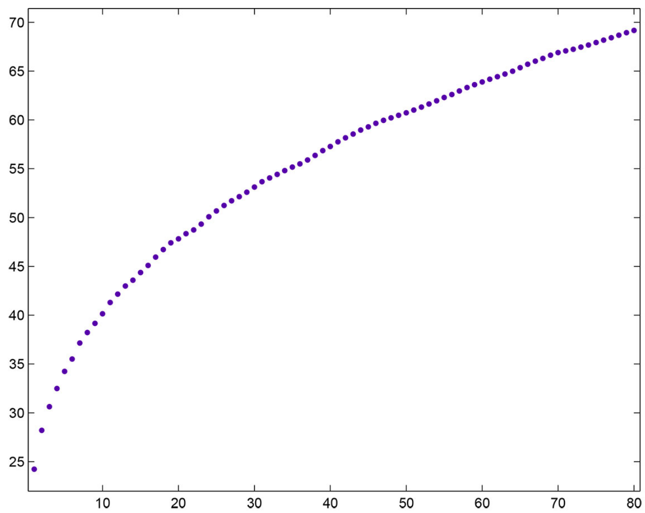

Figure 28

A plot of the p_4 parameters is

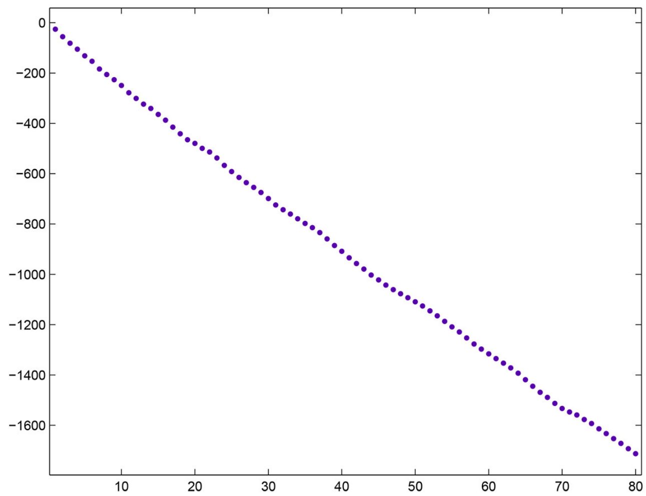

Figure 29

The curve is quadratic except for the oscillations. A plot of the SSE values is

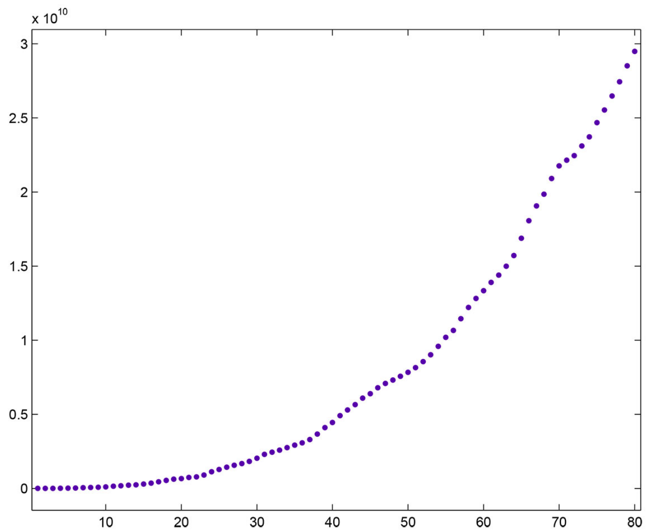

Figure 30

The curve is cubic except for the oscillations. A plot of the RMSE values is

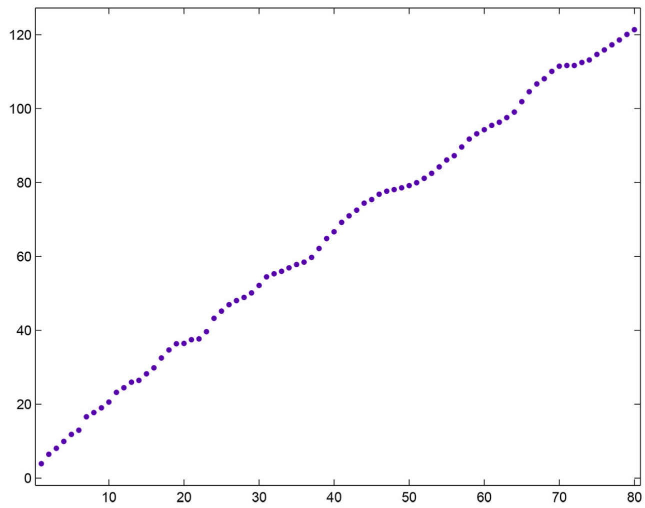

Figure 31

The curve is cubic except for the oscillations.

A plot of the p_1 parameters for $\pi_1(n) - w_n$ versus the p_1 parameters for $li(n) - w_n$ is

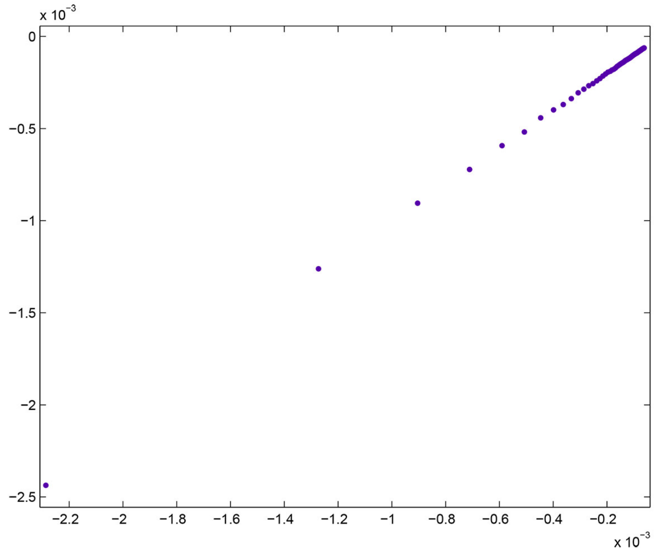

Figure 32

The slope is about 1.042 and the y-intercept is about 0.

A plot of the p_2 parameters for $\pi_1(n) - w_n$ versus the p_2 parameters for $li(n) - w_n$ is

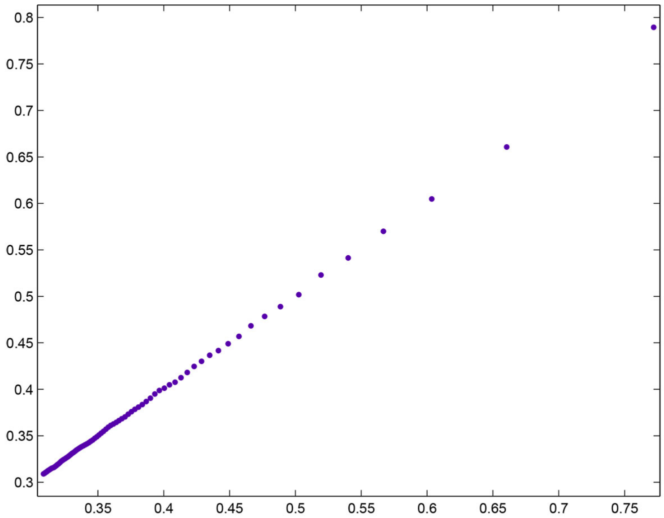

Figure 33

The slope is about 1.015 and the y-intercept is about -0.005.

A plot of the p_3 parameters for $\pi_1(n) - w_n$ versus the p_3 parameters for $li(n) - w_n$ is

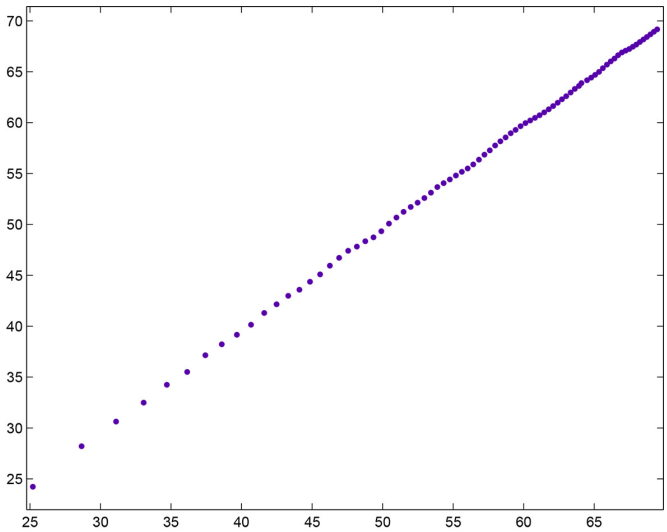

Figure 34

The slope is about 1.007 and the y-intercept is about -0.7429.

A plot of the p_4 parameters for $\pi_1(n) - w_n$ versus the p_4 parameters for $li(n) - w_n$ is

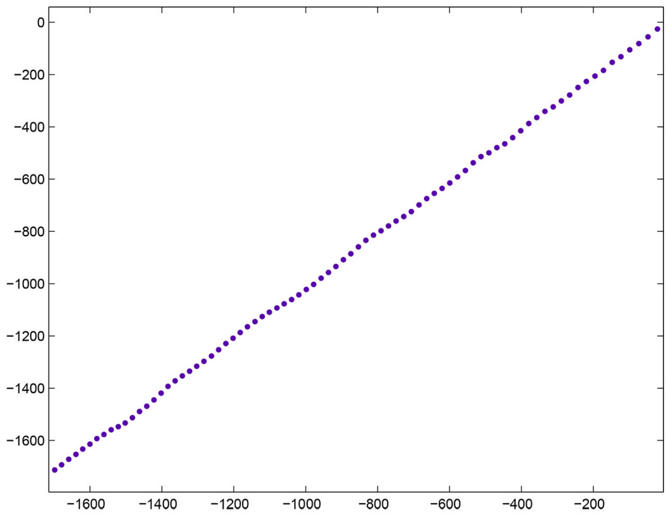

Figure 35

The slope is about 1.006 and the y-intercept is about -7.788.

A plot of the SSE values for $\pi_1(n) - w_n$ versus the SSE values for $li(n) - w_n$ is

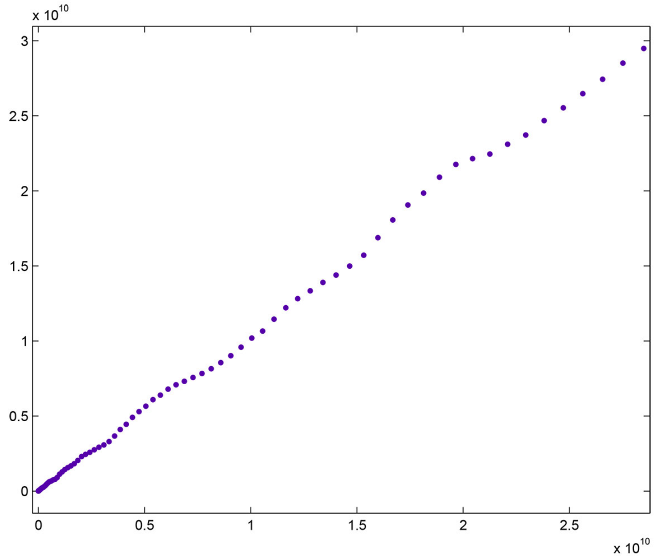

Figure 36

The slope is about 1.046.

A plot of the RMSE values for $\pi_1(n) - w_n$ versus the RMSE values for $li(n) - w_n$ is

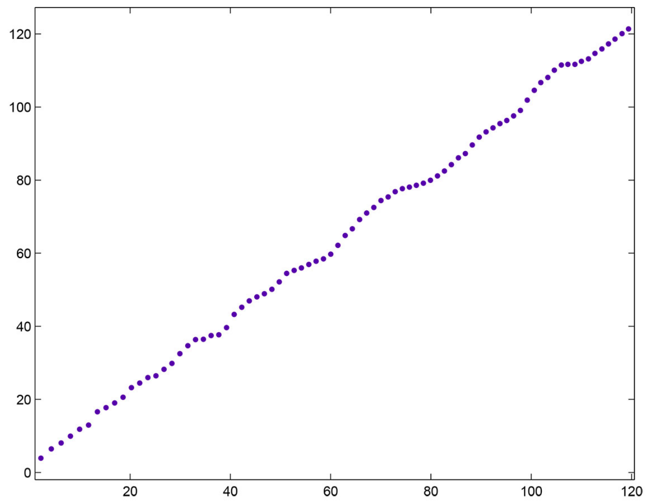

Figure 37

The slope is about 1.005 and the y-intercept is about 1.851.

Other than the oscillations of the parameters for $\pi_1(n) - w_n$, the parameters are about the same. This has implications for the difference between $li(n)$ and $\pi_1(n)$. In 1914, Littlewood [9] proved that $\pi(x) - li(x)$ changes sign infinitely often.

4. RIEMANN'S $R(x)$ FUNCTION

A C program for computing $R(x)$ is given in the Methods section. A plot of $R(n)$ versus $\pi_1(n)$ for $n = 1, 2, 3, \ldots, 1000$ is

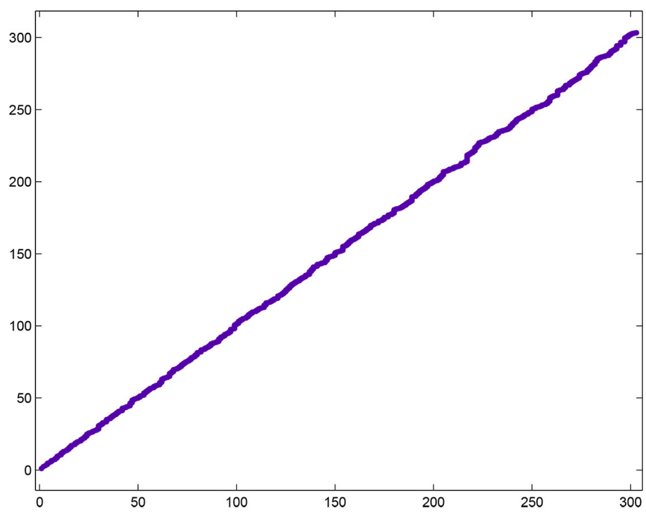

Figure 38

For a linear least-squares fit of the curve, $p1 = 0.9986$ with a 95% confidence interval of (0.9978, 0.9994), $p_2 = 0.162$ with a 95% confidence interval of (0.01725, 0.3068), SSE=1128, R-squared=0.9998, and RMSE=1.063. For larger n values, the slope is usually 1.
A plot of the y-intercepts of the linear least-squares fits for maximum n values of 1000, 2000, 3000,. . .,200000 is

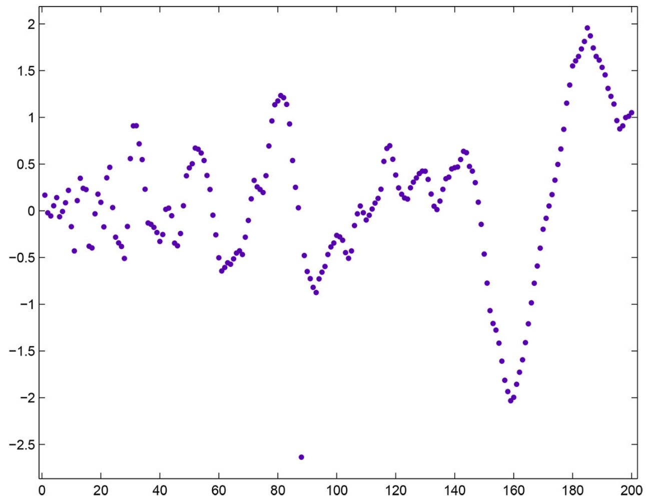

Figure 39

For large positive y-intercepts, the slope may be as low as 0.9996. For negative y-intercepts, the slope is 1.

A plot of the SSE values of the linear least-squares fits is

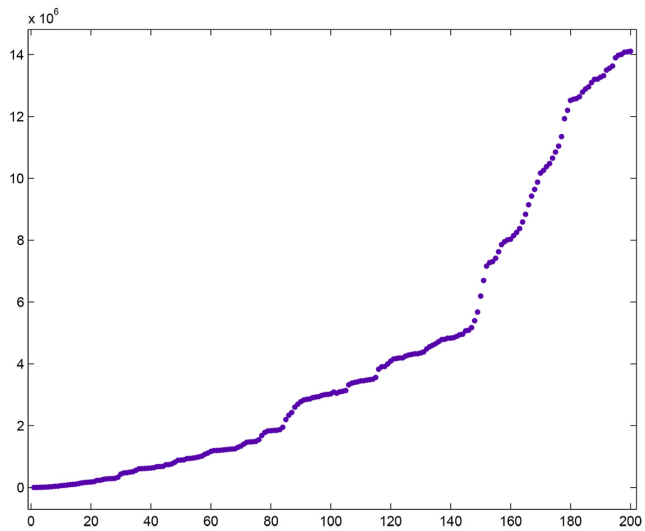

Figure 40

A plot of the RMSE values of the linear least-squares fits is

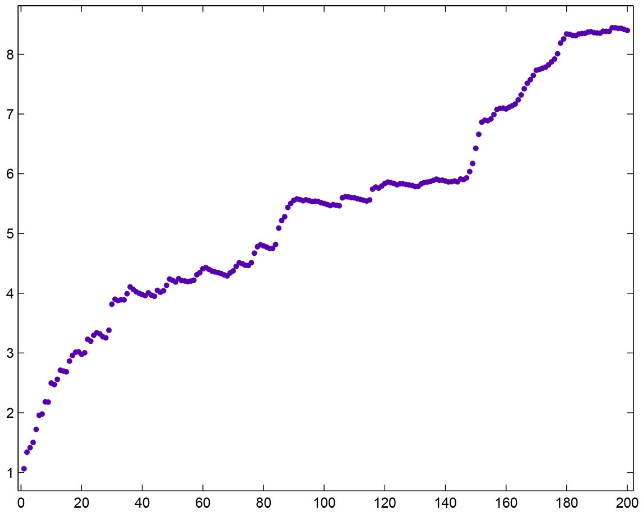

Figure 41

These values are similar to those for w_n but are over twice as large.

A plot of the SSE values of the linear least-squares fits and 6 times the SSE values of the linear least squares fits of the w_n values is

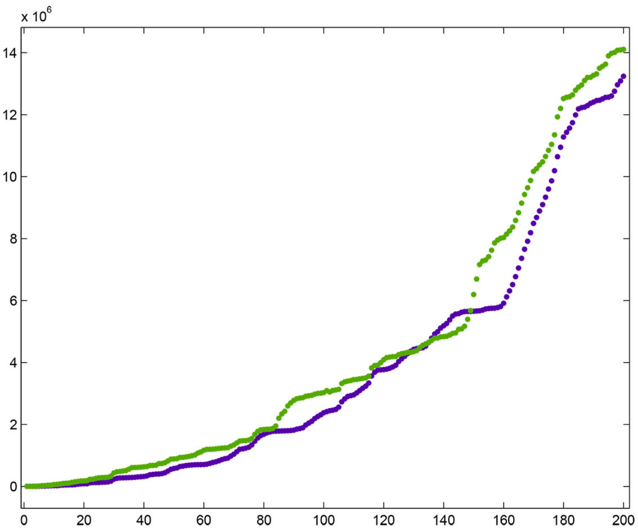

Figure 42

The curves have similar properties. In this sense, w_n is much more accurate than $R(n)$.

5. METHODS

```
    compute w(n)
#include <math.h>
#include <stdio.h>
#include "cheby3hk.h" // 300001 maximum
#include "table5.h"
int mobius(unsigned int a, unsigned int *table, unsigned int tsize);
void main () {
unsigned int h,i,N,count,index;
int r;
unsigned int tsize=114155;
double sum,sum1,temp;
FILE *Outfp;
Outfp = fopen("sortu.dat","w");
if (Outfp==NULL)
    return;
index=1;
count=0;
sum1=0.0;
N=1;
for (h=1; h<=200000; h++) {
    sum=0.0;
    for (i=1; i<=N; i++) {
        if (N==(N/i)*i)
            sum=sum+(zero[i]-zero[i-1])*mobius(i,table,tsize);
        }
    r=(int)(sum/(log(2)-0.01));
    temp=(double)r*log(2);
    sum1=sum1+(double)r*log(2);
    if (h==(h/1000)*1000)
        printf(" %d %.16llf %d \n",h,sum1/log(sum1),count);
    N=N+2;
    for (i=index; i<=tsize; i++) {
        if (table[i-1]<2*h)
            count=count+1;
        else {
            index=i;
            break;
            }
```

```
        }
    fprintf(Outfp,” %.16llf, %d \n”,sum1/log(sum1),count);
    }
fclose(Outfp);
return;
}
```

compute li(x)

```
#include <math.h>
#include <stdio.h>
void main () {
unsigned int h,MAXN;
int j;
double temp,x,f;
FILE *Outfp;
Outfp = fopen(”sortz.dat”,”w”);
if (Outfp==NULL)
    return;
fprintf(Outfp,” %d, %.16llf, \n”,2,1.045164);
for (h=2; h<=75000; h++) {
    MAXN=h*2;
    f=-1e+99;
    x=log(MAXN);
    temp=x-10;
    if (temp<0.0)
        temp=-temp;
    if (temp>=12.0)
        goto L2;
    if (x==0.0)
        goto L4;
    temp=x;
    if (temp<0.0)
        temp=-temp;
    j=(int)(10.0+2.0*temp);
    f=1.0/(double)((j+1)*(j+1));
    L1: f=(f*(double)j*x+1.0)/(double)(j*j);
    j=j-1;
    if (j!=0.0)
        goto L1;
```

```
    temp=x;
    if (temp<0.0)
        temp=-temp;
    f=f*x+log(1.781072418*temp);
    goto L4;
    L2: temp=x;
    if (temp<0.0)
        temp=-temp;
    j=(int)(5.0+20.0/temp);
    f=x;
    L3: f=1.0/(1.0/f-1.0/(double)j)+x;
    j=j-1;
    if (j!=0)
        goto L3;
    f=exp(x)/f;
    L4: printf(" %d %.16llf \n",MAXN,f);
    fprintf(Outfp," %d, %.16llf, \n",MAXN,f);
    }
fclose(Outfp);
return;
}
```

compute R(x)

```
#include <math.h>
#include <stdio.h>
#include "table5.h"
double li(double z);
int mobius(unsigned int a, unsigned int *table, unsigned int tsize);
unsigned int tsize=114155;
void main () {
unsigned int h,N,MAXN;
double sum,x,c;
FILE *Outfp;
Outfp = fopen("rx.dat","w");
if (Outfp==NULL)
    return;
for (h=1; h<=200000; h++) {
    x=(double)(h*2);
    c=2.0;
```

```
    N=1;
    while (x>c) {
        c=c*2.0;
        N=N+1;
        }
    MAXN=N;
    sum=0.0;
    for (N=1; N<=MAXN; N++)
        sum=sum+mobius(N,table,tsize)*li(pow(x,1.0/(double)N))/(double)N;
    printf(" %d %d %d %.16llf \n",2*h,(unsigned int)c,MAXN,sum);
    fprintf(Outfp," %.16llf, \n",sum);
    }
fclose(Outfp);
return;
}
```

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[9] J. E. Littlewood, "Sur la distribution des nombres premiers", *Comptes Rendus*, 1914, **158** : 1869-1872.