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CRITICAL$ve-$weight$m-$DOMINATION ON $S-$VALUED GRAPHS

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Abstract

A mixed domination in a $S-$valued graph is vertex dominating the edges with weight and vice versa. The mixed domination number is the minimum cardinality of any mixed dominating set. The notion of critical $ve-$weight $m-$domination on $S-$valued graphs was discussed here.

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1. INTRODUCTION

The theory of domination in graphs was initiated by Berge[1]. In 2015, Chandramouleeswaran et.all[7] introduced the notion of Semiring-valued graphs(simply called $S-$valued graphs). A mixed domination in a $S-$valued graph is vertex dominating edges with weight and vice versa. The mixed domination number is the minimum cardinality of any mixed dominating set. A graph is critical $ve-$weight$m-$dominating if the removal of any vertex decreases its domination number. This paper gives examples and properties of critical $ve-$weight$m-$dominating graphs and gives a method of constructing them and poses some open problems.

2. PRELIMINARIES

In this section, we recall some basic definitions that are needed for our work.

Definition 2.1 *[2] A semiring* $(S, +, \cdot)$ *is an algebraic system with a non-empty set*

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S together with two binary operations $+$ and $\cdot$ such that

1. *$(S, +, 0)$ is a monoid.*
2. *$(S, \cdot)$ is a semigroup.*
3. *For all $a, b, c \in S$, $a \cdot (b + c) = a \cdot b + a \cdot c$ and $(a + b) \cdot c = a \cdot c + b \cdot c$.*
4. *$0 \cdot x = x \cdot 0 = 0 \,\forall\, x \in S$.*

Definition 2.2 *[2] Let $(S, +, \cdot)$ be a semiring. A Canonical Pre-order $\preceq$ in S defined as follows : for $a, b \in S$, $a \preceq b$ if and only if, there exists an element $c \in S$ such that $a + c = b$.*

Definition 2.3 *[7] Let $G = (V, E \subset V \times V)$ be a given graph with $V, E \neq \phi$. For any semiring $(S, +, \cdot)$, a semiring-valued graph (or a $S-$valued graph), G^S, is defined to be the graph $G^S = (V, E, \sigma, \psi)$ where $\sigma : V \to S$ and $\psi : E \to S$ is defined to be*

$$\psi(x, y) = \begin{cases} min\,\{\sigma(x), \sigma(y)\} & if\ \sigma(x) \preceq \sigma(y) \text{ or } \sigma(y) \preceq \sigma(x) \\ 0 & otherwise \end{cases}$$

for every unordered pair (x, y) of $E \subset V \times V$. We call σ, a $S-$vertex set and ψ, a $S-$edge set of G^S.

Definition 2.4 *[6] A $S-$ valued graph $G^S = (V, E, \sigma, \psi)$ is said to be a $S-$ Star($S-$ Wheel) if its underlying graph G is a Star(Wheel) along with $S-$ values.*

Definition 2.5 *[3] Consider the $S-$valued graph $G^S = (V, E \subset V \times V, \sigma, \psi)$. The open neighbourhood of v_i in G^S is defined as the set*
$N_S(v_i) = \{(v_j, \sigma(v_j)), where(v_i, v_j) \in E, \psi(v_i, v_j) \in S.\}$

Definition 2.6 *[3] The closed neighbourhood of v_i in $G^S = (V, E, \sigma, \psi)$ is defined to be the set $N_S[v_i] = N_S(v_i) \cup \{(v_i, \sigma(v_i))\}$*

Definition 2.7 *[4] Consider the $S-$valued graph $G^S = (V, E, \sigma, \psi)$, and a vertex $v \in V$. The vertex v is said to be a $ve-$ weight $m-$ dominating vertex of an edge e, if $e \in \langle N_S[v] \rangle$ such that $\psi(e) \preceq \sigma(v)$.*

Definition 2.8 *[4] Consider the $S-$valued graph $G^S = (V, E, \sigma, \psi)$.Let $D \subseteq V$. If every edge of G^S is weight $m-$dominated by any vertex in D, then D is said to be a $ve-$ weight $m-$ dominating set.*

Definition 2.9 *[5] Consider the $S-$valued graph $G^S = (V, E, \sigma, \psi)$. The vertex-edge mixed domination number of G^S denoted by $\gamma^S_{VE}(G^S)$ is defined by $\gamma^S_{VE}(G^S) = (|D|_S, |D|)$,where D is the minimal $ve-$weight $m-$dominating set.*

3. CRITICAL$ve-$weight$m-$DOMINATION ON $S-$VALUED GRAPHS

In this section, we introduce the notion of critical vertex - edge mixed domination on $S-$valued graphs, analogous to the notion in crisp graph theory, and prove some simple results.

Definition 3.1 *A vertex $v \in V$ in the $S-$valued graph $G^S = (V, E, \sigma, \psi)$, is said to be critical $ve-$ weight $m-$ dominating vertex, if removal of the vertex v from G^S decreases its domination number $\gamma^S_{VE}(G^S)$.*

Example 3.2 Let $(S = \{0, a, b, c\}, +, \cdot)$ be a semiring with the following Cayley Tables:

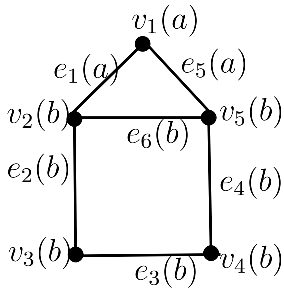
$+$	0	a	b	c
0	0	a	b	c
a	a	a	b	c
b	b	b	b	b
c	c	c	b	c

$\cdot$	0	a	b	c
0	0	0	0	0
a	0	0	a	0
b	0	a	b	c
c	0	0	c	c

Let $\preceq$ be a canonical pre-order in S, given by

$$0 \preceq 0,\ 0 \preceq a,\ 0 \preceq b,\ 0 \preceq c,\ a \preceq a,\ a \preceq b,\ a \preceq c,\ b \preceq b,\ c \preceq b,\ c \preceq c$$

Consider the $S-$ valued graph G^S :



Define $\sigma : V \to S$ by

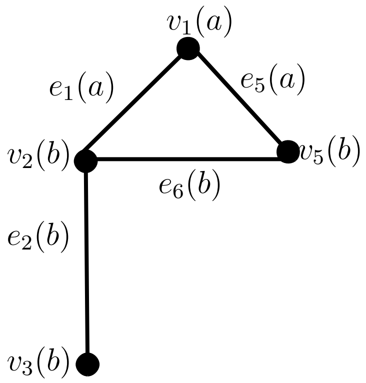
$$\sigma(v_1) = a, \sigma(v_2) = \sigma(v_3) = \sigma(v_4) = \sigma(v_5) = b$$

and $\psi : E \to S$ by

$$\psi(e_1) = \psi(e_5) = a, \psi(e_2) = \psi(e_3) = \psi(e_4) = \psi(e_6) = b$$

Consider a set D_1 =$\{v_2, v_4\}$
Here $N_S[v_2] = \{(v_1, a), (v_2, b), (v_3, b), (v_5, b)\}$.

And the edges $e_1, e_2, e_5 and e_6 \in \langle N_S[v_2] \rangle$,
also $\psi(e_1) = a \preceq b = \sigma(v_2)$
$\psi(e_2) = b \preceq b = \sigma(v_2)$
$\psi(e_5) = a \preceq b = \sigma(v_2)$
$\psi(e_6) = b \preceq b = \sigma(v_2)$
$\therefore$The vertex v_2 is a $ve-$weight $m-$dominating vertex of the edges e_1, e_2, e_5ande_6.
Here $N_S[v_4] = \{(v_3, b), (v_4, b), (v_5, b)\}$.
And the edges $e_3, e_4 \in \langle N_S[v_4] \rangle$,
also $\psi(e_3) = b \preceq b = \sigma(v_4)$
$\psi(e_4) = b \preceq b = \sigma(v_4)$
$\therefore$ The vertex v_4 is a $ve-$weight $m-$dominating vertex of the edges $e_3$ande_4.
Hence all the edges of G^S are $m-$dominated by a vertex in D_1 =$\{v_2, v_4\}$, and therefore D_1 is a$ve-$weight $m-$dominating set.
Similarly D_2=$\{v_3, v_5\}$, D_3 =$\{v_2, v_3, v_4\}$,D_4 =$\{v_2, v_4, v_5\}$, D_5=$\{v_2, v_3, v_5\}$, D_6=$\{v_3, v_4, v_5\}$, D_7=$\{v_2, v_3, v_4, v_5\}$, are all $ve-$weight $m-$dominating sets.
Also $D_1 = \{v_2, v_4\}$, D_2 $\{v_3, v_5\}$ are the minimal $ve-$ weight $m-$ dominating sets.
$\therefore$ The vertex-edge mixed domination number of G^S is $\gamma^S_{VE}(G^S) = (|D_1|_S, |D_1|) = (|D_2|_S, |D_2|) = (b, 2)$.



Here the removal of the vertex v_4, (given in the above figure) decreases the vertex-edge mixed domination number to $\gamma^S_{VE}(G^S) = (b, 1)$, and therefore the vertex v_4 is a critical $ve-$weight $m-$dominating vertices.
Similarly the vertices v_2,v_3,v_5 are critical $ve-$weight $m-$dominating vertices, since the removal of any of these vertices will reduce the vertex-edge mixed domination number to $\gamma^S_{VE}(G^S) = (b, 1)$.

Definition 3.3 *The $S-$valued graph $G^S = (V, E, \sigma, \psi)$ is said to be critical $ve-$weight $m-$dominating graph, if every vertex of G^S is a critical $ve-$weight $m-$dominating vertex.*

Example 3.4

1. For example, every $S-$regular Complete graphK_n^S is a critical $ve-$weight $m-$dominating graph.

2. Any $S-$regular Star graphS_n^S, cannot be a critical $ve-$weight $m-$dominating graph. Since removal of any vertex, except the pole will not decrease its domination number and suppose if the pole is removed then the graph does not exist.

3. Similar to the above case, a $S-$regular Wheel graphW_n^S, cannot be a critical $ve-$weight $m-$dominating graph.

4. Any $S-$regular Complete Bipartite graph $K_{m,n}^S$ is not a critical $ve-$weight $m-$dominating graph. Since the removal of any vertex will not decrease its domination number.

Theorem 3.5 *If G^S is a critical $ve-$weight $m-$dominating graph with e^S edges, then* $p^S \leq (2e^S + 3\gamma^S - \Delta^S)/3$.

Proof : Let us assume G^S has no isolated vertices. Then every vertex will have atleast two adjacent vertices. Let v be a vertex of maximum degree Δ^S and $D = D_v \cup \{v\}$ be a minimum dominating set of G^S. Then each vertex of $N^S(v)$ has atleast two edges incident to the vertices in D(i.e. one incident to v and one to some vertex in D_v.) Each of the remaining vertices in $V - D$ has atleast one more edge incident to vertices in D_v for the total of $p^S - \gamma^S - \Delta^S$ more edges. Since the minimum degree is atleast two, this latter set of vertices contributes atleast $(p^S - \gamma^S - \Delta^S)/2$ more edges. Therefore, $e^S \geq 2\Delta^S + (p^S - \gamma^S - \Delta^S) + (p^S - \gamma^S - \Delta^S)/2$, from which the result follows.
Now suppose G^S has t^S isolated vertices, then $p^S - t^S \leq [2e^S + 3(\gamma^S - t^S) - \Delta^S]/3 = (2e^S + 3\gamma^S - \Delta^S)/3 - t^S$.
Lemma : The $S-$valued graph $H^S.G^S$is a critical $ve-$weight $m-$dominating graph if and only if both H^S and G^S are critical $ve-$weight $m-$dominating graphs.

Theorem 3.6 *If G^S is a critical $ve-$weight $m-$dominating graph if and only if each block of G^S is a critical $ve-$weight $m-$dominating graph. Also, if G^S is a critical $ve-$weight $m-$dominating graph with blocks $G_1^S, G_2^S...G_n^S$ then*

Proof : We apply induction hypothesis on the number of blocks n. The statement being trivial if n=1. Let us assume the statement is true for n blocks.
Then let us consider a $S-$valued graph G^S with blocks $G_1^S, G_2^S...G_n^S, G_{n+1}^S$ such that G_{n+1}^S contains only one cut vertex of G^S.

Then $G^S = H^S.G^S_{n+1}$, where H^S is the subgraph composed of blocks $G^S_1, G^S_2...G^S_n$. Hence the result follows from above Lemma and the induction hypothesis.

Critical $ve-$weight $m-$Domination in Networks:

Critical $ve-$weight $m-$dominating graphs can be used to model multiprocessor networks in which a subset of processors (represented by a $ve-$weight $m-$dominating set) can transmit messages to all remaining processors in a single time unit, i.e. the time for a message to cross a communication line. Such networks have the pleasant characteristics that any processor can be in a minimum set of these ”dominating” processors and the failure of any processor leaves a network which requires one fewer dominating prosessors.

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