Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 21, Number 3 (2025), pp. 319-332

http://www.ripublication.com/gjpam.htm

Generalized Common Fixed Point theorem for Hybrid Mapping in Modified Fuzzy Metric Spaces with Application

Prachi Singh[1], **Jaynendra Shrivas**[*2] **and Heena Soni**[3]

[1,2,3]*Department of Mathematics, Govt. V.Y.T. PG Autonomous College, Durg (C.G.) India 491001*

Email: prachibksingh@gmail.com; jayshrivas95@gmail.com; heenasoni349@gmail.com

Abstract

In this paper, we prove common fixed point theorems in fuzzy metric spaces. We modify the classical definition of $R-$weakly commuting mappings by incorporating an altering distance function and prove several fixed point theorems under more relaxed compatibility conditions. Examples are provided to illustrate the applicability of our results and to demonstrate the independence of introduced conditions.

Keywords: Common fixed point theorems, fuzzy metric space, $R-$weakly commuting mappings, altering distance function, hybrid $R-$commuting mapping.

2010 Mathematics Subject Classification. 47H10, 54H25.

1. INTRODUCTION

The idea of a fuzzy set was first proposed by Zadeh [11] in 1965 as a novel technique to depict ambiguity in daily life. However, the idea of a fuzzy metric space appears to be more appropriate when the uncertainty results from fuzziness rather than randomization, as occurs occasionally when measuring an ordinary length. They can be separated into two groups listed below: The first class of results includes those that consider a fuzzy metric on a set X as a map, where X is the sum of all fuzzy points in the set, and meet certain axioms that are comparable to those of the ordinary metric.

[*]Corresponding Author.

Thus, in such an approach, numerical distances are set up between fuzzy objects. On the other hand, in the second group, we keep those results in which the distance between objects is fuzzy and the objects themselves may or may not be fuzzy.

Following the seminal work of Kramosil and Michalek [4], various notions such as weak compatibility and $R-$weak commutativity have been studied in fuzzy metric settings [1, 3, 4, 5, 6, 7, 8, 9, 10]. Inspired by the work of Manro et al. [6], this paper seeks to advance the theory by introducing hybrid mappings and altering distance functions, leading to broader fixed point results under weaker assumptions.

2. PRELIMINARIES

Now, we begin with some basic definitions.

Definition 2.1. A binary operation $*\colon [0,1]\times[0,1]\to[0,1]$ is said to be continuous $t-$norm if $*$ satisfies the following conditions:

(i) $*$ is commutative and associative;

(ii) $*$ is continuous;

(iii) $a*1=a,\ \forall a\in[0,1]$;

(iv) $a*b\leq c*d$ whenever $a\leq c$ and $b\leq d\,\forall a,b,c,d\in[0,1]$.

Example 2.2. $a*b=min\{a,b\}$ and $a*b=a.b$ are $t-$norms.

Kramosil and Michalek [4], introduced by the concept of fuzzy metric spaces as follows;

Definition 2.3. A 3-tuple $(X,M,*)$ is said to be fuzzy metric space (shortly, FM- space) if X is an arbitrary set, $*$ is a continuous $t-$norm and M is a fuzzy set in $X^2\times[0,\infty)$ satisfying the following conditions:

(i) $M(x,y,0)=0$,

(ii) $M(x,y,t)=1,\ \forall\, t>0$ if and only if $x=y$,

(iii) $M(x,y,t)=M(y,x,t)$, (Symmetry)

(iv) $M(x,y,t)*M(y,z,s)\leq M(x,z,t+s), \forall x,y,z\in X$ and $s,t>0$,

(v) $M(x,y,.)\colon [0,1)\to[0,1]$ is left continuous.

Note that $M(x, y, t)$ can be thought of as the degree of proximity between x and y with respect to t.

We can fuzzify examples of metric spaces into fuzzy metric spaces in a natural way: Let $X = \mathbb{R}$ (the set of all real numbers). Define $*$ as the usual product $a * b = a \times b$ (which is a continuous $t-$norm) and the fuzzy metric M by

$$M(x, y, t) = e^{\frac{-|x-y|}{t}}.$$

Let's check the condition:

(i) $M(x, y, 0) = e^{-\infty} = 0,$

(ii) $M(x, y, t) = 1,\ \forall\, t > 0$ iff $x = y$ because if $x = y,\ |x - y|$ and $e^0 = 1,$

(iii) $M(x, y, t) = e^{\frac{-|x-y|}{t}} = e^{\frac{-|y-x|}{t}} = M(y, x, t)$ so symmetry holds,

(iv) $M(x, z, t + s) = e^{\frac{-|x-z|}{t+s}},$
$M(x, y, t) * M(y, z, s) = e^{\frac{-|x-y|}{t}}.e^{\frac{-|y-z|}{s}} = e^{\left(\frac{-|x-y|}{t}+\frac{-|y-z|}{s}\right)},$
$\because\ |x - z| \leq |x - y| + |y - z|,$
$\therefore\ e^{\frac{-|x-z|}{t+s}} \geq e^{-\left(\frac{|x-y|}{t}+\frac{|y-z|}{s}\right)}$, so the triangular inequality holds.

(v) For fixed x, y the function $t \mapsto M(x, y, t) = e^{\frac{-|x-y|}{t}}$ is continuous from $(0, \infty)$ to $(0, 1]$, and the extension to $[0, \infty)$ by setting $M(x, y, 0) = 0$ is left continuous at $t = 0$.

Definition 2.4. Let $(X, M, *)$ be a fuzzy metric space. Then

(a) a sequence $\{x_n\}$ in X is said to

  (i) be a Cauchy sequence if
  $\lim_{n\to\infty} M(x_{n+p}, x_n, t) = 1,\ \forall t > 0$ and $p \in \mathbb{N}$,

  (ii) be convergent to a point $x \in X$
  if $\lim_{n\to\infty} M(x_n, x, t) = 1,\ \forall t > 0.$

(b) X is said to be complete if every Cauchy sequence in X converges to some point in X.

Example 2.5. Let $X = [0, 1]$ and let $*$ be the continuous $t-$norm defined by $a * b = ab,\ \forall a, b \in [0, 1]$. For each $t \in [0, \infty)$ and $x, y \in X$, define

$$M(x, y, t) = \begin{cases} \frac{t}{t+|x-y|^2}, & if\ t > 0 \\ 0, & if\ t = 0. \end{cases}$$

Clearly, $(X, M, *)$ is a complete fuzzy metric space.

Definition 2.6. A pair of self mappings (f, g) of a fuzzy metric space $(X, M, *)$ is said to be commuting if $M(fgx, gfx, t) = 1,\ \forall x \in X.$

Definition 2.7. A pair of self mappings (f, g) of a fuzzy metric space $(X, M, *)$ is said to be weakly commuting if $M(fgx, gfx, t) \geq M(fx, gx, t),\ \forall x \in X$ and $t > 0.$

In 1994, Mishra et al. [5] introduced the concept of compatible mapping in fuzzy metric space, analogous to the concept of compatible mapping in metric space as follows:

Definition 2.8. A pair of self mappings (f, g) of a fuzzy metric space $(X, M, *)$ is said to be compatible if $\lim_{n\to\infty} M(fgx_n, gfx_n, t) = 1,\ \forall t > 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = u$ for some $u \in X$.

In 1994, Pant [7] introduced the notion of $R-$weakly commuting maps in metric spaces. Later, Vasuki [10] initiated the concept of non-compatibility of mappings in fuzzy metric spaces by introducing the notion of $R-$weakly commuting mappings in fuzzy metric spaces and proved some common fixed point theorems for these maps.

Definition 2.9. A pair of self mappings (f, g) of a fuzzy metric space $(X, M, *)$ is said to be $R-$weakly commuting if there exists some $R > 0$ such that

$$M(fgx, gfx, t) \geq M(fx, gx, \frac{t}{R}),$$

for all $x \in X$ and $t > 0.$

Later on, Pathak et al. [9] improved the notion of $R-$weakly commuting mappings in metric spaces by introducing the notions of $R-$weakly commutativity of type (A_g) and $R-$weakly commutativity of type (A_f). In 2008, Imdad and Ali [1] embarked on the notion of $R-$weakly commutativity of type (A_g) and $R-$weakly commutativity of type (A_f) in fuzzy metric with inspiration from Pathak et al. [9] and they further introduced the notion of $R-$weakly commuting mappings of type (P) in fuzzy metric spaces.

Definition 2.10. A pair of self mappings (f, g) of a fuzzy metric space $(X, M, *)$ is said to be $R-$weakly commuting of type (P) if there exists some $R > 0$ such that

$$M(ffx, ggx, t) \geq M(fx, gx, \frac{t}{R}),$$

for all $x \in X$ and $t > 0.$

Definition 2.11. (**Altering distance function**)[2]: A function $\psi\colon [0, \infty) \to [0, \infty)$ is called an altering distance function if the following conditions are satisfied

(i) $\psi(u)$ is continuous and non decreasing.

(ii) $\psi(t) = 0$ iff $t = 0$.

Inspired by the work of Manro et al. [6], this paper seeks to advance the theory by introducing hybrid mappings and altering distance functions, leading to wider fixed point results under weaker assumptions.

Definition 2.12. (Hybrid $R-$weakly commuting mapping): A pair of self mappings (f, g) of a fuzzy metric space $(X, M, *)$ is said to be hybrid $R-$weakly commuting with respect to ψ (altering distance function) if there exists some $R > 0$ such that

$$\psi(M(fgx, gfx, t)) \geq \psi(M(fx, gx, \frac{t}{R})),$$

for all $x \in X$ and $t > 0$.

Example 2.13. Let $X = [0, 1]$ and define two mappings f and g as follows;

$$f(x) = x + \frac{1}{4}, \;\; for\ x \in [0, 1]$$

and

$$g(x) = x - \frac{1}{4}, \;\; for\ x \in [0, 1].$$

Let's choose the conditions of altering distance function. Consider $\psi(u) = \sqrt{u}$ for $u \in [0, 1]$ so that $\psi(0) = 0$ and $\psi(u) > 0$ for $u > 0$.

Now we check the hybrid $R-$weakly commuting condition as follows:

$$\psi(M(fg(x), gf(x), t)) \geq \psi(M(f(x), g(x), \frac{t}{R})),$$

for the distance function, $M(x, y, t) = |x - y| + t, \forall x, y \in X$ and $\forall t > 0$. So that

$$\begin{aligned} M(fg(x), gf(x), t) = |fg(x) - gf(x)| + t &= |x - x| + t, \\ &= t \end{aligned}$$

and

$$\begin{aligned} M(f(x), g(x), \frac{t}{R}) = |f(x) - g(x)| + \frac{t}{R} &= \left| \left(x + \frac{1}{4}\right) - \left(x - \frac{1}{4}\right) \right| + \frac{t}{R}, \\ &= \frac{1}{2} + \frac{t}{R}. \end{aligned}$$

Now, applying the altering distance function $\psi(u) = \sqrt{u}$, we get

$$\psi(M(fg(x), gf(x), t)) = \psi(t) = \sqrt{t}$$

and

$$\psi(M(f(x), g(x), \frac{t}{R})) = \psi(\frac{1}{2} + \frac{t}{R}) = \sqrt{\frac{1}{2} + \frac{t}{R}},$$

for the inequality,

$$\sqrt{t} \geq \sqrt{\frac{1}{2} + \frac{t}{R}},$$

i.e.,

$$R \geq \sqrt{\frac{t}{t - \frac{1}{2}}},$$

this will hold for sufficiently large R. For instance, choosing $R = 2$ the inequality holds for $t \geq 1$, making (f, g) hybrid $R-$weakly commuting with respect to ψ.

3. MAIN RESULTS

Theorem 3.1. *Let $(X, M, *)$ be a complete fuzzy metric space. Let $f, g\colon X \to X$ be self mappings satisfying*

(i) *f and g are hybrid $R-$weakly commuting with respect to same altering distance function ψ,*

(ii) *there exists $0 < k < 1$ such that*

$$\psi(M(gx, gy, kt)) \geq \psi(M(fx, fy, t)),\ \forall x, y \in X,$$

(iii) $g(X) \subseteq f(X)$.

Then f and g have a unique common fixed point.

Proof. Let $x_0 \in X$. Since $g(X) \subseteq f(X)$, choose $x_1 \in X$ such that $g(x_0) = f(x_1)$. In general, choose x_{n+1} such that $y_n = fx_{n+1} = gx_n$. Then by (ii), we have

$$\begin{aligned}\psi(M(fx_n, fx_{n+1}, t)) &= \psi(M(gx_{n-1}, gx_n, t)) \geq \psi(M(fx_{n-1}, fx_n, \frac{t}{k})),\\ &= \psi(M(gx_{n-2}, gx_{n-1}, \frac{t}{k})) \geq ... \geq \psi(M(fx_0, fx_1, \frac{t}{k^n})).\end{aligned}$$

Therefore, for any p,

$$\begin{aligned}\psi(M(fx_n, fx_{n+p}, t)) &\geq \psi(M(fx_n, fx_{n+1}, \frac{t}{p})) \geq ... \geq \psi(M(fx_{n+p-1}, fx_{n+p}, \frac{t}{p})),\\ &\geq \psi(M(fx_0, fx_1, \frac{t}{pk^n})) \geq ... \geq \psi(M(fx_0, fx_1, \frac{t}{pk^{n+p-1}})).\end{aligned}$$

As $n \to \infty$, $\{fx_n\} = \{y_n\}$ is a Cauchy sequence and so, by completeness of X, $\{y_n\} = \{fx_n\}$ is convergent. Call the limit z, then $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$. As $f(X)$ is complete, so there exists a point p in X such that $fp = z$. Now, from (ii),

$$\begin{aligned}&\psi(M(gp, gx_n, kt)) \geq \psi(M(fp, fx_n, t)),\ \forall t > 0,\\ &\lim_{n\to\infty} \psi(M(gp, gx_n, kt)) \geq \lim_{n\to\infty} \psi(M(fp, fx_n, t)),\\ &\psi(M(gp, z, kt)) \geq \psi(M(fp, z, t)),\\ &\psi(M(gp, z, kt)) \geq \psi(M(z, z, t)),\\ &\psi(M(gp, z, kt)) \geq \psi(1),\\ &M(gp, z, kt) = 1,\\ &gp = z = fp.\end{aligned}$$

Now, using hybrid $R-$weakly commuting condition

$$\begin{aligned}&\psi(M(fgp, gfp, t)) \geq \psi(M(fp, gp, \frac{t}{R})),\ \forall t > 0,\\ &\psi(M(fgp, gfp, t)) \geq \psi(1),\\ &M(fgp, gfp, t) = 1,\\ &fgp = gfp,\\ &gp = fp.\end{aligned}$$

Now, we show that z is a common fixed point of f and g. From (ii),

$$\begin{aligned}&\psi(M(gz, gx_n, kt)) \geq \psi(M(fz, fx_n, t)),\ \forall t > 0,\\ &\lim_{n\to\infty} \psi(M(gz, gx_n, kt)) \geq \lim_{n\to\infty} \psi(M(fz, fx_n, t)),\\ &\psi(M(gz, z, kt)) \geq \psi(M(fz, z, t)),\\ &\psi(M(gz, z, kt)) \geq \psi(M(gz, z, t)),\\ &\psi(M(gz, z, kt)) \geq \psi(1),\\ &M(gz, z, kt) = 1,\\ &gz = z = fz.\end{aligned}$$

Hence z is a common fixed point of f and g. For uniqueness, let w be another fixed point of f and g. Then by (ii),

$$\begin{aligned}
&\psi(M(gz, gw, kt)) \geq \psi(M(fz, fw, t)), \ \forall t > 0,\\
&\psi(M(z, w, kt)) \geq \psi(M(z, w, t)),\\
&M(z, w, kt) \geq M(z, w, t),\\
&z = w.
\end{aligned}$$

Therefore, z is a unique common fixed point of f and g. $\square$

Theorem 3.2. *Let $(X, M, *)$ be a complete fuzzy metric space. Let $f, g\colon X \to X$ be self mappings satisfying conditions (i), (ii) and (iii). If one of $g(X)$ or $f(X)$ is complete, then f and g have a unique common fixed point.*

Proof. From the proof of the above theorem, we conclude that $\{fx_n\} = \{y_n\}$ is a Cauchy sequence in X. Now, suppose that $f(X)$ is a complete subspace of X. Then the subsequence of $\{y_n\}$ must get a limit in $f(X)$. Call it u and $f(v) = u$. As $\{y_n\}$ is a Cauchy sequence that contains a convergent subsequence, the sequence $\{y_n\}$ also converges, which implies the convergence of a subsequence of the convergent sequence. Now, from (ii),

$$\begin{aligned}
&\psi(M(gv, gx_n, kt)) \geq \psi(M(fv, fx_n, t)), \ \forall t > 0,\\
&\lim_{n\to\infty} \psi(M(gv, gx_n, kt)) \geq \lim_{n\to\infty} \psi(M(fv, fx_n, t)),\\
&\psi(M(gv, u, kt)) \geq \psi(M(fv, u, t)),\\
&\psi(M(gv, u, kt)) \geq \psi(M(u, u, t)),\\
&\psi(M(gv, u, kt)) \geq \psi(1),\\
&M(gv, u, kt) = 1,\\
&gv = u = fv,
\end{aligned}$$

which shows that pair (f, g) has a point of coincidence. Since f and g are hybrid $R-$weakly commuting, $fgv = gfv$, so that $fu = gu$. Now, we show that u is a fixed point of f and g. From (ii),

$$\begin{aligned}
&\psi(M(gu, gx_n, kt)) \geq \psi(M(fu, fx_n, t)), \ \forall t > 0,\\
&\lim_{n\to\infty} \psi(M(gu, gx_n, kt)) \geq \lim_{n\to\infty} \psi(M(fu, fx_n, t)),\\
&\psi(M(gu, u, kt)) \geq \psi(M(fu, u, t)),\\
&\psi(M(gu, u, kt)) \geq \psi(M(gu, u, t)),\\
&M(gu, u, kt) \geq M(gu, u, kt),\\
&M(gu, u, kt) = 1,\\
&gu = u = fu.
\end{aligned}$$

So, u is a common fixed point of f and g. For uniqueness, let u^* be another fixed point of f and g. Then by (ii),

$$\begin{aligned}
&\psi(M(gu, gu^*, kt)) \geq \psi(M(fu, fu^*, t)), \ \forall t > 0, \\
&\psi(M(u, u^*, kt)) \geq \psi(M(u, u^*, t)), \\
&M(u, u^*, kt) \geq M(u, u^*, t), \\
&u = u^*.
\end{aligned}$$

Therefore, u is the unique common fixed point of f and g. □

Corollary 3.3. *Let $(X, M, *)$ be a complete fuzzy metric space. Let $f, g\colon X \to X$ be continuous self mappings satisfying*

(i) f *and* g *are weakly commuting,*

(ii) there exists $0 < k < 1$ *such that*
$M(gx, gy, kt) \geq M(fx, fy, t), \ \forall t > 0,$

(iii) $g(X) \subseteq f(X)$,

(iv) $f(X)$ *is closed.*

Then f *and* g *have a unique common fixed point.*

Corollary 3.4. *Let $(X, M, *)$ be a complete fuzzy metric space. Let $f, g\colon X \to X$ be self mappings satisfying*

(i) $f = g$ *(i.e., only one operator),*

(ii) there exists $0 < k < 1$ *such that*
$M(fx, fy, kt) \geq M(x, y, t), \ \forall x, y \in X,$

(iii) $f(X)$ *is complete.*

Then f *has a unique fixed point.*

4. NUMERICAL EXAMPLES

Example 4.1. Let $X = [0, 1]$ and define the fuzzy metric $M\colon X^2 \times (0, \infty) \to [0, 1]$ by

$$M(x, y, t) = \frac{t}{t + |x - y|},$$

then $(X, M, *)$ is a complete fuzzy metric space.

Define $f, g\colon X \to X$ as follows:

$$f(x) = \frac{x}{2},\ g(x) = \frac{x}{4}.$$

Then, $f(X) = [0, 0.5],\ g(X) = [0, 0.25]$ which implies that $g(X) \subseteq f(X)$.

Let's test the contractive condition using fuzzy metric space and $\psi(u) = u^2$ for $u \in [0, 1]$, which is a valid altering distance function. Choose $x = 0.5,\ y = 0.25,\ t = 1$ and $k = \frac{1}{2}$.

So that, $g(0.5) = 0.125,\ g(0.25) = 0.0625$ and $f(0.5) = 0.25,\ f(0.25) = 0.125$.

Left side:

$$\begin{aligned} M(gx, gy, kt) &= \frac{0.5}{0.5 + |0.125 - 0.0625|} \\ &= \frac{0.5}{0.5625} \\ &\approx 0.889 \implies \psi(M(gx, gy, kt)) \approx 0.789. \end{aligned}$$

Right side:

$$\begin{aligned} M(fx, fy, t) &= \frac{1}{1 + |0.25 - 0.125|} \\ &= \frac{1}{0.125} \\ &\approx 0.889 \implies \psi(M(fx, fy, t)) \approx 0.789. \end{aligned}$$

Thus,

$$\psi(M(gx, gy, kt)) \geq \psi(M(fx, fy, t)),$$

So, all the conditions of Theorem 3.1 are satisfied.

Hence, $f(x)$ and $g(x)$ satisfy all the conditions of Theorem 3.1. The unique common fixed point is clearly $x = 0$.

5. APPLICATIONS

Nonlinear integral equation via fuzzy fixed point theorem:

Consider the nonlinear integral equation

$$x(t) = \int_0^T K(t, s, x(s))ds,\ t \in [0, T].$$

- $x \in C([0, T])$, the Banach space of continuous real valued functions on $[0, T]$,
- $K\colon [0, T] \times [0, T] \times \mathbb{R} \to \mathbb{R}$ is a continuous possibly nonlinear kernel.

We aim to prove the existence and uniqueness of a solution $x(t)$ using the fuzzy metric space theory.

Let $X = C([0,T])$ and define the fuzzy metric $M(f,g,t)$ as follows:

$$M(f,g,t) = \frac{t}{t + ||f-g||_\infty},\ t > 0.$$

Define two operators

$$f(x)(t) = \int_0^T K(t,s,x(s))ds,\ t \in [0,T]$$

and

$$g(x)(t) = \int_0^T K(t,s,x(s))ds,\ t \in [0,T].$$

(So here, $f = g$ but we distinguish them for applying the fixed point theorem structure.) Suppose K satisfies the Lipschitz-type fuzzy contraction condition:

$$|K(t,s,x) - K(t,s,y)| \leq \phi(|x-y|),$$

for some increasing function $\phi\colon \mathbb{R}^+ \to \mathbb{R}^+$. This implies,

$$||f(x) - f(y)||_\infty \leq T.\phi(||x-y||_\infty),$$

$$M(f(x), f(y), t) \geq \frac{t}{t + T.\phi(||x-y||_\infty)},$$

we define an altering distance function $\psi(u) = u^2,\ u \in [0,1]$.

Now, we assume that

$$\psi(M(gx, gy, kt)) \geq \psi(M(fx, fy, t)),$$

and f is weakly compatible (even identical here). So the fuzzy fixed point theorem applies. By applying Theorem 3.1, we conclude that there exists a unique solution $x^*(t)$ to the nonlinear integral equation. The solution is stable under fuzzy perturbations (e.g., noisy or vague kernels).

Example 5.1. Consider the nonlinear Fredholm integral equation of the second kind,

$$x(t) = \int_0^1 \frac{1}{1 + (x(s))^2} ds,\ t \in [0,1],$$

where the kernel is:

$$K(t,s,x(s)) = \frac{1}{1 + (x(s))^2}.$$

Let $X = C([0,1])$, the space of continuous real valued functions on $[0,1]$, with the sup norm:

$$||x|| = \sup_{t\in[0,1]} |x(t)|.$$

Define the fuzzy metric:

$$M(x,y,t) = \frac{t}{t+||x-y||},\ t>0.$$

Define the integral operator $f\colon X \to X$ by

$$f(x)(t) = \int_0^1 \frac{1}{1+(x(s))^2}ds,\ t\in[0,1].$$

Note: This function is independent of t, so $f(x)(t) = c(x)$, a constant function.

For any $x \in X,\ x(s)$ is continuous and bounded on $[0,1]$, so

$$0 < \frac{1}{1+||x||^2} \leq f(x)(t) \leq 1.$$

Check the fuzzy contraction condition, let $x, y \in X$. Then

$$|f(x)(t) - f(y)(t)| = \left| \int_0^1 \left(\frac{1}{1+(x(s))^2} - \frac{1}{1+(y(s))^2} \right) ds \right|.$$

Use the inequality

$$\left| \frac{1}{1+x^2} - \frac{1}{1+y^2} \right| = \left| \frac{x^2-y^2}{(1+x^2)(1+y^2)} \right| \leq |x-y|. \left| \frac{x+y}{(1+x^2)(1+y^2)} \right|,$$

since $x(s), y(s) \in [-M, M]$ for some $M > 0$, we can bound the kernel difference

$$|K(x(s)) - K(y(s))| \leq C.|x(s) - y(s)|.$$

Then

$$\begin{aligned}
|f(x)(t) - f(y)(t)| &\leq \int_0^1 C.|x(s) - y(s)ds| \\
&\leq C.||x-y|| \\
&\implies ||f(x) - f(y)|| \leq C.||x-y||.
\end{aligned}$$

Choose $C < 1$, say $C = \frac{1}{2}$, then

$$\begin{aligned}
||f(x) - f(y)|| \leq \frac{1}{2}||x-y|| \implies M(f(x), f(y), t) &= \frac{t}{t+||f(x)-f(y)||} \\
&\geq \frac{t}{t+||x-y||} \\
&\geq M(x,y,t).
\end{aligned}$$

Now, apply altering distance function $\psi(u) = u^2$. Then

$$\psi(M(f(x), f(y), t)) \geq \psi(M(x, y, kt)).$$

All the hypotheses of Theorem 3.1 are satisfied:

- f is a self map on a complete fuzzy metric space,
- The contraction condition holds with respect $\psi(u) = u^2$,
- $f(X)$ is complete (since constant functions are continuous).

Therefore, the integral equation has a unique solution $x^* \in C([0, 1])$ such that

$$x^*(t) = \int_0^1 \frac{1}{1 + (x^*(s))^2} ds,$$

this solution is a constant function $x^*(t) = c^*$ and c^* satisfied

$$c^* = \int_0^1 \frac{1}{1 + (c^*)^2} ds \implies c^*(1 + (c^*)^2) = 1.$$

REFERENCES

[1] M. Imdad and J. Ali, Jungck's Common fixed point theorem and E.A property, *Acta Math. Sinica*, **24(1)**(2008) 87–94.

[2] M. S. Khan, M. Swaleh and S. Sessa, Fuzzy fixed point theorems by altering distances between the points, *Bull. Aust. Math. Soc.*, **30(1)**(1984) 1-9.

[3] S. Kumar and S. K. Garg, Expansion mapping theorems in metric spaces, *Int. J. Contemp. Math. Sci.*, **4**(2009) 1749–1758.

[4] I. Kramosil and J. Michalek, Fuzzy metric and statistical metric spaces, *Kybernetica*, **11**(1975) 326–334.

[5] S. N. Mishra, N. Sharma and S. L. Singh, Common fixed points of maps on fuzzy metric spaces, *Int. J. Math. Math. Sci.*, **17**(1994) 253–258.

[6] S. Manro, S. S. Bhatia and S. Kumar, Common fixed point theorems in fuzzy metric spaces, *Ann. Fuzzy Math. Inform.*, **3(1)**(2012) 151- 158.

[7] R. P. Pant, Common fixed points of non commuting mappings, *J. Math. Anal. Appl.*, **188**(1994) 436–440.

[8] R. P. Pant, A common fixed point theorem under a new condition, *Indian J. Pure Appl. Math.*, **30(2)**(1999) 147–152.

[9] H. K. Pathak, Y. J. Cho and S. M. Kang, Remarks on $R-$Weakly commuting mappings and common fixed point theorems, *Bull. Korean Math. Soc.*, **34(2)**(1997) 247–257.

[10] R. Vasuki, Common fixed points for $R-$weakly commuting maps in fuzzy metric spaces, *Indian J. Pure Appl. Math*, **30**(1999) 419–423.

[11] L. A. Zadeh, Fuzzy sets, *Inform. Control*, **89**(1965) 338–353.