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Common Fixed Point Theorems in B-Metric Spaces with Graph Structures

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Abstract

In this paper, we prove some common fixed point theorems in b-metric spaces that are connected with a directed graph. We use conditions on two mappings and the structure of the graph to show when they have a common fixed point.

Keywords: Weak commutative mappings; multiplicative metric space; common fixed points.

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1. INTRODUCTION

Fixed point theory is an important branch of mathematics with many applications in analysis and other areas. A fixed point is point of invariance of a mapping. In 1986, Jungck [6] introduced the concept of compatible mappings, which helped generalize weak commutativity. Later, he defined weakly compatible mappings [3], which are mappings that commute at their coincidence points. These ideas led to deeper results in fixed point theory.

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Czerwik [4] introduced the concept of b-metric spaces as a generalization of metric spaces, where the triangle inequality is relaxed using a constant. This allowed the development of new fixed point results in more general spaces. Aghajani et al. [1] extended these ideas by studying generalized weak contractive mappings in partially ordered b-metric spaces.

The use of graphs in fixed point theory was proposed by Echenique, who gave a constructive proof of Tarski's fixed point theorem using graphs. Graph theory, as explained in books by Gross and Yellen [5] and Pathak and Chouhan [8], plays a useful role in connecting elements of a set through edges and paths, which can be either directed or undirected.

Akram et al. [2] and Ozturk and Girgin [7] have also contributed to the study of fixed point theorems involving graphs, showing how graph structures can be used to prove the existence of fixed points under certain conditions. Mustafa et al. [9] presented common fixed point results in b-metric spaces using graphs, which inspired further work in this direction.

In this paper, we prove some common fixed point theorems in b-metric spaces equipped with a directed graph. We also discuss conditions under which mappings have a unique common fixed point, extending the results of previous studies.

2. PRELIMINARIES

Definition 2.1. [4] Let X be a nonempty set. A function $d : X \times X \to R^+$ satisfying the following conditions:
(i) $d(x, y) \geq 0$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(iii) $d(x, y) \leq d(x, z) + d(z, y)\ \forall\ x, y \in X$,

Definition 2.2. [4] A b-metric space is a generalization of a metric space, where the triangle inequality is relaxed using a positive constant $s \geq 1$.
A function $d : X \times X \to \mathbb{R}^+$ is called a b-metric if it satisfies:
(B1) $d(x, y) = 0$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;
(B2) $d(x, y) = d(y, x),\ \forall\ x, y \in X$,
(B3) $d(x, z) \leq s[d(x, y) + d(y, z)], \forall\ x,\ y \in X$.

Definition 2.3. [3] Let T and G be a self mappings of a set X. If $Tv = Gv = v$ for some $v \in X$, then v is called a fixed point of T and G, and v is called a fixed point of T and G.

Definition 2.4. [6] Let (X, d) be a metric space and (x_n) be a sequence in X. The sequence is called a Cauchy sequence if it holds that for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $d(x_n, x_m) < \epsilon$ for all $m, n > N$.

Definition 2.5. [6] We call a metric space X complete if every Cauchy sequence in it is convergence to $x \in X$.

Definition 2.6. The mappings $T, g : X \to X$ are weakly compatible if they commute at their coincidence point, i.e; $T(gx) = g(Tx)$ whenever $gx = Tx$.

Definition 2.7. [5] Given a diagraph G. Then a path from the vertex x to the vertex y in G of length $n(n \in N)$ is a sequence $(x_i)_{i=0}^{n}$ of $n+1$ vertices such that $x_0 = x, x_n = y$ and $(x_{i-1}, x_i) \in \ E(G)\ for\ i = 1, 2, ...n$.

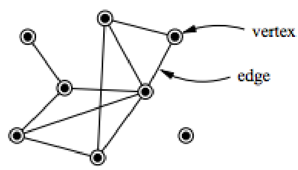

Figure 1: Graph

Definition 2.8. [8] A graph $G = (V, E)$ consists of a set of objects $V = \{v_1, v_2, ...\}$, whose elements are called vertices and an another set $E = e_1, e_2, ...$, whose element are called edges, $e_k = (v_i, v_j)$.

Definition 2.9. [8] A graph consist the direction of edges then it is called directed graph.

Definition 2.10. [8] A graph which have no direction then this is called undirected graph.

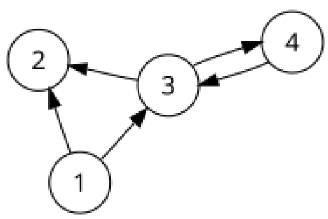

Figure 2: Directed graph

Proposition 2.11. *[9] Let T and g be weakly compatible self maps of a non-empty set X. If T and g have a unique point of coincidence $u = Tv = gv$, Then u is the unique common fixed point of T and g.*

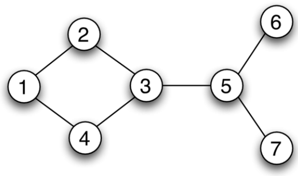

Figure 3: Undirected graph

3. MAIN RESULTS

In this section we will present some common fixed point results in b-metric spaces with a graph:

Theorem 3.1. *Let (X, d) be a b-metric space (with $s \geq 1$) endowed with a graph G and the mappings $T, g : X \to X$ satisfy*

$$d(Tx, Ty) \leq \lambda d(gx, Tx) + \mu[max\{d(gy, Ty), d(gx, Ty).d(gy, Tx), d(gx, gy)\}] \tag{3.1}$$

for all $x, y \in X$ with $(gx, gy) \in E(G)$

1. *Suppose $T(X) \subseteq g(X)$ and $g(X)$ is complete subspace of X. Then*

2. *If $GgT \neq \phi$ and the property $G_{(T,gx_n)}$ is satisfied, then T and g have a point of coincidence in X.*

3. *Moreover if x and y are point of coincidence of T and g in X implies $(x, y) \in E(G)$ then T and g have a unique point in X.*

4. *Further if T and g are weakly compatible then, T and g have a unique common fixed point in X.*

Proof. Assume that $GgT \neq \varnothing$. Then there exists $x_0 \in GgT$. Since $T(X) \subseteq g(X)$, there exists $x_1 \in X$ such that

$$gx_1 = Tx_0.$$

Again, we can find $x_2 \in X$ such that

$$gx_2 = Tx_1.$$

Continuing in this manner, we construct a sequence (gx_n) such that

$$gx_n = Tx_{n-1}, \quad n = 1, 2, 3, \ldots$$

and $(gx_n, gx_m) \in E(G)$ for all m, n.

Suppose that $gx_n = gx_{n+1}$ for some $n \in \mathbb{N}$. Then $gx_n = Tx_n$, which implies that x_n is a coincidence point.

Therefore, we may assume that

$$gx_n \neq gx_{n+1}, \quad \forall\, n \in \mathbb{N}.$$

Now we show that the sequence (gx_n) is Cauchy in $g(X)$. For this, put $x = x_{n-1}$ and $y = x_n$. Since $gx_n = Tx_{n-1}$, applying condition (1) for $n \in \mathbb{N}$, we obtain:

$$\begin{aligned}
d(gx_n, gx_{n+1}) &= d(Tx_{n-1}, Tx_n) \\
&\leq \lambda d(gx_{n-1}, Tx_{n-1}) + \mu\, [max\{d(gx_n, Tx_n), d(gx_{n-1}, Tx_n), \\
&\quad d(gx_n, Tx_{n-1}), d(gx_{n-1}, gx_n)\}] \\
&= \lambda d(gx_{n-1}, gx_n) + \mu[max\{d(gx_n, gx_{n+1}), d(gx_{n-1}, gx_{n+1}), \\
&\quad d(gx_n, gx_n), d(gx_{n-1}, gx_n)\}].
\end{aligned} \tag{3.2}$$

By (B3) of b-metric axioms, we have

$$d(gx_{n-1}, gx_{n+1}) \leq s\, [d(gx_{n-1}, gx_n) + d(gx_n, gx_{n+1})],$$

from (3.2)

$$\begin{aligned}
d(gx_n, gx_{n+1}) &\leq \lambda\, d(gx_{n-1}, gx_n) + \mu \cdot \max\{d(gx_n, gx_{n+1}), \\
&\quad s\,(d(gx_{n-1}, gx_n) + d(gx_n, gx_{n+1}))\,,\ d(gx_n, gx_n),\ d(gx_{n-1}, gx_n)\} \\
&= \lambda\, d(gx_{n-1}, gx_n) + \mu \cdot \max\{d(gx_n, gx_{n+1}), \\
&\quad s\,(d(gx_{n-1}, gx_n) + d(gx_n, gx_{n+1}))\,,\ 0,\ d(gx_{n-1}, gx_n)\}\,.
\end{aligned} \tag{3.3}$$

From inequality (3.3) following cases arises-
Case(I) If max = $d(gx_n, gx_{n+1})$ then from (3.3) ,

$$\begin{aligned}
d(gx_n, gx_{n+1}) &\leq \lambda\, d(gx_{n-1}, gx_n) + \mu s\, d(gx_n, gx_{n+1}) \\
\text{or} \quad (1 - \mu s)\, d(gx_n, gx_{n+1}) &\leq \lambda\, d(gx_{n-1}, gx_n) \\
\implies d(gx_n, gx_{n+1}) &\leq \beta\, d(gx_{n-1}, gx_n),
\end{aligned} \tag{3.4}$$

where $\beta = \frac{\lambda}{1-\mu s}$, continuing this process

$$\begin{aligned} d(gx_n, gx_{n+1}) &\leq \beta^2\, d(gx_{n-2}, gx_{n-1}) \\ &\leq \cdots \\ &\leq \beta^n\, d(gx_0, gx_1). \end{aligned} \tag{3.5}$$

Now, for $m, n \in \mathbb{N}$, such that $m > n$, and by (B3) of b-metric axioms, we have

$$\begin{aligned} d(gx_n, gx_m) &\leq sd(gx_n, gx_{n+1}) + sd(gx_{n+1}, gx_m) \\ &\leq sd(gx_n, gx_{n+1}) + s^2 d(gx_{n+1}, gx_{n+2}) + s^2 d(gx_{n+2}, gx_m) \\ &\leq sd(gx_n, gx_{n+1}) + s^2 d(gx_{n+1}, gx_{n+2}) + s^3 d(gx_{n+2}, gx_{n+3}) + \\ &\qquad ... + s^{m-n-1} d(gx_{m-2}, gx_{m-1}) + s^{m-n-1} d(gx_{m-1}, gx_m) \\ &\leq sd(gx_n, gx_{n+1}) + s^2 d(gx_{n+1}, gx_{n+2}) + s^3 d(gx_{n+2}, gx_{n+3}) + \\ &... + s^{m-n-1} d(gx_{m-2}, gx_{m-1}) + s^{m-n} d(gx_{m-1}, gx_m) \; as \; s \geq 1. \end{aligned} \tag{3.6}$$

Hence, equ (3.4) and (3.6) gives,

$$\begin{aligned} d(gx_n, gx_m) &\leq s\beta^n d(gx_0, gx_1) + s^2\beta^{n+1} d(gx_0, gx_1) + s^3\beta^{n+2} d(gx_0, gx_1) + \cdots \\ &\quad + s^{m-n}\beta^{m-1} d(gx_0, gx_1) \\ &= s\beta^n \left[1 + (s\beta) + (s\beta)^2 + \cdots + (s\beta)^{m-n-1}\right] d(gx_0, gx_1) \\ &\leq s\beta^n \sum_{i=0}^{\infty} (s\beta)^i\, d(gx_0, gx_1) \\ &= \frac{s\beta^n}{1 - s\beta}\, d(gx_0, gx_1) \to 0 \; as \; n \to \infty. \end{aligned} \tag{3.7}$$

Therefore (gx_n) is Cauchy sequence and by completeness of $g(X)$, there is $u \in g(X)$, such that $(gx_n) \to u = g(v)$, for some $v \in X$.

Case(2): If $max = sd(gx_{n-1}, gx_n)$, then from (3.3) ,

$$\begin{aligned} d(gx_n, gx_{n+1}) &\leq \lambda d(gx_{n-1}, gx_n) + \mu s d(gx_{n-1}, gx_n) \\ &\leq \beta d(gx_{n-1}, gx_n), \end{aligned}$$

taking $\lambda + \mu s \leq \beta$, where $\beta = \frac{\lambda}{1-\mu s}$ continuing this process

$$
\begin{aligned}
d(gx_n, gx_{n+1}) &\leq \beta^2\, d(gx_{n-2}, gx_{n-1}) \\
&\vdots \\
&\leq \beta^n\, d(gx_0, gx_1) \\
&= \beta^n\, d(gx_0, gx_1) \xrightarrow[n\to\infty]{} 0.
\end{aligned} \tag{3.8}
$$

After this, we will write the equation (3.5) to (3.7) and the Cauchy sequence will be proved.

Case(3): If $max = sd(gx_n, gx_{n+1})$

$$
\begin{aligned}
d(gx_n, gx_{n+1}) &\leq \lambda d(gx_{n-1}, gx_n) + \mu[sd(gx_n, gx_{n+1})] \\
(1 - \mu s)\, d(gx_n, gx_{n+1}) &\leq\ \lambda d(gx_{n-1}, gx_n) \\
d(gx_n, gx_{n+1}) &\leq \frac{\lambda}{1 - \mu s} d(gx_{n-1}, gx_n)
\end{aligned}
$$

taking $\lambda + \mu \geq 1$,

$d(gx_n, gx_{n+1}) \leq \beta d(gx_{n-1}, gx_n)$, where $\beta = \frac{\lambda}{1-\mu s}$.

Continuing this process

$$
\begin{aligned}
d(gx_n, gx_{n+1}) &\leq \beta^2\, d(gx_{n-2}, gx_{n-1}) \\
&\vdots \\
&\leq \beta^n\, d(gx_0, gx_1) \xrightarrow[n\to\infty]{} 0.
\end{aligned} \tag{3.9}
$$

After this, we will write the equation (3.5) to (3.7) and the Cauchy sequence will be proved.

case(4): If $max = d(gx_{n-1}, gx_n)$, then from equation (3),

$$
\begin{aligned}
&d(gx_n, gx_{n+1}) \leq \lambda d(gx_{n-1}, gx_n) + \mu[d(gx_{n-1}, gx_n)] \\
&d(gx_n, gx_{n+1} \leq (\lambda + \mu) d(gx_{n-1}, gx_n) \\
&d(gx_n, gx_{n+1}) \leq \beta . d(gx_{n-1}, gx_n), \qquad here\ \beta = \lambda + \mu \\
&d(gx_n, gx_{n+1}) \leq \beta^n d(gx_0, gx_1)
\end{aligned} \tag{3.10}
$$

Therefore (gx_n) is a Cauchy sequence and by completeness of $g(X)$ there is $u \in g(X)$, such that $(gx_n) \to u = g(v)$, for some $v \in X$.

As, $x_0 \in GgT$, then $(gx_n, gx_m) \in E(G)$, for $m,\ n = 1,\ 2,\ 3....$ and so $(gx_n,\ gx_{n+1}) \in E(G)$. By property $G(_T,\ gx_n)$ sequence of undirected edge property there is a subsequence (gx_{ni}) of (gx_n), such that $(gx_{ni},\ u) \in E(G)$.

Now, by using $(B3)$, we have

$$d(Tv, gv) \leq sd(Tv, Tx_{ni}) + sd(Tx_{ni}, gv) \tag{3.11}$$

Now, by using $(B3)$, we have

$$d(Tv, gv) \leq sd(Tv, Tx_{ni}) + sd(Tx_{ni}, gv) \tag{3.12}$$

But condition (1) implies,
$d(Tv, Tx_{ni}) \leq$
$\lambda d(gv, Tv) + \mu \, max[d(gx_{ni}, Tx_{ni}), d(gv, Tx_{ni}), d(gx_{ni}, Tv), d(gv, gx_{ni})]$
Thus (3.12) becomes,

$$\begin{aligned} d(Tv, gv) \leq s\lambda d(gv, Tv) + s\mu \, max[d(gx_{ni}, Tx_{ni}), d(gv, Tx_{ni}), \\ d(gx_{ni}, Tv), d(gv, gx_{ni})] + sd(Tx_{ni}, gv) \end{aligned} \tag{3.13}$$

Replacing Tx_{ni} by gx_{ni+1} in (3.13), we get,

$$\begin{aligned} d(Tv, gv) \leq s\lambda d(gv, Tv) + s\mu \, max\{d(gx_{ni}, gx_{ni+1}), d(gv, gx_{ni+1}), \\ d(gx_{ni}, gv), d(gv, gx_{ni})\}] + sd(gx_{ni+1}, gv) \end{aligned} \tag{3.14}$$

If

$$\max = d(gx_{n_i}, gx_{n_i+1}),$$

then from equation (3.14) we have

$$d(Tv, gv) \leq s\lambda \, d(gv, Tv) + s\mu \, d(gx_n, gx_{n_i+1}) + s \, d(gx_{n_i+1}, gv).$$

This can be rewritten as

$$d(Tv, gv) \leq \frac{s\mu}{1 - s\lambda} \, d(gx_n, gx_{n_i+1}) + \frac{s}{1 - s\lambda} \, d(gx_{n_i+1}, gv). \tag{3.15}$$

Using equation (3.15), we get

$$d(Tv, gv) \leq \frac{s\mu}{1 - s\lambda} \, \beta^{n_i} d(gx_0, gx_1) + \frac{s}{1 - s\lambda} \, d(gx_{n_i+1}, gv).$$

Taking the limit as $i \to \infty$ and using the facts that

$$\lim_{i \to \infty} d(gv, gx_{n_i}) = 0, \qquad \frac{s\mu}{1 - s\lambda} < 1,$$

we obtain

$$d(Tv, gv) \leq 0.$$

Hence,

$$d(Tv, gv) = 0,$$

which implies that $Tv = gv = u$. Therefore, v is a coincidence point of T and g, and u is a point of coincidence.

Similarly, for cases (2), (3), and (4), we can also prove that $Tv = gv = u$.

Finally, if T and g are weakly compatible, then by Proposition (2.11), T and g have a unique common fixed point.

□

Proposition 3.2. *Let T and g be weakly compatible self maps of a non empty set X. If T and g have a unique point of coincidence $u = Tv = gv$ then u is the unique common fixed point of* t*and g.*

Corollary 3.3. *Let (X, d) be a b-metric space (with $s \geq 1$) endowed with a graph G, and let $T, g : X \to X$ be mappings satisfying*

$$d(Tx, Ty) \leq \lambda\, d(gx, Tx) + \mu \max\{d(gy, Ty),\, d(gx, Ty)\}, \quad \forall\, x, y \in X \text{ with } (gx, gy) \in E(G).$$

Suppose that $T(X) \subseteq g(X)$ and that $g(X)$ is a complete subspace of X. Then:

1. *If $G_{gT} \neq \varnothing$ and the property $G_{(T, g_{x_n})}$ is satisfied, then T and g have a point of coincidence in X.*

2. *Moreover, if any two points of coincidence x and y of T and g satisfy $(x, y) \in E(G)$, then T and g have a* unique *point of coincidence in X.*

3. *Furthermore, if T and g are weakly compatible, then T and g have a* unique *common fixed point in X.*

Corollary 3.4. Let (X, d) be a b-metric space (with $s \geq 1$) endowed with a graph G, and let the mappings $T, g : X \to X$ satisfy

$$d(Tx, Ty) \leq \lambda\, d(gx, Tx) + \mu \max\{d(gy, Tx),\, d(gx, Ty)\},$$

for all $x, y \in X$ with $(gx, gy) \in E(G)$.

Suppose that $T(X) \subseteq g(X)$ and $g(X)$ is a complete subspace of X. Then:

1. If $GgT \neq \varnothing$ and the property $G_{(T, g_{x_n})}$ is satisfied, then T and g have a point of coincidence in X.

2. Moreover, if x and y are points of coincidence of T and g in X and $(x, y) \in E(G)$, then T and g have a unique point of coincidence in X.

3. Further, if T and g are weakly compatible, then T and g have a unique common fixed point in X.

Corollary 3.5. Let (X, d) be a complete b-metric space (with $s \geq 1$) endowed with a graph G, and let the mapping $T : X \to X$ satisfy

$$d(Tx, Ty) \leq \lambda\, d(x, Tx) + \mu\, \max\{d(y, Tx),\ d(x, Ty)\},$$

for all $x, y \in X$ with $(x, y) \in E(G)$. Then:

1. If $GgT \neq \varnothing$ and the property $G_{(T,x_n)}$ is satisfied, then T has a fixed point in X.

2. Moreover, if x and y are two fixed points of T in X and $(x, y) \in E(G)$, then $x = y$.

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