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Fixed Point Results for Monotone $\vartheta-$ Nonexpansive Mapping in Ordered Hyperbolic Metric Space

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Abstract

In this paper, we introduce monotone ϑ-nonexpansive mapping in hyperbolic metric space equipped with a partial order. We prove some strong and $\triangle-$ convergence theorems and approximate the fixed point for ϑ-nonexpansive mapping in hyperbolic metric space. Further, we provide a numerical example to illustrate the convergence of proposed iterative algorithm by ϑ-nonexpansive mapping. Furthermore, as an application, we have demonstrated the solution of nonlinear integral equations. Our results extend some of the existing results in the literature ([17], [23], [34], [35]).

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1. INTRODUCTION

One of the fundamental results of fixed point theory is the Banach contraction principle [7]. The theory of fixed point has been utilized broadly to prove the existence of solutions for different nonlinear functional equations; several attempts to obtain fixed points in partially ordered sets have been made in the past few years. In 2004, Ram and Reurings [27] found applications of linear and non-linear matrix equations in partially ordered sets. In 2005, Nieto and Lopez [21] used the same approach to further extend the results of fixed-point theory in partially ordered metric spaces. They proved the existence and uniqueness of a solution for a first-order ordinary differential equation with periodic boundary conditions.

The study of monotone nonexpansive mappings has increased rapidly in the past few years to obtain fixed points in partially ordered sets. In 2015, Bachar and Khamsi [8] introduced monotone nonexpansive mappings that are defined on a partially ordered Banach space. For more details in this direction, one may refer to [11, 17, 32].

In the modern era, extensive research is being conducted on the convergence of the fixed point approximation method in hyperbolic space. Many authors have investigated various extensions and generalizations of nonexpansive mappings; one may refer to [2, 24, 25, 29, 33–35], among others.

Several iteration processes have been studied with various types of mappings and spaces. Many researchers compared their iteration processes with others to establish fast convergence and identify both strong and weak convergence theorems, as well as fixed points ([1, 4, 9, 12, 13, 20, 22, 26, 28, 31]).

We recall some well-defined iterative algorithms as follows:
In 2007, Agarwal et al. [3] introduced S- iterative algorithm as follows:

$$\begin{cases} \varkappa_1 = \varkappa \in X \\ \varkappa_{n+1} = (1-\alpha_n)Q\varkappa_n + \alpha_n \nu_n, \\ \nu_n = (1-\beta_n)\varkappa_n + \beta_n Q\varkappa_n, \quad n \in \mathbb{N} \end{cases} \tag{1}$$

where $\{\alpha_n\}, \{\beta_n\}$ is a sequence of real numbers in (0,1).

In 2014, J. Srivastava [30] introduced Picard-S iterative algorithm as follows:

$$\begin{cases} \varkappa_1 \in X, \\ \varkappa_{n+1} = Q\nu_n, \\ \nu_n = (1-\alpha_n)Q\varkappa_n + \alpha_n Q z_n, \\ z_n = (1-\beta_n)\varkappa_n + \beta_n Q\varkappa_n, n \in \mathbb{N}. \end{cases} \tag{2}$$

Where $\alpha_n, \beta_n \in (0,1)$.

In 2018, K. Ullah and M. Arshad [33] introduced M- iterative algorithm as follows:

$$\begin{cases} \varkappa_1 = \varkappa \in X, \\ \nu_n = (1-\alpha_n)\varkappa_n + \alpha_n Q\nu_n, \\ z_n = Q\nu_n, \\ \varkappa_{n+1} = Qz_n, \quad n \in \mathbb{N}. \end{cases} \tag{3}$$

Where $\alpha_n \in (0,1)$.

In 2020, T. Abdeljawad et al. [6] introduced JA -iterative algorithm as follows:

$$\begin{cases} \varkappa_1 \in X, \\ \nu_n = (1-\beta_n)\varkappa_n + \beta_n Q\varkappa_n, \\ z_n = Q\nu_n, \\ \varkappa_{n+1} = (1-\alpha_n)Q\varkappa_n + \alpha_n Qz_n, \quad n \in \mathbb{N}. \end{cases} \tag{4}$$

Where $\alpha_n, \beta_n \in (0,1)$.

In 2020, Ali and Ali [5] introduced F- iterative algorithm as follows:

$$\begin{cases} \varkappa_1 \in X, \\ \nu_n = Q((1-\alpha_n)\varkappa_n + \alpha_n Q\varkappa_n), \\ z_n = Q\nu_n, \\ \varkappa_{n+1} = Qz_n, \quad n \in \mathbb{N}. \end{cases} \tag{5}$$

Where $\alpha_n \in (0,1)$.

In 2021 Kalsoom et al. [17] introduced a new iterative algorithm as follows:

$$\begin{cases} \varkappa_1 = \varkappa \in X, \\ \varkappa_{n+1} = Q((1-\alpha_n)Q\varkappa_n + \alpha_n Q\nu_n), \\ \nu_n = (1-\beta_n)z_n + \beta_n Qz_n, \\ \varkappa_n = (1-\gamma_n)\varkappa_n + \gamma_n Q\varkappa_n, n \in \mathbb{N}. \end{cases} \tag{6}$$

Where α_n, β_n and $\gamma_n \in (0,1)$. They proved the existence and convergence results in hyperbolic space.

$$\begin{cases} \varkappa_1 \in X, \\ \varkappa_{n+1} = Q((1-\alpha_n)\nu_n + \alpha_n Q(\nu_n)), \quad n \in \mathbb{N}, \\ \nu_n = Q(Q(z_n)), \\ z_n = Q(Q(\varkappa_n)), \end{cases} \tag{7}$$

where $\{\alpha_n\}$ in (0,1).

Motivated by [17], we introduce an iterative algorithm (7) to prove some strong and $\triangle$-convergence results for monotone ϑ-nonexpansive mappings within the framework of a uniformly convex hyperbolic metric space, we provide a numerical example to illustrate the convergence of new iterative algorithm. Additionally, we compare the rate of convergence of iterative algorithms ([3,5,6,17,30,33]) with that of leading proposed iterative algorithm (7).

2. PRELIMINARIES

In this section, we present some definitions and preliminary results required to establish subsequent outcomes.

Definition 2.1. *[18] A hyperbolic space $(\mathbb{H}, d, W)$ is a metric space $(\mathbb{H}, d)$ together with a convexity mapping $W : \mathbb{H} \times \mathbb{H} \times [0,1] \to \mathbb{H}$ such that for all $u, v, w, z \in \mathbb{H}$ and $\lambda, \mu \in [0,1]$, we have*

1. $d(u, W(v,z,\lambda)) \leq \lambda d(u,v) + (1-\lambda) d(u,z)$;
2. $d(W(v,z,\lambda), W(v,z,\mu)) = |\lambda - \mu| d(v,z)$;
3. $W(v,z,\lambda) = W(v,z,(1-\lambda))$;
4. $d(W(v,z,\lambda), W(u,t,\lambda)) \leq \lambda d(v,z) + (1-\lambda) d(u,t)$.

Definition 2.2. *[11] A hyperbolic space $\mathbb{H}$ is said to be uniformly convex if for any $r > 0$ and $\epsilon \in (0,2]$, there exists $\delta \in (0,1]$ such that for all $u, v, t \in \mathbb{H}$, $d(W(u,v,\frac{1}{2},t)) \leq (1-\delta) r$, if $d(u,t) \leq r$, $d(v,t) \leq r$ and $d(u,v) \geq \epsilon r$.*

Definition 2.3. *[11] Let X be a nonempty subset of a hyperbolic space $(\mathbb{H}, d, W)$ and $\{\varkappa_n\}$ be a bounded sequence in X. For $\varkappa \in \mathbb{H}$, there is a continuous functional $r(., \{\varkappa_n\}) : X \to [0, \infty)$ defined by*

$$r(\varkappa, \{\varkappa_n\}) = \limsup_{n \to \infty} d(\varkappa_n, \varkappa).$$

The asymptotic radius $r(X, \{\varkappa_n\})$ of $\{\varkappa_n\}$ with respect to X is given by

$$r(X, \{\varkappa_n\}) = \inf\{r(\varkappa, \{\varkappa_n\}) : \varkappa \in X\}.$$

A point $\varkappa \in X$ is said to be an asymptotic center of the sequence $\{\varkappa_n\}$ with respect to X, if

$$r(\varkappa, \{\varkappa_n\}) = \inf\{r(y, \{\varkappa_n\}) : y \in X\}.$$

The set of all asymptotic centers of $\{\varkappa_n\}$ with respect to X is denoted by $A(X, \{\varkappa_n\})$.

Definition 2.4. *[25] Assume that $(\mathbb{H}, d, W)$ is hyperbolic metric space and $\{\varkappa_n\}$ is bounded sequence. Then $\{\varkappa_n\}$ is said to $\triangle-$convergence to $\varkappa \in b$, if $\varkappa$ is unique asymptotic center of every $\{\varkappa_{n_k}\}$ where $\{\varkappa_{n_k}\}$ is subsequence of $\{\varkappa_n\}$.*

Definition 2.5. *[11] Assume that X is a nonempty subset of a hyperbolic metric space $(\mathbb{H}, d)$. Then a function $Q : X \to [0, \infty)$ is called a type function, if there is a bounded sequence $\{\varkappa_n\}$ in $\varkappa$ such that $Q(\varkappa) = \lim_{n\to\infty}(\varkappa_n, \varkappa)$, for any $\varkappa \in X$.*

Definition 2.6. *A bounded sequence $\{\varkappa_n\}$ in X is said to be $\triangle-$convergence at a point $\varkappa \in X$, if $\varkappa$ is the unique and a type function generated by every subsequence $\{\varkappa_{n_k}\}$ of $\{\varkappa_n\}$ attains its infimum at $\varkappa$.*

Definition 2.7. *[11] Let $(\mathbb{H}, d)$ denote a metric space with the partial order $\preceq$. A self mapping Q on $\mathbb{H}$ is referred as monotone if $Q(a) \preceq Q(b)$ whenever $a \preceq b$, for all $a, b \in \mathbb{H}$.*

Lemma 2.1. *[11] Assume that $(\mathbb{H}, d, Q)$ is a uniformly convex hyperbolic metric space with monotone modulus of uniform convexity r and $\varkappa^* \in \mathbb{H}$ and $\{\alpha_n\}$ is a sequence such that $0 < a \leq \alpha_n \leq b < 1$. If $\{s_n\}$ and $\{t_n\}$ are sequences in $\mathbb{H}$ such that $\limsup_{n\to\infty} d(s_n, \varkappa^*) \leq r$, $\limsup_{n\to\infty} d(t_n, \varkappa^*) \leq r$ and $\limsup_{n\to\infty} d(\alpha_n s_m \oplus (1-\alpha_n)t_n, \varkappa^*) = r$, for $r \geq 0$. Then, $\lim_{n\to\infty} d(s_n, t_n) = 0$.*

Lemma 2.2. *[19] A bounded sequence $\{\varkappa_n\}$ is a complete uniformly convex hyperbolic metric space $\mathbb{H}$ with the monotone modulus of uniform convexity Q has a unique asymptotic center concerning every nonempty closed convex subset X to $\mathbb{H}$.*

3. MAIN RESULTS

Let us first introduce monotone $\vartheta-$ nonexpansive mapping in partially ordered hyperbolic metric space $(\mathbb{H}, d, \preceq)$ as follows:

Definition 3.1. *Let X be a nonempty subset of ordered hyperbolic metric space $(\mathbb{H}, d, \preceq)$. A self map Q on X is said to be monotone $\vartheta-$ nonexpansive mapping and* *there exists* $\vartheta \in [0, \frac{1}{2})$. *Such that*

$$\frac{1}{2}d(\varkappa, Q\varkappa) \leq d(\varkappa, \nu) \Rightarrow d(Q\varkappa, Q\nu) \leq \vartheta d(\varkappa, Q^2\varkappa) + \vartheta d(\nu, Q^2\nu) + (1 - 4\vartheta)d(\varkappa, \nu) \tag{8}$$

for all $\varkappa, \nu \in X$ *with* $\varkappa \preceq \nu$.

Remark 3.1. *Obviously, the class of monotone* $\vartheta-$*nonexpansive mapping reduces monotone nonexpansive mapping if* $\vartheta = 0$.

Lemma 3.1. *Let* X *be nonempty subset ordered hyperbolic metric space* $(\mathbb{H}, d, \preceq)$. *Let* $Q : X \to X$ *be a monotone* ϑ*-nonexpansive mapping with* $Fix(Q) \neq \emptyset$. *Then* Q *is monotone quasi-nonexpansive and for all* $\varkappa, \nu \in X,\ \vartheta \in [0, \frac{1}{2})$ *with* $\varkappa \preceq \nu$,

$$d(Q\varkappa, Q\nu) \leq d(\varkappa, \nu) + \left(\frac{\vartheta}{1 - 2\vartheta}\right)[d(\nu, Q\nu) + d(\varkappa, Q\varkappa) + d(Q\varkappa, Q^2\varkappa) + d(Q\nu, Q^2\nu)]. \tag{9}$$

Proof. Let $Fix(Q) \neq \emptyset$ is quasi-nonexpansive, for all $p \in Fix(Q)$. From the Definition 3.1, we have

$$\begin{aligned} d(Q\varkappa, p) = d(Q\varkappa, Qp) &\leq \vartheta d(\varkappa, Q^2\varkappa) + \vartheta d(p, Q^2 p) + (1 - 4\vartheta)d(\varkappa, p) \\ &\leq \vartheta d(\varkappa, Q\varkappa) + \vartheta d(p, Qp) + (1 - 4\vartheta)d(\varkappa, p) \\ &\leq d(\varkappa, p). \end{aligned}$$

This implies that Q is monotone quasi-nonexpansive mapping with $\varkappa \preceq p$.

Furthermore, for all $\varkappa, \nu \in X$, some $\vartheta \in [0, \frac{1}{2})$ with $\varkappa \preceq \nu$, we obtain

$$d(Q\varkappa, Q\nu) \leq d(\varkappa, \nu) + \left(\frac{\vartheta}{1 - 2\vartheta}\right)[d(\nu, Q\nu) + d(\varkappa, Q\varkappa) + d(Q\varkappa, Q^2\varkappa) + d(Q\nu, Q^2\nu)],$$

for all $\varkappa, \nu \in X$, $\vartheta \in [0, \frac{1}{2})$ with $\varkappa \preceq \nu$,

$$\begin{aligned} d(Q\varkappa, Q\nu) &\leq \vartheta d(\varkappa, Q^2\varkappa) + \vartheta d(\nu, Q^2\nu) + (1 - 4\vartheta)d(\varkappa, \nu) \\ &\leq \vartheta[d(\varkappa, Q\nu) + d(Q\nu, Q^2\varkappa)] + \vartheta[d(\nu, Q\varkappa) + d(Q\varkappa, Q^2\nu)] + (1 - 4\vartheta)d(\varkappa, \nu) \\ d(Q\varkappa, Q\nu) &\leq d(\varkappa, \nu) + \left(\frac{\vartheta}{1 - 2\vartheta}\right)[d(\nu, Q\nu) + d(\varkappa, Q\varkappa) + d(Q\varkappa, Q^2\varkappa) + d(Q\nu, Q^2\nu)]. \end{aligned}$$

Hence, for all $\varkappa, \nu \in X$, $\vartheta \in [0, \frac{1}{2})$ with $\varkappa \preceq \nu$,

$$d(Q\varkappa, Q\nu) \leq d(\varkappa, \nu) + \left(\frac{\vartheta}{1-2\vartheta}\right)[d(\nu, Q\nu) + d(\varkappa, Q\varkappa) + d(Q\varkappa, Q^2\varkappa) + d(Q\nu, Q^2\nu)].$$

This completes the proof. □

Lemma 3.2. *Let X be a nonempty subset of uniformly convex ordered hyperbolic metric space $(\mathbb{H}, d, \preceq)$. If $Q : X \to X$ is monotone $\vartheta-$nonexpansive mapping, then $Fix(Q)$ is closed. Furthermore, if $\mathbb{H}$ is strictly convex, then X is convex and $Fix(Q)$ is also convex.*

Proof. Let $\{\varkappa_n\}$ be a sequence in $Fix(Q)$ which converges to $\varkappa \in X$.

By continuity of metric, we have

$$\begin{aligned}
\lim_{n\to\infty} d(Q\varkappa_n, \varkappa) = \lim_{n\to\infty} d(Q\varkappa_n, Q\varkappa) &\leq \lim_{n\to\infty} \{\vartheta d(\varkappa_n, Q^2\varkappa_n) \\
&\quad + \vartheta d(\varkappa, Q^2\varkappa) + (1-4\vartheta)d(\varkappa_n, \varkappa)\} \\
&\leq \vartheta \lim_{n\to\infty} \{d(\varkappa_n, \varkappa) + d(Q\varkappa, Q\varkappa_n)\} + \vartheta \lim_{n\to\infty} \{d(\varkappa, \varkappa_n) \\
&\quad + d(Q\varkappa_n, Q\varkappa)\} + (1-4\vartheta) \lim_{n\to\infty} d(\varkappa_n, \varkappa) \\
&\leq \vartheta \lim_{n\to\infty} d(Q\varkappa, Q\varkappa_n) + \vartheta \lim_{n\to\infty} d(Q\varkappa_n, Q\varkappa) \\
&\quad + (1-2\vartheta) \lim_{n\to\infty} d(\varkappa_n, \varkappa) \\
&\leq \lim_{n\to\infty} d(\varkappa_n, \varkappa).
\end{aligned} \tag{10}$$

Since $(1-2\vartheta) > 0 \; where \; \vartheta \in [0, 1)$, hence, $Fix(Q)$ is closed.

Now, we assume that $\mathbb{H}$ is strictly convex and X is convex. We show that $Fix(Q)$ is convex.

Let $s, t \in Fix(Q)$ and $\varkappa \in X$ with $s \neq t$. Since

$$d(s, Qs) = 0 \leq d(s, \varkappa),$$

we obtain

$$d(Qs, Q\varkappa) \leq \vartheta d(s, Q^2 s) + \vartheta d(\varkappa, Q^2\varkappa) + (1-4\vartheta)d(s, \varkappa).$$

$$d(Qs, Q\varkappa) \leq d(s, \varkappa).$$

By similar argument, we get

$$d(t, Qt) = 0 \leq d(t, \varkappa),$$

we obtain

$$d(Qt, Q\varkappa) \leq \vartheta d(t, Q^2 t) + \vartheta d(\varkappa, Q^2 \varkappa) + (1 - 4\vartheta) d(t, \varkappa).$$

$$d(Qt, Q\varkappa) \leq d(t, \varkappa).$$

Let $\varkappa = \lambda s + (1 - \lambda)t \in X$, for $\lambda \in [0, 1)$. Then

$$\begin{aligned} d(s, t) &\leq d(s, Q\varkappa) + d(Q\varkappa, t) \\ &\leq d(s, \varkappa) + d(\varkappa, t) \\ &= d(s, \lambda s + (1 - \lambda)t) + d(\lambda s + (1 - \lambda)t, t) \\ &\leq d(s, t) \end{aligned}$$

From strict convexity of $\mathbb{H}$, there exists $\mu \in [0, 1)$ such that $Q(\nu) = \mu s + (1 - \mu)\varkappa$.

Now,

$$\begin{aligned} d(s, Q(\varkappa)) &\leq d(s, \varkappa) \\ \Rightarrow d(s, \mu s + (1 - \mu)t) &\leq d(s, \lambda s + (1 - \lambda)t) \\ \Rightarrow (1 - \mu) d(s, t) &\leq (1 - \lambda) d(s, t). \end{aligned}$$

Hence, we have $(1 - \mu) \leq (1 - \lambda)$ and $\mu \leq \lambda$, which implies that $\mu = \lambda$. Thus, $\varkappa \in Fix(Q)$, which implies that $Fix(Q)$ is convex. □

We now apply iterative algorithm (7) for monotone $\vartheta-$nonexpansive mappings.

Lemma 3.3. *Let $(\mathbb{H}, d, \preceq)$ be a uniformly convex partially ordered hyperbolic metric space and X a nonempty closed convex subset of $\mathbb{H}$. Let $Q : X \to X$ be a monotone mapping and $\varkappa_1 \in X$ be such that $\varkappa_1 \preceq Q\varkappa_1$ (or $Q\varkappa_1 \preceq \varkappa_1$). Then, for sequence $\{\varkappa_n\}$ defined by (7), we have*

(a) $\varkappa_n \preceq Q\varkappa_n \preceq \varkappa_{n+1}$ (or $\varkappa_{n+1} \preceq Q\varkappa_n \preceq \varkappa_n$);

(b) $\varkappa_n \preceq p$ (or $p \preceq \varkappa_n$) provided $\{\varkappa_n\}$ $\triangle-$ convergence to a point $p \in X, \forall n \in \mathbb{N}$.

Theorem 3.1. *Let X be a nonempty closed convex subset of uniformly convex partially ordered hyperbolic metric space $(\mathbb{H}, d, \preceq)$. Let $Q : X \to X$ be a monotone mapping and $\{\varkappa_n\}$ defined in (7) is a bounded sequence with $\varkappa_n \leq l$ for all $l \in X$ such that*

$$\lim_{n\to\infty} d(\varkappa_n, Q\varkappa_n) = 0. \tag{11}$$

Then, $Fix(Q) \neq \emptyset$.

Proof. Since $\{\varkappa_n\}$ be a bounded sequence such that

$$\lim_{n\to\infty} \inf d(\varkappa_n, Q\varkappa_n) = 0. \tag{12}$$

Then, there exists a subsequence $\{\varkappa_{n_k}\}$ such that

$$\lim_{k\to\infty} \inf d(\varkappa_{n_k}, Q\varkappa_{n_k}) = 0. \tag{13}$$

By Lemma (3.3), we have

$$\varkappa_1 \preceq \varkappa_{n_k} \preceq \varkappa_{n_{k+1}}.$$

Define

$$X_k = \{l \in X : \varkappa_k \preceq l\}.$$

Clearly, for every $k \in \mathbb{N}$, X_k is closed convex. As $l \in X_k$. It shows that $X_k \neq \emptyset$.

Define

$$X_\infty = \bigcap_{k=1}^{\infty} X_k \neq \emptyset.$$

Then, X_∞ is a closed convex subset of X. Let $l \in X_\infty$, then

$$\varkappa_{n_k} \preceq l, \quad \forall k \in \mathbb{N}.$$

As we know, Q is a mapping which is monotone, then

$$\varkappa_{n_k} \preceq Q\varkappa_{n_k} \preceq Ql,$$

which implies that $Q(X_\infty) \subset X_\infty$.

Let a type function $\xi : X_\infty \to [0, \infty)$ generated by $\{\varkappa_{n_k}\}$ such that

$$\xi(l) = \limsup_{k\to\infty} d(\varkappa_{n_k}, l).$$

Then, there exists a unique $u \in X_\infty$ such that

$$\xi(u) = \inf\{\xi(l) : l \in X_\infty\}.$$

By definition of the type function,

$$\xi(Q(u)) = \limsup_{k\to\infty} d(\varkappa_{n_k}, Qu).$$

By using Lemma 3.1

$$\begin{aligned} d(Q\varkappa_{n_k}, Qu) \leq d(\varkappa_{n_k}, u) + \left(\frac{\vartheta}{1-2\vartheta}\right)[d(u, Qu) + d(\varkappa_{n_k}, Q\varkappa_{n_k}) \\ + d(Q\varkappa_{n_k}, Q^2\varkappa_{n_k}) + d(Qu, Q^2u)]. \end{aligned}$$

From the boundedness of the sequence $\{\varkappa_{n_k}\}$ and $\lim_{n\to\infty} d(\varkappa_{n_k}, Q\varkappa_{n_k}) = 0$, we have

$$\limsup_{k\to\infty} d(Q\varkappa_{n_k}, Qu) \leq \limsup_{k\to\infty} d(\varkappa_{n_k}, u),$$

implies $\xi(Qu) = \xi(u)$. This shows that $Q(u) = u$, and hence, $Fix(Q) \neq \emptyset$. □

Lemma 3.4. *Let X be a nonempty subset of uniformly convex partially ordered hyperbolic metric space $(\mathbb{H}, d, \preceq)$ and $Q : X \to X$ a monotone $\vartheta-$ nonexpansive mapping. Then, for each $\varkappa, \nu \in X$ with $\varkappa \preceq \nu$,*
(a) $d(Q\varkappa, Q^2\varkappa) \leq d(\varkappa, Q\varkappa)$.
(b) Either $\frac{1}{2}d(\varkappa, Q\varkappa) \leq d(\varkappa, \nu)$ or $\frac{1}{2}d(Q\varkappa, Q^2\varkappa) \leq d(Q\varkappa, \nu)$

Proof. For all $\varkappa \in X$, we have that $\frac{1}{2}d(\varkappa, Q\varkappa) \leq d(\varkappa, Q\varkappa)$, which implies that

$$\begin{aligned} d(Q\varkappa, Q^2\varkappa) &\leq \vartheta d(\varkappa, Q^2\varkappa) + \vartheta d(Q\varkappa, Q(Q^2\varkappa)) + (1-4\vartheta)d(\varkappa, Q\varkappa) \\ &\leq \vartheta d(\varkappa, Q\varkappa) + \vartheta d(Q\varkappa, Q^2\varkappa) + \vartheta d(Q\varkappa, \varkappa) + \vartheta d(Q\varkappa, Q(Q^2\varkappa)) \\ &\quad + (1-4\vartheta)d(\varkappa, Q\varkappa) \qquad (14) \\ &\leq \vartheta d(Q\varkappa, Q^2\varkappa) + \vartheta d(Q\varkappa, Q^2\varkappa) + (1-2\vartheta)d(\varkappa, Q\varkappa) \\ &\leq d(\varkappa, Q\varkappa) \end{aligned}$$

Since $(1-2\vartheta)>0, \vartheta \in [0,\frac{1}{2})$. Hence, part (a) is satisfied.

Now, we will prove part (b); we argue with contradiction, and suppose

$$\begin{aligned} \frac{1}{2}d(\varkappa, Q\varkappa) &> d(\varkappa,\nu) \; and \\ \frac{1}{2}d(Q\varkappa, Q^2\varkappa) &> d(Q\varkappa,\nu) \end{aligned} \tag{15}$$

By triangle inequality,

$$\begin{aligned} d(\varkappa, Q\varkappa) &\le d(\varkappa,\nu) + d(Q\varkappa,\nu) \\ &< \frac{1}{2}d(\varkappa, Q\varkappa) + \frac{1}{2}d(Q\varkappa, Q^2\varkappa) \\ &< \frac{1}{2}d(\varkappa, Q\varkappa) + \frac{1}{2}d(\varkappa, Q\varkappa) \\ &< d(\varkappa, Q\varkappa), \end{aligned}$$

which is a contradiction. Thus, we obtain the desired result.

□

Theorem 3.2. *Let X be a nonempty, closed and convex subset of uniformly convex partially ordered hyperbolic metric space $(\mathbb{H}, d, \preceq)$ and $Q : X \to X$ a monotone $\vartheta-$ nonexpansive mapping. Then, $Fix(Q) \neq \emptyset$ iff $\{\varkappa_n\}$ is a sequence which is also bounded for some $\varkappa \in X$ provides that $\varkappa_n \preceq p$ for some $\varkappa \in X$, and $\varkappa \preceq Q\varkappa$.*

Proof. Let $\{\varkappa_n\}$ be a sequence for some $\varkappa \in X$. As we know that Q is monotone and $\varkappa \preceq Q\varkappa$, so we get

$$Q\varkappa \preceq Q^2\varkappa.$$

In the similar manner, we get

$$Q^2\varkappa \preceq Q^3\varkappa \preceq Q^4\varkappa .... \preceq Q^n\varkappa \preceq Q^{n+1}\varkappa \preceq ..$$

Define $\{\varkappa_n\} = Q^n\varkappa, \quad \forall n \in \mathbb{N}$. Then, the asymptotic center of $\{\varkappa_n\}$ with respect to X is $A(X, \{\varkappa_n\}) = \{p\}$ where p is unique and $\varkappa_n \preceq \varkappa$ for all $n \in \mathbb{N}$. Now, we claim that $\{d(\varkappa_{n+1}, \varkappa_{n+2})\}$ is a non-increasing sequence, that is,

$$d(\varkappa_{n+1}, \varkappa_{n+2}) \le d(\varkappa_n, \varkappa_{n+1}).$$

Since

$$\frac{1}{2}d(\varkappa_n, Q\varkappa_n) \le d(\varkappa_n, \varkappa_{n+1}),$$

we obtain that

$$\begin{aligned} d(\varkappa_{n+1}, \varkappa_{n+2}) &= d(Q\varkappa_n, Q\varkappa_{n+1}) \\ &\leq \vartheta d(\varkappa_n, Q^2\varkappa_n) + \vartheta d(\varkappa_{n+1}, Q^2\varkappa_{n+1}) + (1-4\vartheta) d(\varkappa_n, \varkappa_{n+1}) \\ &\leq \vartheta\{d(\varkappa_n, \varkappa_{n+1}) + d(Q\varkappa_{n+1}, Q\varkappa_n)\} + \vartheta\{d(\varkappa_{n+1}, \varkappa_n) + d(Q\varkappa_n, Q\varkappa_{n+1})\} \\ &\quad + (1-\vartheta) d(\varkappa_n, \varkappa_{n+1}) \\ &\leq \vartheta d(Q\varkappa_{n+1}, Q\varkappa_n) + \vartheta d(Q\varkappa_n, Q\varkappa_{n+1}) + (1-2\vartheta) d(\varkappa_n, \varkappa_{n+1}) \\ d(\varkappa_n, Q\varkappa_{n+1}) &\leq d(\varkappa_n, \varkappa_{n+1}). \end{aligned}$$

Now, we claim that

$$\frac{1}{2} d(\varkappa_n, \varkappa_{n+1}) \leq d(\varkappa_n, p) \tag{16}$$

and

$$\frac{1}{2} d(\varkappa_{n+1}, \varkappa_{n+2}) \leq d(\varkappa_{n+1}, p). \tag{17}$$

To prove this, we consider the contradiction

$$d(\varkappa_n, p) < \frac{1}{2} d(\varkappa_n, \varkappa_{n+1}) \tag{18}$$

and

$$d(\varkappa_{n+1}, p) < \frac{1}{2} d(\varkappa_{n+1}, \varkappa_{n+2}). \tag{19}$$

By using triangle inequality

$$\begin{aligned} d(\varkappa_n, \varkappa_{n+1}) &\leq d(\varkappa_n, p) + d(\varkappa_{n+1}, p) \\ &< \frac{1}{2} d(\varkappa_n, \varkappa_{n+1}) + \frac{1}{2} d(\varkappa_{n+1}, \varkappa_{n+2}) \\ &< d(\varkappa_n, \varkappa_{n+1}). \end{aligned}$$

Which is not possible, so (16) and (17) is satisfied.

In the first case

$$\begin{aligned} &\frac{1}{2} d(\varkappa_n, \varkappa_{n+1}) \leq d(\varkappa_n, p) \; and \\ &\frac{1}{2} d(\varkappa_n, Q\varkappa_n) \leq d(\varkappa_n, p) \end{aligned}$$

$$
\begin{aligned}
d(Q\varkappa_n, Qp) &\leq \vartheta d(\varkappa_n, Q^2p) + \vartheta d(p, Q^2p) + (1-4\vartheta)d(\varkappa_n, p)\\
&\leq \vartheta\{d(\varkappa_n, p) + d(Qp, Q\varkappa_n)\} + \vartheta\{d(p, \varkappa_n) + d(Q\varkappa_n, Qp)\}\\
&\quad + (1-4\vartheta)d(\varkappa_n, p)\\
&\leq \vartheta d(Qp, Q\varkappa_n) + \vartheta d(Q\varkappa_n, Qp) + (1-2\vartheta)d(\varkappa_n, p)\\
d(Q\varkappa_n, Qp) &\leq d(\varkappa_n, p).
\end{aligned}
$$

Taking lim sup on both sides, we get

$$\limsup_{n\to\infty} d(Q\varkappa_n, Qp) \leq \limsup_{n\to\infty} d(\varkappa_n, p)$$

$$Qp = p.$$

Similarly, in the second case

$$
\begin{aligned}
\frac{1}{2}d(\varkappa_{n+1}, \varkappa_{n+2}) &\leq d(\varkappa_{n+1}, p)\\
\frac{1}{2}d(\varkappa_{n+1}, Q\varkappa_{n+1}) &\leq d(\varkappa_{n+1}, p)
\end{aligned}
$$

$$
\begin{aligned}
d(Q\varkappa_{n+1}, Qp) &\leq \vartheta d(\varkappa_{n+1}, Q^2\varkappa_{n+1}) + \vartheta d(p, Q^2p) + (1-4\vartheta)d(\varkappa_{n+1}, p)\\
&\leq \vartheta\{d(\varkappa_{n+1}, p) + d(Qp, Q\varkappa_{n+1})\} + \vartheta\{d(p, \varkappa_{n+1}) + d(Q\varkappa_{n+1}, Qp)\}\\
&\quad + (1-4\vartheta)d(\varkappa_{n+1}, p)\\
&\leq \vartheta d(Qp, Q\varkappa_{n+1}) + \vartheta d(Q\varkappa_{n+1}, Qp) + (1-2\vartheta)d(\varkappa_{n+1}, p)\\
d(Q\varkappa_{n+1}, Qp) &\leq d(\varkappa_{n+1}, p).
\end{aligned}
$$

Taking lim sup on both sides, we get

$$\limsup_{n\to\infty} d(Q\varkappa_{n+1}, Qp) \leq \limsup_{n\to\infty} d(\varkappa_{n+1}, p)$$

$$Qp = p.$$

Conversely, $Fix(Q) \neq \emptyset$, then there exists $\varkappa \in Fix(Q)$ and $Q^n\varkappa = \varkappa$, $\forall n \in \mathbb{N}$; i.e. $\{Q^n(\varkappa)\}$, is a constant sequence and hence, it is bounded and this completes the proof. $\square$

We now prove some convergence results for monotone $\vartheta-$ nonexpansive mapping.

Theorem 3.3. *Let X be a nonempty, closed and convex subset of $\mathbb{H}$, $Q : X \to X$ a $\vartheta-$ nonexpansive mapping with $Fix(Q) \neq \emptyset$ and $\{\varkappa_n\}$ be an iterative scheme generated* *by (7) with the real sequence* $\{\alpha_n\} \in (0, 1)$*. Then* $\{\varkappa_n\}$ *is* $\triangle-$ *convergence to all fixed point of* Q*.*

Proof. The proof is divided into three parts.

Part 1. For any $p \in Fix(Q)$, we have

$$\lim_{n\to\infty} d(\varkappa_n, p) \quad exists. \tag{20}$$

Since $p \in Fix(Q)$, by Lemma 3.1, Q is a quasi-nonexpansive map, i.e.,

$$d(Q\varkappa, p) \leq d(\varkappa, p), \tag{21}$$

for all $\varkappa \in X$ and each $p \in Fix(Q)$.

Now using (7), we have

$$\begin{aligned} d(z_n, p) &= d(Q(Q(\varkappa_n)), p) \\ &\leq d(\varkappa_n, p). \end{aligned} \tag{22}$$

From (7) and (22), we obtain

$$\begin{aligned} d(\nu_n, p) &= d(Q(Q(z_n)), p) \\ &\leq d(z_n, p) \\ &\leq d(\varkappa_n, p). \end{aligned} \tag{23}$$

Finally, using (7) and (23), we get

$$\begin{aligned} d(\varkappa_{n+1}, p) &= d((1-\alpha_n)Q(\nu_n) + \alpha_n Q(\nu_n), p) \\ &\leq (1-\alpha_n)d(Q(\nu_n), p) + \alpha_n(Q(\nu_n), p) \\ &\leq d(\nu_n, p) \\ &\leq d(\varkappa_n, p). \end{aligned} \tag{24}$$

Then, by (24), $\{d(\varkappa_n, p)\}$ is a non-increasing sequence of real numbers that is bounded below. Hence, it implies the desired outcome (20).

Part 2. Next, we prove that

$$\lim_{n\to\infty} d(\varkappa_n, Q(\varkappa_n)) = 0. \tag{25}$$

From (25), we have $\lim_{n\to\infty} d(\varkappa_n, p)$ exists for each $p \in Fix(Q)$. Thus, we take

$$\lim_{n\to\infty} d(\varkappa_n, p) = \tau. \tag{26}$$

By (23) and (26), we obtain

$$\limsup_{n\to\infty} d(\nu_n, p) \le \lim_{n\to\infty} d(\varkappa_n, p) = \tau. \tag{27}$$

Since Q is a quasi-nonexpansive, we get

$$\limsup_{n\to\infty} d(Q\varkappa_n, p) \le \lim_{n\to\infty} d(\varkappa_n, p) = \tau. \tag{28}$$

On the other hand, by (7), we have

$$\begin{aligned} \limsup_{n\to\infty} d(\varkappa_{n+1}, p) &\le \lim_{n\to\infty} d(Q\nu_n, p) \\ &\le \lim_{n\to\infty} d(\nu_n, p). \end{aligned} \tag{29}$$

Which implies that

$$d(\varkappa_{n+1}, p) \le d(\nu_n, p). \tag{30}$$

Therefore

$$\tau \le \liminf_{n\to\infty} d(\nu_n, p). \tag{31}$$

By (27) and (31), we obtain

$$d(\nu_n, p) = \tau. \tag{32}$$

From (31), we have

$$\tau = d(\varkappa_{n+1}, p) = \lim_{n\to\infty} d(\varkappa_{n+1}, p) = \lim_{n\to\infty} Q(d((1-\alpha_n)\nu_n + \alpha_n Q\nu_n, p). \tag{33}$$

Finally, from (27), (29), and (33), and applying Lemma 2.1, we obtain the required results (25).

Part 3. We are ready to establish the $\triangle-$ convergence of $\{\varkappa_n\}$. Because we have seen that the sequence $\{\varkappa_n\}$ is bounded, it essentially has a unique asymptotic center $A(X, \{\varkappa_n\}) = \{\varkappa\}$. By Lemma 2.2, let $\{\nu_n\}$ be any subsequence of $\{\varkappa_n\}$ such that $A(X, \{\nu_n\}) = \{\nu\}$. Then, by (25), we get

$$\lim_{n\to\infty} d(\nu_n, Q\nu_n) = 0. \tag{34}$$

We want to show that ν is an fixed point of Q. By Lemma 3.1, we have

$$\begin{aligned} A(Q\nu, \{\nu_n\}) &= \limsup_{n\to\infty} d(\nu_n, Q\nu) \\ &\le d(\nu_n, \nu) + \left(\frac{\vartheta}{1-2\vartheta}\right)[d(\nu, Q\nu) + d(\nu_n, Q\nu_n) + d(Q\nu_n, Q^2\nu_n) + d(Q\nu, Q^2\nu)] \\ &= \limsup_{n\to\infty} d(\nu_n, \nu) \\ &= A(\nu, \{\nu_n\}). \end{aligned} \tag{35}$$

This implies that $Q(\nu) \in A(X, \{\nu_n\})$.

Now uniqueness of the asymptotic center suggests $Q(\nu) = \nu$, that is $\nu \in Fix(Q)$. Subsequently, we assert the fixed point ν stands as the unique asymptotic center for any subsequence $\{\nu_n\}$ derived from $\{\varkappa_n\}$. Conversely, let us suppose that $p \neq \nu$. By (20), we deduce that $\lim_{n\to\infty} d(\varkappa_n, \nu)$ exists. Now, keeping the uniqueness of the asymptotic center in mind, we can see that

$$\begin{aligned} \limsup_{n\to\infty} d(\nu_n, \nu) &\le \limsup_{n\to\infty} d(\nu_n, p) \\ &\le \limsup_{n\to\infty} d(\varkappa_n, p) \\ &< \limsup_{n\to\infty} d(\varkappa_n, \nu) \\ &= \limsup_{n\to\infty} d(\nu_n, \nu). \end{aligned} \tag{36}$$

However, this is a contradiction. Thus, $p \in Fix(Q)$ is the unique asymptotic center for each subsequence $\{\nu_n\}$ of $\{\varkappa_n\}$. This proves that $\{\varkappa_n\}$ is $\triangle-$ convergence to fixed point of Q. □

Theorem 3.4. *Let X be a nonempty, closed and convex subset of $\mathbb{H}$, $Q : X \to X$ a $\vartheta-$ nonexpansive mapping with $Fix(Q) \neq \emptyset$ and $\{\varkappa_n\}$ be an iterative scheme generated by (7) with the real sequence $\{\alpha_n\} \in (0, 1)$. Then $\{\varkappa_n\}$ is strongly convergence to all fixed point of Q.*

Proof. Assume that $\liminf_{n\to\infty} d(\varkappa_n, Fix(Q)) = 0$. From Theorem 3.3 part 2, $\lim_{n\to\infty} d(\varkappa_n, Fix(Q)) = 0$ exists, so $\lim_{n\to\infty} d(\varkappa_n, Fix(Q)) = 0$ exists.

First, we will prove that $Fix(Q)$ is closed. For this, assume that $\{\nu_n\}$ is sequence in $Fix(Q)$ convergent to limit $\nu \in X$.

Since

$$d(\varkappa_n, Q(\nu_n)) = 0 \leq d(\varkappa_n, \nu) \quad \forall n \in \mathbb{N},$$

we have

$$\begin{aligned} d(\nu_n, Q(\nu)) &= d(Q(\nu_n), Q(\nu)) \\ &\leq \vartheta d(\nu_n, Q^2(\nu_n)) + \vartheta d(\nu, Q^2(\nu)) + (1-4\vartheta) d(\nu_n, \nu). \end{aligned}$$

As $\vartheta \in [0, \frac{1}{2})$,

$$\limsup_{n\to\infty} d(\nu_n, Q(\nu)) \leq \limsup_{n\to\infty} d(\nu_n, \nu) = 0.$$

Therefore, $\{\nu_n\}$ converges strongly to $Q(\nu)$, which implies that $Q(\nu) = \nu$. Hence, $Fix(Q)$ is closed.

Since $\lim_{n\to\infty} d(\varkappa_n, Fix(Q)) = 0$, assume that $\{\varkappa_{n_k}\}$ is a subsequence of $\{\varkappa_n\}$ such that $d(\varkappa_{n_k}, \nu_k) \leq \frac{1}{2^k}$, for all $k \geq 1$, where $\{\nu_k\}$ is sequence in $Fix(Q)$. Due to Theorem 3.3 part 2, we get

$$d(\varkappa_{n+1}, \nu_k) \leq d(\varkappa_{n_k}, \nu_k) \leq \frac{1}{2^k}.$$

Hence by triangular inequality,

$$d(\nu_{k+1}, \nu_k) \leq d(\nu_{k+1}, \varkappa_{n_{k+1}}) + d(\varkappa_{n_{k+1}}, \nu_k) \leq \frac{1}{2^{k-1}},$$

which shows that $\{\nu_k\}$ is Cauchy sequence. Since $Fix(Q)$ is closed, $\{\nu_k\}$ converges to fixed point $\nu \in Fix(Q)$. Now,

$$d(\varkappa_{n_k}, \nu) \leq d(\varkappa_{n_k}, \nu_k) + d(\nu_k, \nu),$$

as $k \to \infty$, $\{\varkappa_{n_k}\}$ converges strongly to ν.

Due to Theorem 3.3 part 2 $\lim_{n\to\infty} d(\varkappa_n, \nu)$ exists, so that $\{\varkappa_n\}$ converges strongly to ν.

The converse part of this theorem is trivial, hence the proof is completed. □

Next, we discussed the convergence behaviour of some iterative algorithms with a table and graphical representation.

Example 1. *Let $\mathbb{H} = \mathbb{R}$ with the usual norm and $X = [6, 9]$ and the mapping $Q : X \to X$ be defined by*

$$Q^2\varkappa = Q\varkappa = \begin{cases} \frac{\varkappa+42}{7}, & if\ \varkappa < 9 \\ 6, & if\ \varkappa = 9. \end{cases} \tag{37}$$

For any $\varkappa \in X$; take $\alpha_n = \frac{1}{2}$, $\beta_n = \frac{1}{2}$, $\gamma_n = \frac{1}{2}$ and $\vartheta = \frac{1}{7}$. The fixed point of 7, and take initial point $\varkappa_1 = 6$. Then, Q is a monotone $\vartheta-$nonexpansive mapping.

Table Convergence of some iterative algorithms for proposed iterative algorithm (7).

Steps	Agrawal et al.	Srivastava	Abdeljawad et al.	Kalsoom et al.	Ullah and Arshad	Ali and Ali	Proposed
$\varkappa_1$	6.00000000	6.00000000	6.00000000	6.00000000	6.00000000	6.00000000	6.00000000
$\varkappa_2$	6.45918367	6.98396501	6.98896293	6.98646397	6.98833819	6.99833403	6.99996600
$\varkappa_3$	6.70751770	6.99974288	6.99987818	6.99981678	6.99999841	6.99999722	**7.00000000**
$\varkappa_4$	6.84182080	6.99999588	6.99999866	6.99999752	6.99999998	**7.00000000**	7.00000000
$\varkappa_5$	6.91445410	6.99999993	6.99999999	6.99999997	**7.00000000**	7.00000000	7.00000000
$\varkappa_6$	6.95373538	**7.00000000**	**7.00000000**	**7.00000000**	7.00000000	7.00000000	7.00000000
$\varkappa_7$	6.97497934	7.00000000	7.00000000	7.00000000	7.00000000	7.00000000	7.00000000
$\varkappa_8$	6.98646842	7.00000000	7.00000000	7.00000000	7.00000000	7.00000000	7.00000000
$\varkappa_9$	6.99268190	7.00000000	7.00000000	7.00000000	7.00000000	7.00000000	7.00000000
$\varkappa_{10}$	6.99604225	7.00000000	7.00000000	7.00000000	7.00000000	7.00000000	7.00000000
$\varkappa_{11}$	6.99785959	7.00000000	7.00000000	7.00000000	7.00000000	7.00000000	7.00000000
$\varkappa_{12}$	6.99884243	7.00000000	7.00000000	7.00000000	7.00000000	7.00000000	7.00000000
$\varkappa_{13}$	6.99937397	7.00000000	7.00000000	7.00000000	7.00000000	7.00000000	7.00000000
$\varkappa_{14}$	6.99966143	7.00000000	7.00000000	7.00000000	7.00000000	7.00000000	7.00000000
$\varkappa_{15}$	6.99981690	7.00000000	7.00000000	7.00000000	7.00000000	7.00000000	7.00000000
$\varkappa_{16}$	6.99990097	7.00000000	7.00000000	7.00000000	7.00000000	7.00000000	7.00000000
$\varkappa_{17}$	6.99994645	7.00000000	7.00000000	7.00000000	7.00000000	7.00000000	7.00000000
$\varkappa_{18}$	6.99997104	7.00000000	7.00000000	7.00000000	7.00000000	7.00000000	7.00000000
$\varkappa_{19}$	6.99998434	7.00000000	7.00000000	7.00000000	7.00000000	7.00000000	7.00000000
$\varkappa_{20}$	6.99999153	7.00000000	7.00000000	7.00000000	7.00000000	7.00000000	7.00000000
$\varkappa_{21}$	6.99999542	7.00000000	7.00000000	7.00000000	7.00000000	7.00000000	7.00000000
$\varkappa_{22}$	6.99999752	7.00000000	7.00000000	7.00000000	7.00000000	7.00000000	7.00000000
$\varkappa_{23}$	6.99999866	7.00000000	7.00000000	7.00000000	7.00000000	7.00000000	7.00000000
$\varkappa_{24}$	6.99999928	7.00000000	7.00000000	7.00000000	7.00000000	7.00000000	7.00000000
$\varkappa_{25}$	6.99999961	7.00000000	7.00000000	7.00000000	7.00000000	7.00000000	7.00000000
$\varkappa_{26}$	6.99999979	7.00000000	7.00000000	7.00000000	7.00000000	7.00000000	7.00000000
$\varkappa_{27}$	6.99999989	7.00000000	7.00000000	7.00000000	7.00000000	7.00000000	7.00000000
$\varkappa_{28}$	6.99999994	7.00000000	7.00000000	7.00000000	7.00000000	7.00000000	7.00000000
$\varkappa_{29}$	6.99999997	7.00000000	7.00000000	7.00000000	7.00000000	7.00000000	7.00000000
$\varkappa_{30}$	6.99999998	7.00000000	7.00000000	7.00000000	7.00000000	7.00000000	7.00000000
$\varkappa_{31}$	6.99999999	7.00000000	7.00000000	7.00000000	7.00000000	7.00000000	7.00000000
$\varkappa_{32}$	6.99999999	7.00000000	7.00000000	7.00000000	7.00000000	7.00000000	7.00000000
$\varkappa_{33}$	**7.00000000**	7.00000000	7.00000000	7.00000000	7.00000000	7.00000000	7.00000000

Summary of the comparison Table as below:

Iterations	No. of iterate
Agrawal et al.	33
Srivastava	6
Abdeljawad et al.	6
Kalsoom et al.	6
Ullah and Arshad	5
Ali and Ali	4
Proposed iteration	3

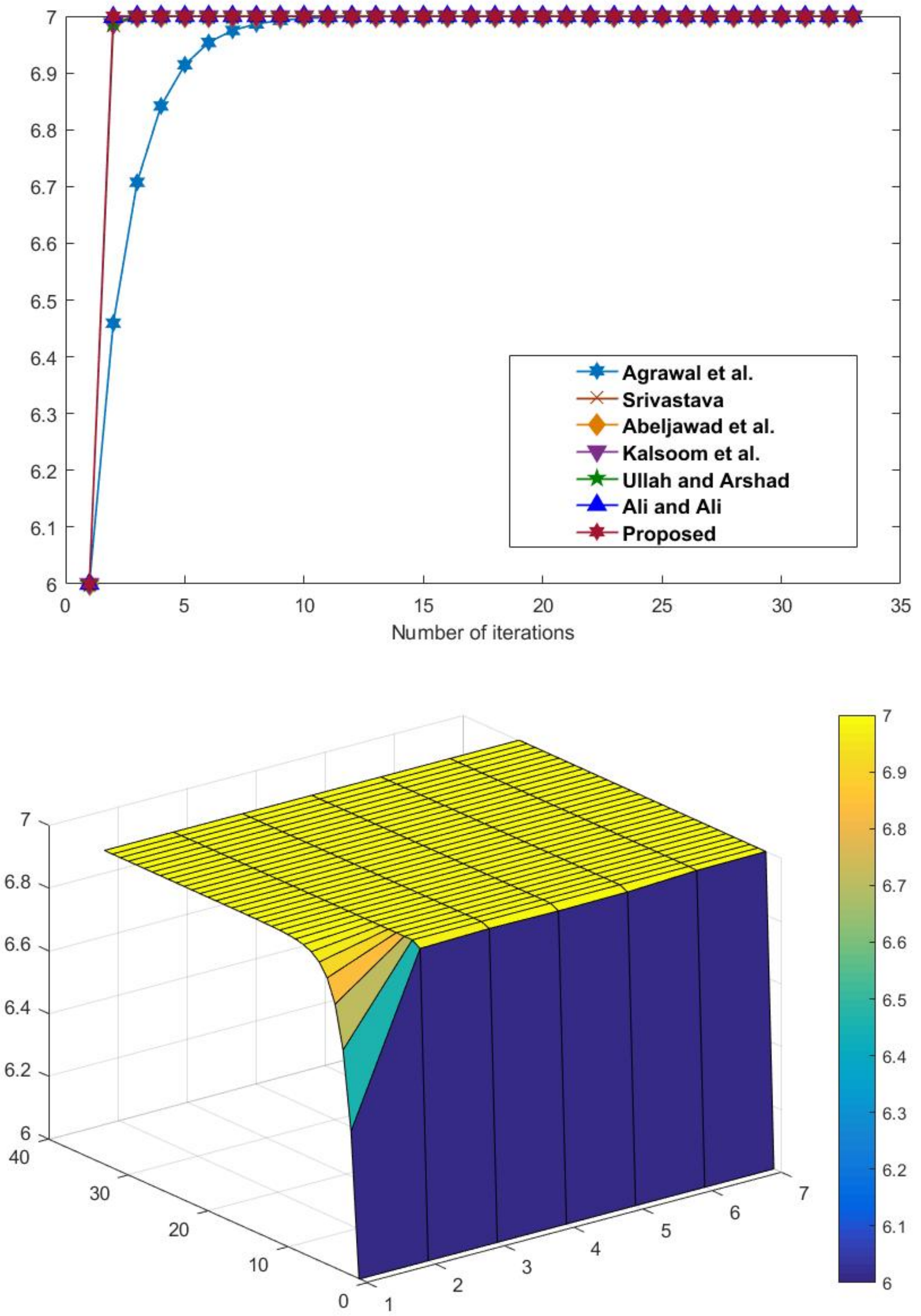

Figure 1: Convergence behaviour of Agrawal et al., Srivastava, Abdeljawad et al., Kalsoom et al., Ullah and Arshad, Ali and Ali, and Proposed algorithms towards the fixed point of the mapping Q.

4. APPLICATION

We will discuss how to apply our findings to nonlinear integral equations. In this section, we present the existence of solution theorems.

Theorem 4.1. *Suppose that X be a nonempty compact subset of ordered Banach space. A mapping $Q : X \to X$ is a monotone $\vartheta-$nonexpansive mapping. Suppose that there is an a fixed point of sequence $\{\varkappa_n\}$. Then Q has a fixed point.*

Proof. Take the following nonlinear functional integral equation. Fredholm integral equation as an example.

$$\zeta(x) = g(x) + \lambda \int_0^x K(x,y)h(y,\zeta(y))dy, \tag{38}$$

for all $x, y \in [0,1]$ where λ is positive constant and $g : [0,1] \to \mathbb{R}^+$, $h : [0,1] \times \mathbb{R} \to \mathbb{R}^+$, $K : [0,1] \times [0,1] \to \mathbb{R}^+$.

Consider the following assumptions:
(A_1) The function $g : [0,1] \to [0,1]$ is continuous;

(A_2) The function $K : [0,1] \times [0,1] \to \mathbb{R}^+$ are continuous such that for all $x, y \in [0,1]$.

$$\int_0^x K(x,y)dy \leq K;$$

(A_3) The functions $h : [0,1] \times \mathbb{R} \to \mathbb{R}$ are continuous and these is a constant C such that for all $y \in [0,1]$, $\zeta_1, \zeta_2 \in T$,

$$|h(x,\zeta_1) - h(x,\zeta_2)| \leq C_1|\zeta_1(x) - \zeta_2(x)|;$$

(A_4) $\lambda CK = 1$. □

Theorem 4.2. *Assume that $X = C[0,1]$ is space of continuous function on [0,1] and S is a compact subset of X where supremum norm is defined by $\|\zeta_1 - \zeta_2\| = C_1 \sup_{\tau \in I} |\zeta_1(x) - \zeta_2(x)|$ and assumptions from (A_1) to (A_4) are true. The mapping $Q : S \to S$ is defined by*

$$Q(\zeta(x)) = g(x) + \lambda \int_0^x K(x,y)h(y,\zeta(y))dy. \tag{39}$$

Then the nonlinear FIE (38) has a solution in $X = C[0,1]$.

Proof. Assume that $\zeta_1, \zeta_2 \in S$. Then

$$\begin{aligned}
&|Q(\zeta_1(x)) - Q(\zeta_2(x))| \\
&\quad = \left| g(x) + \lambda \int_0^x K(x,y)h(y,\zeta_1(y))dy - g(x) - \lambda \int_0^x K(x,y)h(y,\zeta_2(y))dy \right| \\
&\quad \le \lambda \int_0^x K(x,y)|h(y,\zeta_1(y)) - h(y,\zeta_2(y))|dy \\
&\quad \le \lambda \int_0^x K(x,y)C|\zeta_1(y) - \zeta_2(y)|dy.
\end{aligned}$$

On taking supremum both sides, we get

$$\|Q(\zeta_1) - Q(\zeta_2)\| \le \lambda KC\|\zeta_1 - \zeta_2\| = \|\zeta_1 - \zeta_2\|.$$

This shows that the map Q is satisfy $\vartheta-$ nonexpansive for $\vartheta = 0$, than all conditions of Theorem 4.1 are satisfied, hence Q has fixed point. Therefore FIE (38) has a solution in S. $\square$

Example 2. *Let us consider the following nonlinear FIE : For $x, y \in [0,1]$,*

$$\zeta(x) = (x+1) + \int_0^x [x^2(y+2)] \quad 2|\zeta(y)|dy. \tag{40}$$

If we take $\lambda = \frac{1}{5}$, $f(x) = x+1$; $K(x,y) = x^2(y+1)$, $h(y,\zeta(y)) = 2|\zeta(y)|$. Then (40) will be in form of (38).

It is clear that function $g(x) = x+1$, $\forall x \in [0,1]$ is continuous.

For each $x, y \in [0,1]$,.

$$\int_0^1 K(x,y)dt = \int_0^1 x^2(y+2)dy \le \frac{5}{2}.$$

For $\zeta_1, \zeta_2 \in x, y \in [0,1]$,

$$\begin{aligned}
|h(y,\zeta_1(y)) - h(y,\zeta_2(y))| &= \Big| 2|\zeta_1(y)| - 2|\zeta_2(y)| \Big| \\
&= 2\Big| |\zeta_1(y)| - |\zeta_2(y)| \Big| \\
&\le 2|\zeta_1(y) - \zeta_2(y)|.
\end{aligned}$$

Since all assumptions of Theorem 4.2 are satisfied with $C = 2$ and $\lambda CK = 1$. Therefore nonlinear FIE (40) has a solution.

5. CONCLUSION

In this paper, we established a monotone ϑ-nonexpansive mapping within a hyperbolic metric space. We employed these mappings to prove some strong and $\triangle$-convergence theorems, as well as to identify approximated fixed points. Additionally, we discussed the solutions to nonlinear integral equations. We compute comparison of proposed iterative algorithm (7) to known iterative algorithms ([3, 5, 6, 17, 30, 33]) in fact that the iterative sequence generated by the proposed iterative algorithm converges faster than to iterative sequence generated by known iterative algorithm as shown in Example 1 above.

REFERENCES

[1] M. Abbas, T. Nazir, "A New Faster Iteration Process Applied to Constrained Minimization and Feasibility Problems", *Mat. Vesn.* 66 (2014), 223–234.

[2] K. Aoyama, F. Kohsaka, "Fixed point theorem for non-expansive mappings in Banach spaces" *Nonlinear Anal.* , 74 (2011) 4387–4391.

[3] R. P. Agarwal, D. O'Regan, D. R. Sahu, "Iterative Construction of Fixed Points of Nearly Asymptotically Nonexpansive Mappings", *J. Nonlinear Convex Anal.*, **8**, (2007), 61-79.

[4] R. P. Agarwal, D. O'Regan, and D. Sahu, "Fixed Point Theory for Lipschitzian-Type Mappings with Applications", *Springer*, 6 (2009).

[5] J. Ali, F. Ali, "A new iterative scheme to approximating fixed points and the solution of a delay differential equation", *J. Nonlinear Convex Anal.*, 9 (2020), 1251–1263.

[6] T. Abdeljawad, K. Ullah, and J. Ahmad, "Iterative Algorithm for Mappings Satisfying (β_γ, μ) Condition", *Journal of Function Spaces*, 2020 (2020), 1-7.

[7] S. Banach, "Sur les opérations dans les ensembles abstraits et leurs applications aux équations intégrales", *Fundam. Math.* 3 (1922), 133–181.

[8] M. Bachar and M. A. Khamsi, "On common approximate fixed points of monotone nonexpansive semigroups in Banach spaces", *Fixed Point Theory and Applications*, 2015 (2015).

[9] R. Chugh, V. Kumar, S. Kumar, "Strong Convergence of a New Three Step Iterative Scheme in Banach Spaces", *Amer.J. Comput. Math.* 2 (2012), 345–357.

[10] B. A. B. Dehaish and M. A. Khamsi, "Mann iteration process for monotone nonexpansive mappings", *Fixed Point Theory and Applications* 2015 (2015).

[11] B. A. B. Dehaish and M. A. Khamsi, "Browder and Göhde fixed point theorem for monotone nonexpansive mappings", *Fixed Point Theory and Applications*, 2016 (2016).

[12] F. Gürsoy, V. Karakaya, "A Picard-S Hybrid Type Iteration Method for Solving a Differential Equation With Retarded Argument, math.FA, (2014).

[13] S. Ishikawa, "Fixed points by a new iteration method", *Proceedings of American Mathematical Society*, 44 (1974), 147–150.

[14] M. A. Khamsi and A. R. Khan, "On monotone nonexpansive mappings in $L_1([0,1])$", *Fixed Point Theory and Applications*, 2015 (2015).

[15] W. A. Kirk and B. Panyanak, "A concept of convergence in geodesic spaces", *Nonlinear Analysis*, 68 (2008), 3689–3696.

[16] A. Kalsoom, M. Rashid, T. Sun et al., "Fixed points of monotone total asymptotically nonexpansive mapping in hyperbolic space via new algorithm," *Journal of Function Spaces*, 2021 (2021), 10.

[17] A. Kalsoom et al., "Fixed point approximation of monotone nonexpansive mappings in hyperbolic spaces", *Journal of Function Spaces*, 2021 (2021).

[18] U. Kohlenbach, "Some logical metatheorems with applications in functional analysis", *Transactions of the American Mathematical Society*, 357(1) (2005), 89 – 128.

[19] L. Leustean, "Nonexpansive iterations in uniformly convex $W-$hyperbolic spaces", *Nonlinear Analysis and Optimization I: Nonlinear Analysis Publisher: American Mathematical Society*, 2010.

[20] W. R. Mann, "Mean Value Methods in Iterations", *Proc. Am. Math. Soc.*, 4 (1953), 506- 510.

[21] J.J. Nieto, R. Rodriguez-Löpez, "Contractive mapping theorems in partially ordered sets and application to ordinary differential equations", Order 22 (2005), 223–239.

[22] M. A. Noor, "New approximation schemes for general variational inequalities", *Journal of Mathematical Analysis and Applications*, 251 (2000), 217 –229.

[23] R. Pant, R. Pandey, "Existence and convergence results for a class of nonexpansive type mappings in hyperbolic spaces", *Appl. Gen. Topol.*, 20 (2019), 281–295.

[24] R. Pant, R. Shukla, "Approximating fixed points of generalized-non-expansive mappings in Banach spaces", *Numer. Funct. Anal. Optim.*, 2017, 38, 248–266.

[25] R. Pandey, R. Pant, V. Rakocevic, R. Shukla, "Approximating fixed points of a general class of nonexpansive mappings in Banach spaces with applications", *Result Math.*, 74 (2019), 1–24.

[26] E. Picard, "Memoire sur la theorie des equations aux derivees partielles et la methode des approximations successives," *Journal de Mathematiques Pures et Appliquees,* 6 (1890) 145–210.

[27] A.C.M. Ran, M.C.B. Reurings, "A fixed point theorem in partially ordered sets and some applications to matrix equations", *Proc. Am. Math. Soc.*, 132 (2004), 1435–1443.

[28] D.R. Sahu, A. Petru¸sel, "Strong Convergence of Iterative Methods by Strictly Pseudocontractive Mappings in Banach Spaces, *Nonlinear Anal.: Theory Methods Appl.*, 74 (2011), 6012–6023.

[29] T. Suzuki, "Fixed point theorems and convergence theorems for some generalized non-expansive mapping", *J. Math. Anal. Appl.*, 340 (2008), 1088–1095.

[30] J. Srivastava, "Introduction of new Picard–S hybrid iteration with application and some results for nonexpansive mappings", *Arab Journal of Mathematical Sciences*, 28 (2021), 61-76.

[31] B.S. Thakur, D. Thakur, M. Postolache, "A New Iterative Scheme for Numerical Reckoning Fixed Points of Suzuki's Generalized Nonexpansive Mappings", *Appl. Math. Comput.*, 275 (2016), 147–155.

[32] I. Uddin, C. Garodia and J. J. Nietro, "Mann iteration for monotone nonexpansive mappings in ordered $CAT(0)$ space with an application to integral equations", *Journal of Inequalities and Applications*, 2018 (2018), 1-13.

[33] K. Ullah and M. Arshad, "Numerical Reckoning Fixed Points for Suzukis Generalized Nonexpansive Mappings via New Iteration Process", *Filomat*, 32(1) (2018).

[34] K. Ullah, J. Ahmad and M. de la Sen, "On generalized nonexpansive maps in Banach spaces", *Comput.* 8 (2020), 61.

[35] K. Ullah, M.S.U. Khan and M. de la Sen, "Fixed point results on multi-valued generalized (α, β)-nonexpansive mappings in Banach spaces", *Algorithms*, 14 (2021), 223.