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Lagarias' Inequality Pertaining to the Riemann Hypothesis and the Sum of Reciprocals of the Primes

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Abstract

An inequality analogous to Lagarias' inequality is introduced. An alternate form of Lagarias' inequality is derived using Dirichlet inverses. A relationship between the Dirichlet inverse of the sum of divisors function and highly abundant numbers is investigated. A function having some properties similar to those of the sum of divisors function is introduced.

Keywords Riemann hypothesis, harmonic numbers, sum of reciprocals of primes, Dirichlet inversion, highly abundant numbers.

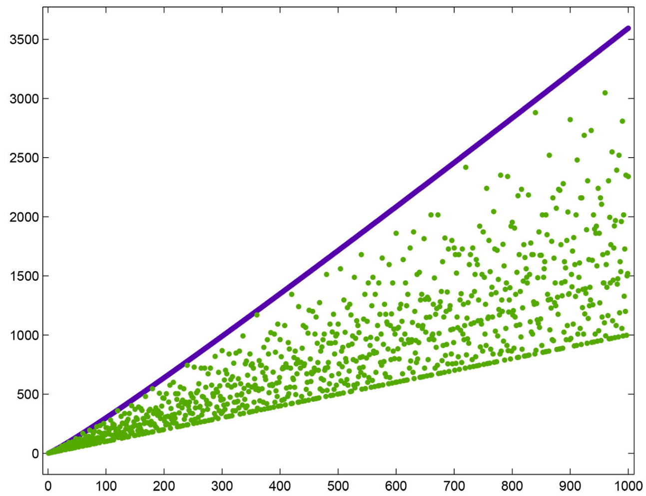
1. INTRODUCTION

Lagarias' [1] Theorem 1.1 is

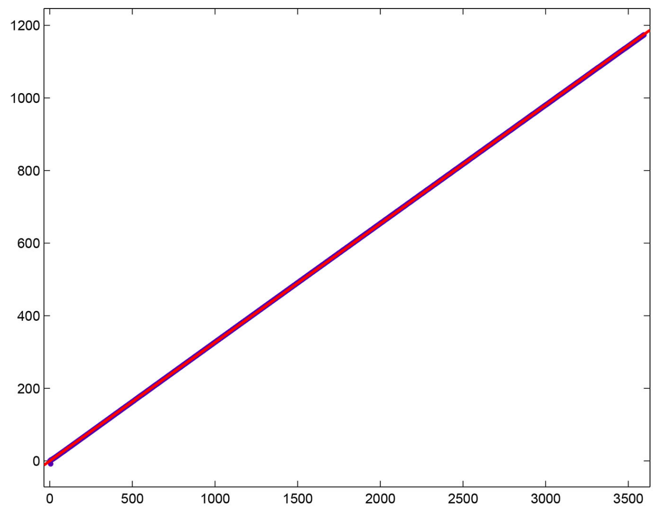
Theorem 1. *The inequality* $\sum_{d|n} d \leq H_n + \exp(H_n)\log(H_n)$ *(with equality only for* $n = 1$*) is equivalent to the Riemann hypothesis.*

The function $\sigma(n) = \sum_{d|n} d$ is the sum of divisors function and $H_n = \sum_{j=1}^{n} \frac{1}{j}$ is called the n-th harmonic number.

Let $\alpha(n)$ denote $H_n + \exp(H_n)\log(H_n)$. A plot $\alpha(n)$ and $\sigma(n)$ for $n \leq 1000$ is



Let R_n denote the sum of the reciprocals of the primes up to the nth prime. Let $\beta(n)$ denote $R_n + \exp(R_n)\log(R_n)$. A plot of $\beta(n)/\log(\beta(n))$ versus $\alpha(n)$ for $n \leq 1000$ is



For a linear least-squares fit of the curve, $p_1 = 0.3269$ with a 95% confidence interval of (0.3269, 0.3269), $p_2 = -0.206$ with a 95% confidence of $(-0.284, -0.1281)$, SSE=422.1, R-squared=1, and RMSE=0.6503.

For a linear least-squares fit of the curve where $n \leq 1000000$, $p_1 = 0.3099$ with a 95% confidence interval of (0.3099, 0.3099), $p_2 = 3503$ with a 95% confidence interval of (3499, 3507), SSE=$1.035 \cdot 10^{12}$, R-squared=1, and RMSE=1018. Solving for $\alpha(n)$ gives about $\frac{\beta(n)-3503}{.3099}$.

2. THE DIRICHLET INVERSES OF $\sigma(n)$ AND $\alpha(n)$AND GENERALIZED MÖBIUS INVERSION

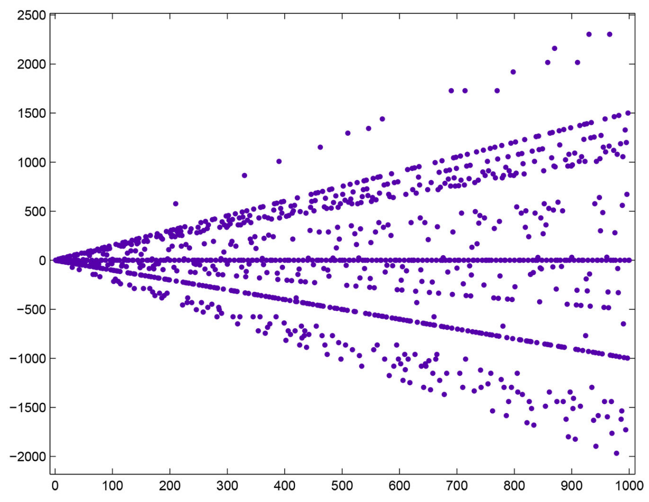
Theorem 2.8 of Apostol's book is

Theorem 2. *If f is an arithmetical function with $f(1) \neq 0$ there is a unique arithmetical function f^{-1}, called the Dirichlet inverse of f, such that $f * f^{-1} = f^{-1} * f = I$. Moreover, f^{-1} is given by the recursion formulas $f^{-1}(1) = 1/f(1)$, $f^{-1}(n) = \frac{-1}{f(1)} \sum_{d|n, d<n} f(\frac{n}{d}) f^{-1}(d)$ for $n > 1$.*

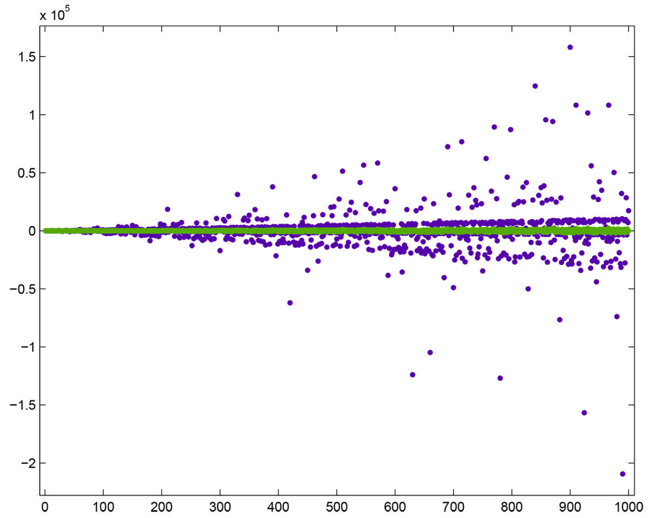
Theorem 2.22 (generalized inversion formula) in Apostol's book is

Theorem 3. *If α has a Dirichlet inverse α^{-1}, then the equation (10) $G(x) = \sum_{n \leq x} \alpha(n) F(\frac{x}{n})$ implies (11) $F(x) = \sum_{n \leq x} \alpha^{-1} G(\frac{x}{n})$. Conversely, (11) implies (10).*

Let $\sigma'(n)$ denote the Dirichlet inverse of $\sigma(n)$. A plot of $\sigma'(n)$ for $n \leq 1000$ is

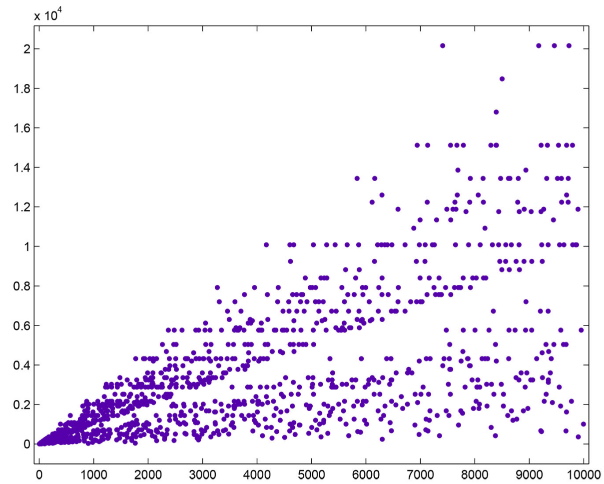


Let $\alpha'(n)$ denote the Dirichlet inverse of $\alpha(n)$. A plot of $\sigma'(n)$ and $\alpha'(n)$ for $n \leq 1000$ is

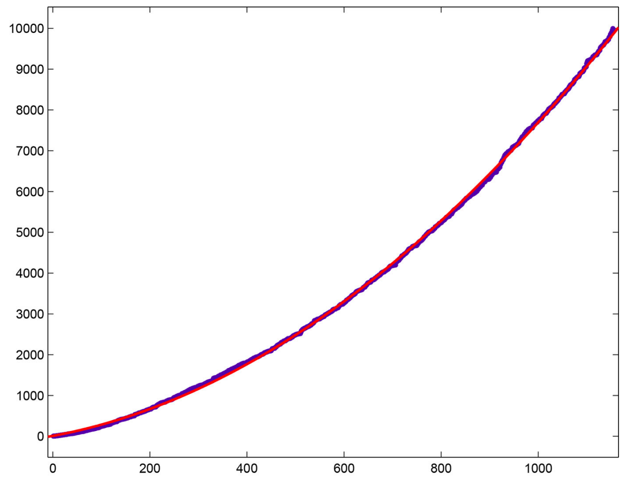


The absolute value of $\sigma'(n)$ is less than the absolute value of $\alpha'(n)$. This is the alternate form of Lagarias' inequality.

For a highly abundant number, $\sigma(n) > \sigma(m)$ for all $m < n$. For a superabundant number, $\sigma(n)/n > \sigma(m)/m$ for all $m < n$. Colossally abundant numbers are those numbers for which there is a positive constant ϵ such that $\frac{\sigma(n)}{n^{1+\epsilon}} \geq \frac{\sigma(k)}{k^{1+\epsilon}}$ for all $k > 1$. A plot of the $|\sigma'(n)|$ values that are highly abundant versus the corresponding n values for $n \leq 10000$ is

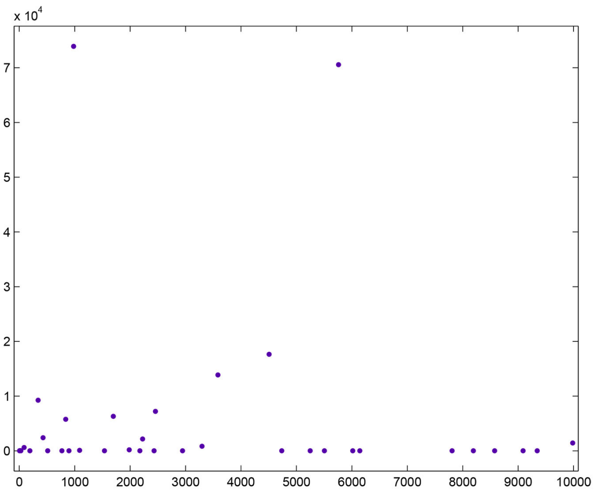


There are 1154 values. Some $|\sigma'(n)|$ value equals every highly abundant number less than 10000. A plot of the corresponding n values is

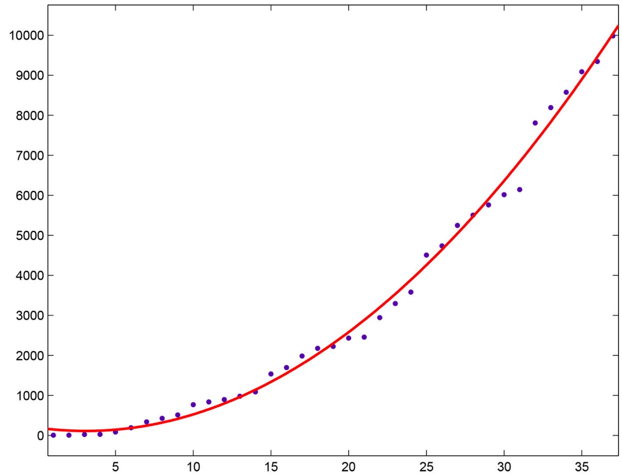


For a quadratic least-squares fit of the curve, $p_1 = 0.00548$ with a 95% confidence interval of (0.005454, 0.005505), $p_2 = 2.207$ with a 95% confidence interval of (2.176, 2.238), $p_3 = 9.383$ with a 95% confidence interval of (1.725, 17.04), SSE=$2.241 \cdot 10^6$, R-squared=0.9998, and RMSE=44.12.

A plot of the rounded $|\alpha'(n)|$ values that are highly abundant versus the corresponding n values for $n \leq 10000$ is



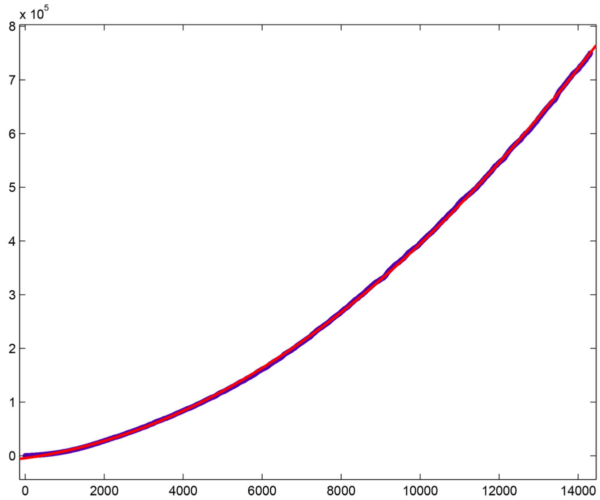
There are 37 values. A plot of the corresponding n values is



For a quadratic least-squares fit of the curve, $p_1 = 8.621$ with a 95% confidence interval of (7.84, 9.402), $p_2 = -53.03$ with a 95% confidence interval of $(-83.63, -22.43)$, $p_3 = 194.4$ with a 95% confidence interval of (-57.78, 446.6), SSE=$1.938 \cdot 10^6$, R-squared=0.9945, and RMSE=238.1.

The highly abundant numbers less than or equal to 250000 are 2, 4, 6, 8, 10, 12, 16, 18, 20, 24, 30, 36, 42, 48, 60, 72, 84, 90, 96, 108, 120, 144, 168, 180, 210, 216, 240, 288, 300, 336, 360, 420, 480, 504, 540, 600, 630, 660, 672, 720, 840, 960, 1008, 1080, 1200, 1260, 1440, 1560, 1620, 1680, 1800, 1920, 1980, 2016, 2100, 2160, 2340, 2400, 2520, 2880, 3024, 3120, 3240, 3360, 3600, 3780, 3960, 4200, 4320, 4620, 4680, 5040, 5760, 5880, 6120, 6240, 6300, 6720, 7200, 7560, 7920, 8400, 8820, 9240, 10080, 10920, 11340, 11760, 11880, 12240, 12600, 13440, 13860, 15120, 16380, 16800, 17640, 18480, 19800, 20160, 21840, 22680, 23760, 25200, 27720, 30240, 32760, 35280, 36960, 37800, 39600, 40320, 41580, 42840, 43680, 45360, 47520, 47880, 49140, 50400, 52920, 54600, 55440, 60480, 64680, 65520, 69300, 70560, 73080, 73920, 75600, 80640, 83160, 85680, 90720, 92400, 95760, 98280, 100800, 105840, 109200, 110880, 120120, 120960, 126000, 128520, 131040, 138600, 146160, 147840, 151200, 161280, 163800, 166320, 180180, 181440, 184800, 191520, 194040, 196560, 207900, 211680, 214200, 218400, 221760, 239400, 240240, and 249480. For $n \leq 100000$, the largest $|\sigma'(n)|$ value equal to a highly abundant number is 221760. Except for $n = 180180$, 207900, 214200, and 218400, there are $|\sigma'(n)|$ values equal to every highly abundant number less than 221760.

For $n \leq 2000000$, there are 242 highly abundant numbers, the largest being 1995840. For $n \leq 750000$, there are $|\sigma'(n)|$ values equal to every highly abundant number less than 1995840 except 1580040, 1607760, 1627920, 1638000, 1769040, 1801800, 1884960, 1940400, and 1965600. A plot of the corresponding n values is



There are 14311 values. For a quadratic least-squares fit of the curve, $p_1 = 0.002984$ with a 95% confidence interval of (0.002983, 0.002986), $p_2 = 10.06$ with a 95% confidence interval of (10.03, 10.09), $p_3 = -4421$ with a 95% confidence interval of (-4508, -4335), SSE=$4.428 \cdot 10^{10}$, R-squared=0.9999, and RMSE=1759.

The average order of $\sigma(n)$ is $\frac{\pi^2}{6}n$. More precisely,

$$\sum_{j=1}^{n} \sigma(j) = \frac{\pi^2}{12}n^2 + O(n \cdot \log(n)) \tag{1}$$

as $n \to \infty$. See Hardy and Wright's [3] Theorem 324. The maximal order of $\sigma(n)$ is

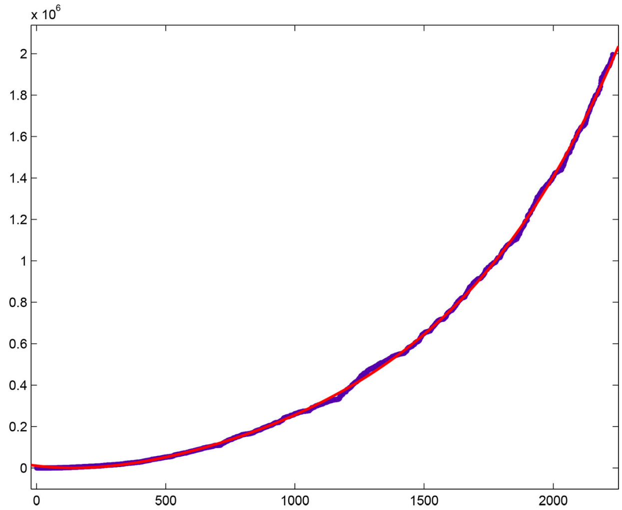
$$\limsup_{n\to\infty} \frac{\sigma(n)}{n \log\log n} = e^{\gamma} \tag{2}$$

where γ is Euler's constant. This was proved by Gronwall. See Hardy and Wright's Theorem 323, Sect.18.3 and 22.9. An asymptotic upper bound of $\sigma(n)$ derived by Robin [4] is

$$\sigma(n) < e^{\gamma} n \log\log n + .6482 \frac{n}{\log\log n}. \tag{3}$$

Robin's criterion states that the Riemann hypothesis is true if and only if $\sigma(n) < e^{\gamma} n \log\log n$ for all $n \geq 5041$. Ramanujan [5] derived upper and lower bounds for the order of generalized highly composite numbers, assuming the Riemann hypothesis. His bounds imply that Robin's criterion holds for all sufficiently large n. The results of Alaoglu and Erdös [6] in their paper are unconditional and mainly concern the exponents in highly composite and superabundant numbers. Robin showed that if the Riemann hypothesis is false, there will necessarily exist a counterexample to the above criterion which is a colossally abundant number.

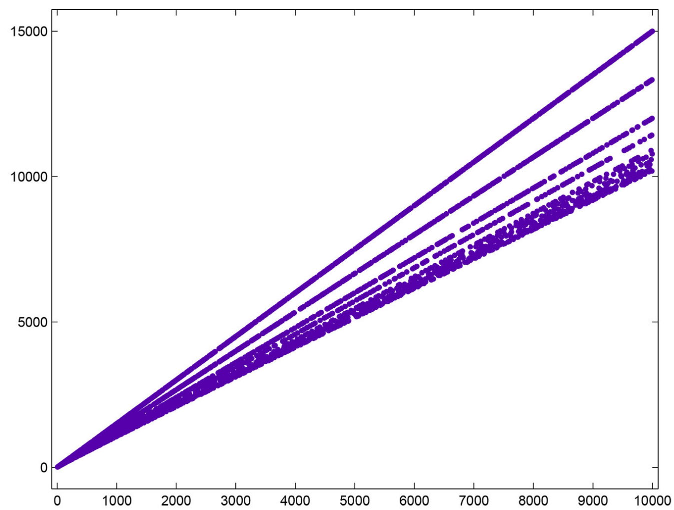
The above quadratic curves of the n values of $|\sigma'(n)|$ values equal to highly abundant numbers are relevant to this. A plot of the n values of $|\sigma'(n)|$ values equal to superabundant numbers for $n \leq 2000000$ is



There are 2239 values. For a quartic least-squares fit of the curve, $p_1 = -0.0003018$ with a 95% confidence interval of $(-0.0003173, -0.0002864)$, $p_2 = 0.5845$ with a 95% confidence interval of (0.5615, 0.6075), $p_3 = -147.3$ with a 95% confidence interval of $(-159.9, -134.7)$, $p_4 = 1.089 \cdot 10^4$ with a 95% confidence interval of (8860, $1.292 \cdot 10^4$), SSE=$2.115 \cdot 10^{11}$, R-squared=0.9997, and RMSE=9.751.
The first few colossally abundant numbers are 2, 6, 12, 60, 120, 360, 2520, 5040, 55440, 720720, 1441440, 4324320,.... Colossally abundant numbers are best described using the Dirichlet inverse of the $\rho(n)$ function to be defined in the next section.

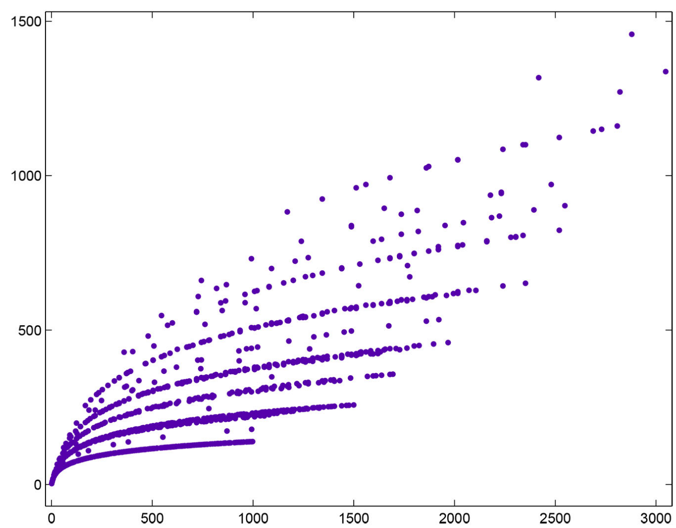
Some empirical results are
(1) $\sigma'(n) = -\sigma(n)$ at prime n.
(2) $\sigma'(n) = n$ at prime-squared n.
(3) $\sigma'(n) = 0$ at prime-cubed n.

Let p and q denote distinct primes. A plot of $\sigma'(n)$ at $n = pq$ versus n for $n \leq 1000$ is

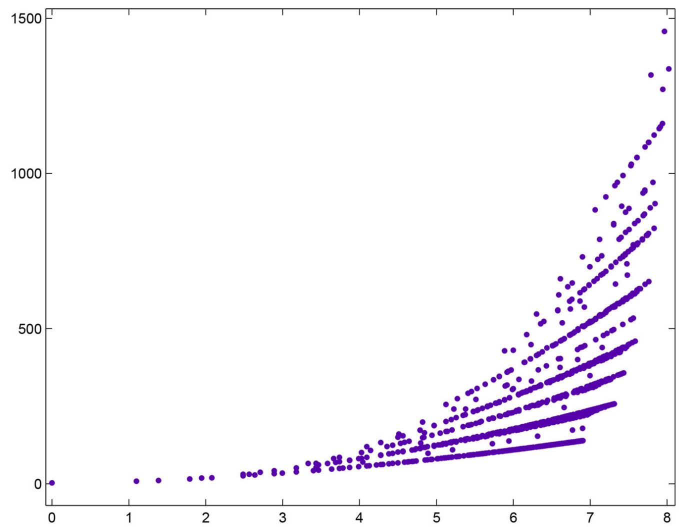


3. A FUNCTION RELATED TO $\sigma(n)$

Let $\rho(n)$ denote $\sum_{d|n}(\exp(R(d)))^2$. A plot of $\rho(n)$ versus $\sigma(n)$ for $n \leq 1000$ is

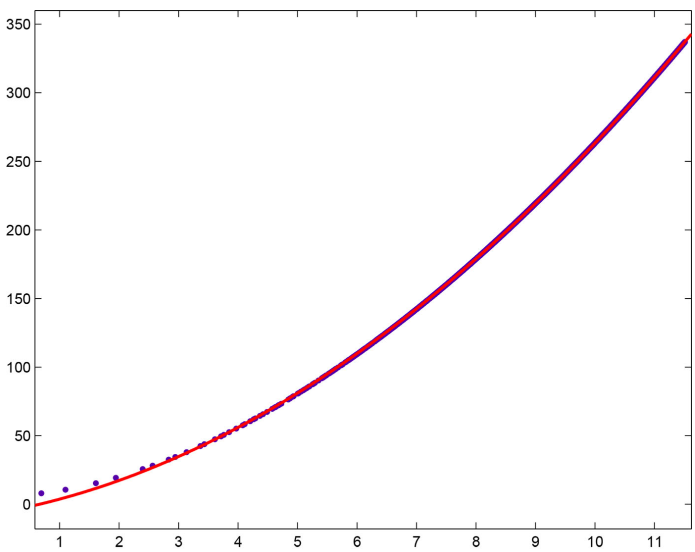


The values consist of logarithmic curves. A plot of $\rho(n)$ versus $\log(\sigma(n))$ is



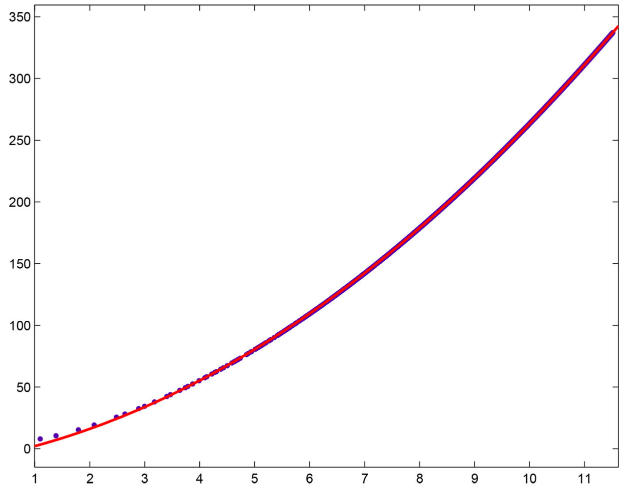
The values consist of quadratic curves.

A plot of $\rho(n)$ versus $\log(n)$ for prime n less than 100000 is



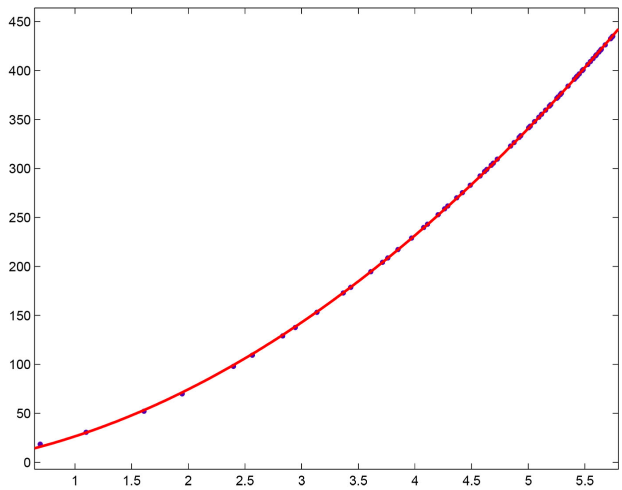
For a quadratic least-squares fit of the curve, $p_1 = 1.903$ with a 95% confidence interval of (1.901, 1.904), $p_2 = 7.917$ with a 95% confidence interval of (7.897, 7.938), $p_3 = -6.113$ with a 95% confidence interval of (-6.207, -6.019), SSE=206.6, R-squared=1, and RMSE=0.1468.

A plot of $\rho(n)$ versus $\log(\sigma(n))$ for prime n less than 100000 is



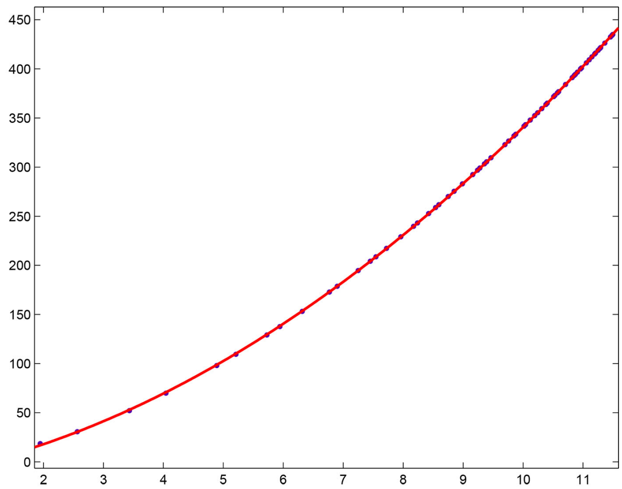
For a quadratic least-squares fit of the curve, $p_1 = 1.882$ with a 95% confidence interval of (1.882, 1.883), $p_2 = 8.311$ with a 95% confidence interval of (8.297, 8.324), $p_3 = -8.013$ with a 95% confidence interval of (-8.075, -7.95), SSE=86.79, R-squared=1, and RMSE=0.09514.

A plot of $\rho(n)$ versus $\log(n)$ for prime-squared n less than 100000 is



For a quadratic least-squares fit of the curve, $p_1 = 10.2$ with a 95% confidence interval of (10.12, 10.29), $p_2 = 17.36$ with a 95% confidence interval of (16.7, 18.01), $p_3 = -1.038$ with a 95% confidence interval of (-2.211, 0.135), SSE=22.2, R-squared=1, and RMSE=0.5983.

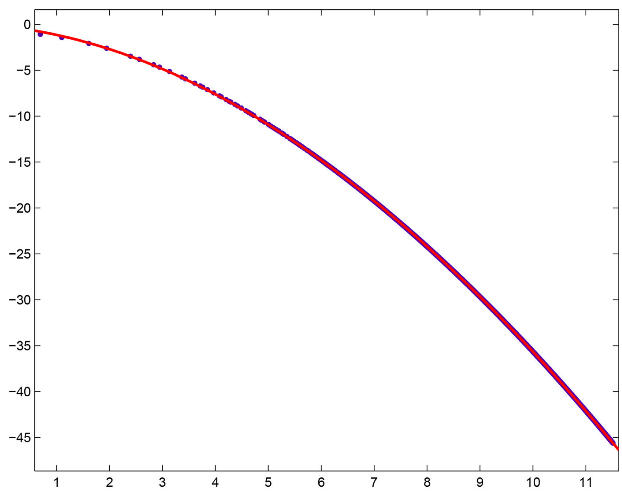
A plot of $\rho(n)$ versus $\log(\sigma(n))$ for prime-squared n less than 100000 is



For a quadratic least-squares fit of the curve, $p_1 = 2.422$ with a 95% confidence interval of (2.407, 2.436), $p_2 = 11.28$ with a 95% confidence interval of (11.06, 11.5), $p_3 = -14.37$ with a 95% confidence interval of $(-15.18, -13.56)$, SSE=8.158, R-squared=1, and RMSE=0.3627.

Similar results are obtained for other prime-power n. For n that are the product of two distinct primes, a group of quadratic curves is obtained.

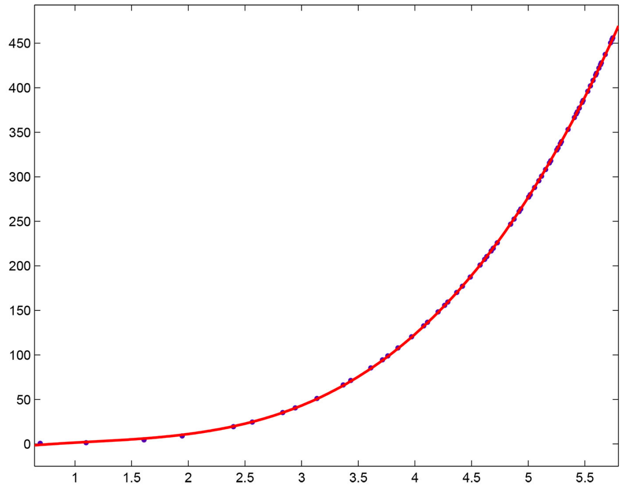
Let $\rho'(n)$ denote the Dirichlet inverse of $\rho(n)$. A plot of $\rho'(n)$ versus $\log(n)$ at prime n less than 100000 is



For a cubic least-squares fit of the curve, $p_1 = 0.002401$ with a 95% confidence interval of (0.002387, 0.002416), $p_2 = -0.316$ with a 95% confidence interval of $(-0.3163,$

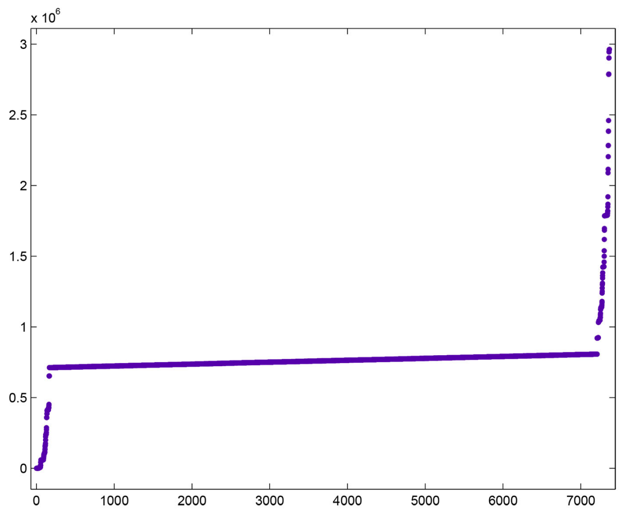
-0.3156), $p_3 = -0.6249$ with a 95% confidence interval of (-0.6277, -0.6221), $p_4 = -0.2069$ with a 95% confidence interval of (-0.214, -0.1997), SSE=0.3098, R-squared=1, and RMSE=0.005685.

A plot of $\rho'(n)$ versus $\log(n)$ at prime-squared n less than 100000 is



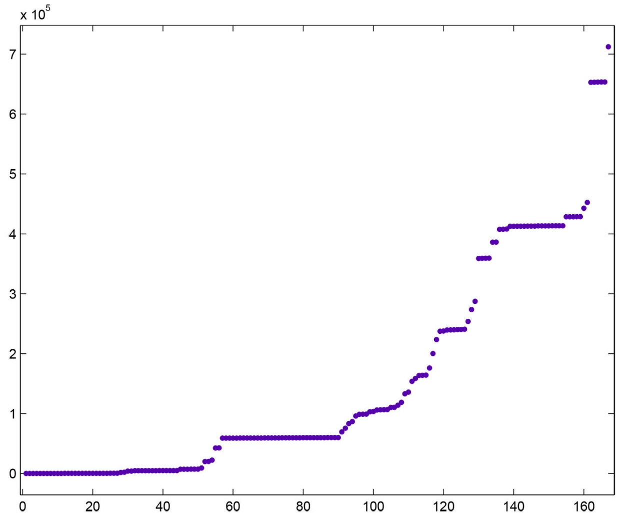
For a cubic least-squares fit of the curve, $p_1 = 4.287$ with a 95% confidence interval of (4.227, 4.348), $p_2 = -14.57$ with a 95% confidence of (-15.21, -13.93), $p_3 = 23.38$ with a 95% confidence interval of (21.35, 25.41), $p_4 = -11.61$ with a 95% confidence interval of (-13.51, -9.713), SSE=17.88, R-squared=1, and RMSE=0.5414.

A plot of the n values of rounded $|\rho'(n)|$ values equal to colossally abundant numbers for $n \leq 3000000$ is

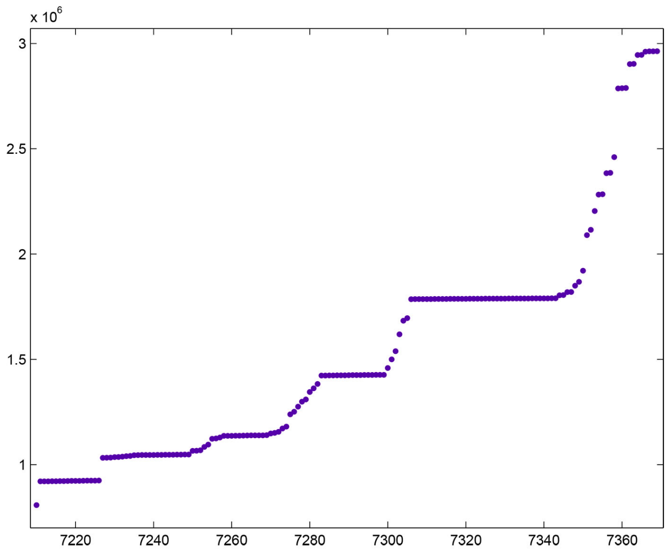


There are 7369 values. For a linear least-squares fit of the curve from $n = 167$ to 7210, $p_1 = 13.54$ with a 95% confidence interval of (13.54, 13.54), $p_2 = 7.101 \cdot 10^5$ with a 95% confidence interval of $(7.101 \cdot 10^5, 7.101 \cdot 10^5)$, SSE=$1.049 \cdot 10^8$, R-squared=1, and RMSE=122.1. The corresponding rounded $|\rho'(n)|$ value is 60. Except for 720720 and 1441440, the superabundant numbers equal some rounded $|\rho'(n)|$ value.

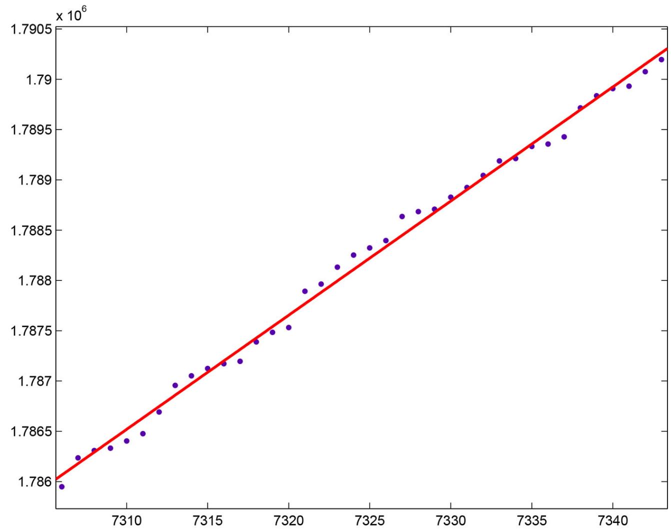
A plot of the n values from 1 to 167 is



A plot of the n values from 7210 to 7369 is



A plot of the n values from 7306 to 7343 is



The corresponding $|\rho'(n)|$ value is 2520. For a linear least-squares fit of the curve, $p_1 = 113.5$ with a 95% confidence interval of (110.7, 116.3), $p_2 = 9.568 \cdot 10^5$ with a 95% confidence interval of $(9.365 \cdot 10^5, 9.772 \cdot 10^5)$, SSE=$3.094 \cdot 10^5$, R-squared=0.9948, and RMSE=92.71.

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