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Soft Near Vector Spaces

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Abstract

We introduce the concept of soft near vector spaces as a natural generalization of soft vector spaces over near vector spaces. Using soft set theory, we formalize the definition, provide illustrative examples, and establish fundamental results concerning operations such as intersection, union, Cartesian products and sums.

Keywords: Bi-intersection, Near vector space, Soft near vector space, Soft set, Soft vector space.

Introduction

The concept of soft sets was first introduced by Molodtsov in 1999[2] as a general mathematical tool for dealing with uncertainty, and it has since been applied to various algebraic structures. Following this, the notion of soft groups was proposed by Aktas and Çağman (2007)[4] and the concept of soft vector spaces was developed by Maji, Biswas, and Roy (2003)[6].
Near vector spaces [1], defined over near fields, generalize the concept of vector spaces by relaxing distributive properties. This paper aims to blend these two frameworks by defining soft near vector spaces and exploring their properties. While the framework of soft vector spaces has been addressed in earlier studies, the notion of soft near vector spaces remains largely unexplored in the existing literature. This paper seeks to bridge this gap by initiating a systematic study of soft near vector spaces, laying the groundwork for further theoretical developments and potential applications.

Preliminaries

Definition 2.1 (Soft Set): Let U be an initial universe and A a set of parameters. A pair (F, A) is called a **soft set** over U if F is a mapping from A to the power set $P(U)$.

Definition 2.2: For two soft sets (F, A) and (G, B) over a common universe U, we say that (F, A) is a **soft subset** of (G, B), denoted by $(F, A) \tilde{\subseteq} (G, B)$, if it satisfies:
i) $A \subset B$;
ii) ii) $\forall \varepsilon \in A, F(\varepsilon)$and $G(\varepsilon)$ are identical approximations.

Definition 2.3: The **bi-intersection** of two soft sets (F, A) and(G, B) over a common universe U is defined to be the soft set (H, C), where $C = A \cap B$ and $H : C \rightarrow P(U)$ is a mapping given by $H(x) = F(x) \cap G(x)$ for all $x \in C$. This is denoted by $(F, A) \tilde{\cap} (G, B) = (H, C)$.

Definition 2.4 : Let (F, A) and (G, B) be two soft sets over a common universe U such that $A \cap B = \emptyset$. The **restricted intersection** of (F, A) and (G, B) is denoted by $(F, A) \Cap (G, B)$, and is defined as $(F, A) \Cap (G, B) = (H, C)$, where $C = A \cap B$ and for all $c \in C, H(c) = F(c) \cap G(c)$.

Definition 2.5: Let (F, A) and (G, B) be two soft sets over a common universe U such that $A \cap B = \emptyset$. The **restricted union** of (F, A) and (G, B) is denoted by $(F, A) \cup_R (G, B)$, and is defined as $(F, A) \cup_R (G, B) = (H, C)$, where $C = A \cap B$ and for all $c \in C$, $H(c) = F(c) \cup G(c)$.

Definition 2.6: Let (F, A) and (G, B) be two soft sets over a common universe U. The **extended intersection** of (F, A) and (G, B) is defined to be the soft set (H, C), where $C = A \cup B$ and for all $e \in C$,

$$\mathrm{H(e)} = \begin{cases} \mathrm{F(e)} & \text{if } \mathrm{e} \in \mathrm{A \backslash B} \\ \mathrm{G(e)} & \text{if } \mathrm{e} \in \mathrm{B \backslash A} \\ \mathrm{F(e) \cap G(e)} & \text{if } \mathrm{e} \in \mathrm{A \cap B}. \end{cases}$$

This relation is denoted by $(F, A) \sqcap_\varepsilon (G, B) = (H, C)$.

Definition 2.7: Let (F, A) and (G, B) be two soft sets over a common universe U. The **union** of (F, A) and (G, B) is defined to be the soft set (H, C) satisfying the following conditions:
(i) $C = A \cup B$;
(ii) for all $e \in C$,

$$\mathrm{H(e)} = \begin{cases} \mathrm{F(e)} & \text{if } \mathrm{e} \in \mathrm{A \backslash B} \\ \mathrm{G(e)} & \text{if } \mathrm{e} \in \mathrm{B \backslash A} \\ \mathrm{F(e) \cup G(e)} & \text{if } \mathrm{e} \in \mathrm{A \cap B}. \end{cases}$$

This relation is denoted by $(F, A) \tilde{\cup} (G, B) = (H, C)$.

Definition 2.8: A set-valued function $F : A \rightarrow P(V)$ can be defined as $F(x) = \{y \in V \mid (x, y) \in R\}$ for all $x \in A$. Then the pair (F, A) is a soft set over V, which is derived from the relation R. For a soft set (F, A), the set $Supp(F, A) = \{x \in A \mid F(x) \neq \emptyset\}$ is called the **support** of the soft set (F, A).

Definition 2.9: Let (F, A) be a non-null soft set over a vector space. Then (F, A) is called a **soft vector space** over V if $F(x)$ is a subvector space of V for all $x \in Supp(F, A)$.

Definition 2.10: A non-empty set N with two binary operations $+$ and $\cdot$ is called a **near-field** if
(i) $(N, +)$ is an abelian group,
(ii) $(N\backslash 0, \cdot)$ is a group, and
(iii) Only one distributive law holds, ie, either left distributive or right distributive.

Definition 2.11: Let V be a non-empty set and N be a near-field. Then V is called a near vector space on N if $(V, +)$ is an abelian group and scalar multiplication $N \times V \rightarrow V$ satisfies the usual axioms analogous to those of a vector space.

Soft Near Vector Spaces

A near group is an algebraic structure that generalizes groups by relaxing certain axioms, allowing for applications in areas where strict group properties are not necessary. By combining the ideas of soft sets and near vector space, we arrive at the notion of a soft near vector space .This concept enables the study of uncertain or parameter-dependent sub vector space structures within near vector space.

Definition 3.1: Let (V, N) be a near vector space and A, a set of parameters.
A soft near vector space over V is a soft set (F, A) such that for each $a \in A$, $F(a)$ is a near subspace of V .

Examples 3.2:

Example 3.2.1: Let $K = R$ be considered as a right near-field with usual addition and multiplication, but only right distributivity assumed:
$(a + b) \cdot \lambda = a \cdot \lambda + b \cdot \lambda$, for all $a, b, \lambda \in R$.

$$(x, y) \cdot \lambda = (x\lambda, y\lambda).$$

Let $E = \{e_1, e_2, e_3\}$ and $F: E \rightarrow R^2$ be defined by $F(e_1) = \{(x, 0) \mid x \in R\}, F(e_2) = \{(0, y) \mid y \in R\}, F(e_3) = \{(x, x) \mid x \in R\}$ .Then (F, E) is a soft near vector space in R^2.

Example 3.2.2 : Let $V = R$ over near-field R (right distributive).
Let $= \{e_1, e_2, e_3\}$, and define: $F(e_1) = R$, $F(e_2) = \{0\}$, $F(e_3) = \{x \in R \mid x \geq 0\}$ gives a near subspace in R.

Example 3.2.3: Let $V = R^3$ over near-field R (right distributive).
Take parameter set $E = e_1, e_2, e_3$, and define:
$F(e_1) = \{(x, 0, 0) \mid x \in R\}, F(e_2) = \{(0, y, y) \mid y \in R\}, F(e_3) = \{(x, x, x) \mid x \in R\}$ Near subspaces in R^3.

Example 3.2.4: Let $V = C(R)$ the set of all real-valued continuous functions on R, with pointwise addition and scalar multiplication (right distributive).

Let $E = \{ e_1, e_2\}$, and define:
$F(e_1) = \{f \in V \mid f(-x) = f(x)\}$ (Even functions),
$F(e_2) = \{f \in V \mid f(-x) = -f(x)\}$ (Odd functions).
Both $F(e_1)$ and $F(e_2)$ are near subspaces of V. Hence (F, E) is a soft near vector space over R.

Example 3.2.5 : let $V = R[x]$, the set of all real polynomials. Let $E = \{e_1, e_2\}$, and define:
$F(e_1) = \{p(x) \in V \mid deg(p) \leq 2\}$, $F(e_2) = \{p(x) \in V \mid p(0) = 0\}$ are near subspaces of V. Hence (F, E) is a soft near vector space over R.

Theorem 3.3 (Arbitrary Intersection preserves softness):. Let (F, A) and (G, B) be soft near vector spaces over (V, N). Then the soft set $(F, A) \sqcap_{\varepsilon} (G, B)$ is a soft near vector space over V .
Proof :
By definition, we can write$(F, A) \cap_{\varepsilon} (G, B)$= (H, C), where $C = A \cup B$ and

$$H(x) = \begin{cases} F(x) & if\ x \in A \setminus B, \\ G(x) & if\ x \in B \setminus A, \\ F(x) \cap G(x) & if\ x \in A \cap B\ for\ all\ x \in C. \end{cases}$$

Suppose that (H, C) is a non-null soft set over G. Let $x \in Supp(H, C)$.
If $x \in A \setminus B$, then $H(x) = F(x) \neq \emptyset$ is a near subspace of V;
If $x \in B \setminus A$, then $H(x) = G(x) \neq \emptyset$ is a near subspace of V;
and if $x \in A \cap B, H(x) = F(x) \cap G(x) \neq \emptyset$. Thus $\emptyset \neq F(x)$ and $\emptyset \neq G(x)$ are both near subspaces of V, and so is their intersection. It follows that (H, C) is a soft near vector space over V.

Theorem 3.4 (Restricted Intersection preserves softness):. Let (F, A) and (G, B) be soft near vector spaces over V then the restricted intersection $(F, A) \Cap (G, B)$ is a soft near vector space over V.

Proof: Let $(F, A) \Cap (G, B) = (H, C)$, where $H(x) = F(x) \cap G(x)$ for all $x \in C$, $A \cap B \neq \emptyset$. Suppose that (H, C) is a non-null soft set over V. If $x \in Supp(H, C)$, then $H(x) = F(x) \cap G(x) \neq \emptyset$. It follows that $\emptyset \neq F(x)$ and $\emptyset \neq G(x)$ are both near subspaces of V. Hence $H(x)$ is a near subspace of V for all $x \in Supp(H, C)$. Thus, (H, C) is a soft near vector space over V.

Theorem 3.5 : If A and B are disjoint, then $(F, A)\ \tilde{\cup}\ (G, B)$ is a soft near vector space over (V, N) .
Proof :
Let $(F, A)\ \tilde{\cup}\ (G, B) = (T, A \cup B)$, where

$$T(x)= \begin{cases} F(x) & if\ x \in A \backslash B \\ G(x) & if\ x \in B \backslash A, \\ F(x) \cup G(x) & if\ x \in A \cap B \end{cases}$$

Since $A \cap B = \emptyset$, it follows that either $x \in A \backslash B\ or\ x \in B \backslash A$ for all $x \in A \cup B$.

If $x \in A\backslash B$, then $T(x) = F(x)$ is a near sub vector space of V and if $x \in B\backslash A$, then $T(x) = G(x)$ is a subvector space of V. Thus, $(T, A \cup B)$ is a soft near vector space over V.

Theorem 3.6 (Restricted union criterion): Let V be a near vector space over a near-field K. Suppose (F, A) and (G, B) are soft near vector spaces over V .Then their restricted union $(F, A) \cup_R (G, B) = (H, A \cap B)$ is a soft near vector space iff for every $e \in A \cap B$,

$$F(e) \subseteq G(e) \text{ or } G(e) \subseteq F(e).$$

Proof :
('*if*' direction)
Fix $e \in A \cap B$. if$F(e) \subseteq G(e)$ (or symmetrically $G(e) \subseteq F(e)$), then

$$H(e) = F(e) \cup G(e) = G(e)$$

('*Only if*' direction)
Assume $(H, A \cap B)$ is a soft near vector space. Suppose for some $e \in A \cap B$, neither $F(e) \subseteq G(e)$ nor $G(e) \subseteq F(e)$.Then there exists $x \in F(e)\backslash G(e), y \in G(e)\backslash F(e)$.

- If $+y \in F(e)$, then $y = (x + y) - x \in F(e)$ (closure under addition and additive inverse in the near subspace$F(e)$), contradicting y ∉F(e).
- If $x + y \in G(e)$, then $y = (x + y) - y \in G(e)$ contradicting $x \notin G(e)$.

Both lead to contradictions. Therefore for each $e \in A \cap B$, one of the inclusions must hold.

Corollary: Restricted union need not be a soft near vector space.
Example: Let V =R^2 (a vector space, hence a near vector space), and take a single parameter e. Define
$F(e) = \{(t, 0): t \in R\}, G(e) = \{(0, t) : t \in R\}$ both are near subspaces of V.Their restricted union $H(e) = F(e) \cup G(e)$ is not closed under addition, hence not a near subspace.

Theorem 5 (Finite point wise sums preserve softness): Let (F, E) and (G, E) be soft near vector spaces over the same E.
Define $(F + G, E)$ by

$$(F + G)(e) = F(e) + G(e) = x + y : x \in F(e), y \in G(e).$$

Then $(F + G, E)$ is a soft near vector space. More generally, any finite pointwise sum $\sum_{j=1}^{n} \mathrm{F_j}$ is soft near.

Proof. For each fixed e, the sum of near subspaces $F(e) + G(e)$ is the smallest near subspace containing $(e)\, U\, G(e)$; it is closed under $+$, inverses, and right scalars. Parameterwise, we obtain softness.

Further Work
We can define soft linear independence, soft span, and soft basis in the context of near vector spaces, and explore conditions for dimension invariance.

Conclusion

We have introduced and explored soft near vector spaces, providing foundational definitions, examples, and basic properties. This work can be extended to study linear transformations, quotient structures, and applications in algebraic coding theory.

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