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Solution of Bernoulli Nonlinear Differential Equation through Sadik Transform

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Abstract

In this paper we have solved Bernoulli Differential equation

$u'(t)+p(t)u(t)=Q(t)u^n(t)$

for the conditions $p(t)=0$ and $Q(t)=t^m,\ m\geq 0$ as well as $p(t)=t^s$ and $Q(t)=t^m\quad s,m\geq 0$ using Sadik Transform along with ADM. Sadik transform is now very powerful tool because after applying the Sadik transform we can have choices whether to proceed through Sadik transform or by any other integral transform whose kernel is exponential form just by fixing different values of alpha and beta according to a convenience and situation of the problem.

Key words: Sadik Transform, Bernoulli Differential equation, ADM.

Introduction

In mathematics, an integral transform maps an equation from its domain into another domain of new variable where it might be converted and solved algebraically and easily than in the original domain. Different Integral transform have been successfully used for two centuries in solving many problems in applied mathematics, mathematical physics, Physical chemistry and engineering science. The solution is back to the original domain using the inverse integral transform. There are so many integral transform are discovered to solve Differential equations, Partial Differential Equations, integral equations etc. which claim their superiority over each other. Recently the Sadik transform has been introduced by Sadikali Latif Shaikh in 2018. The Sadik transform is nothing but generalization of the Laplace Transform, Sumudu Transform, Elzaki Transform, Kamal Transform, Abood Transform, Tarig Transform, Mohand Transform etc. and all those integral transforms whose kernels are of exponential type or similar to the kernel of the Laplace transform.

Prelimiraries
In this section, we recall some notions and definitions that will be used through this paper.
Sadik Transform: If $u(t)$ piecewise continuous on the interval $0 \le t \le A$ for any $A > 0$, and $|u(t)| \le K$, where $t \ge M$, for any real and positive constant K and M, then Sadik transform $\mathbb{S}(\upsilon^{\alpha}, \beta) = \mathbb{S}[u(t)]$ is defined by

$$\mathbb{S}(\upsilon^{\alpha}, \beta) = \mathbb{S}[u(t)] = \frac{1}{\upsilon^{\beta}} \int_0^{\infty} e^{-\upsilon^{\alpha} t} u(t) dt$$

where, υ is complex variable, α is any non-zero real numbers, and β is any real number[2].

Properties of Sadik transform

(1) $\mathbb{S}\left(t^n\right) = \dfrac{\upsilon^{-\beta} n!}{\upsilon^{(n+1)\alpha}}$ (2) $\mathbb{S}\left(\sin(at)\right) = \dfrac{a\upsilon^{-\beta}}{\upsilon^{2\alpha} + a^2}$

(3) $\mathbb{S}\left(\cos(at)\right) = \dfrac{\upsilon^{\alpha}.\upsilon^{-\beta}}{\upsilon^{2\alpha} + a^2}$ (4) $\mathbb{S}\left(e^{at}\right) = \dfrac{\upsilon^{-\beta}}{\upsilon^{\alpha} - a}$

(5) $\mathbb{S}\left(\sinh(at)\right) = \dfrac{a\upsilon^{-\beta}}{\upsilon^{2\alpha} - a^2}$ (6) $\mathbb{S}\left(\cosh(at)\right) = \dfrac{\upsilon^{\alpha}.\upsilon^{-\beta}}{\upsilon^{2\alpha} - a^2}$

Sadik transform on differential coefficients

$$\mathbb{S}\left[x'(t)\right] = \upsilon^{\alpha}\mathbb{S}\left(\upsilon^{\alpha}, \beta\right) - \upsilon^{-\beta}x(0),$$

$$\mathbb{S}[x^{(n)}(t)] = \upsilon^{n\alpha}\mathbb{S}\left(\upsilon^{\alpha}, \beta\right) - \sum_{k=0}^{n-1} \upsilon^{k\alpha-\beta} x^{(n-1)-k}(0)$$

Adomian decomposition method
Adomian decomposition method introduces the solution by decomposing $u(t)$ to an infinite series $u(t) = \sum_{n=0}^{\infty} u_n(t)$ and the nonlinear term Nu by the infinite series $N\left[u(t)\right] = \sum_{n=0}^{\infty} A_n$ where A_n are the Adomian polynomials which are generated for each nonlinearity and can be found by the formula $A_i = \dfrac{1}{i!}\left[\dfrac{d_i}{d\lambda^i} N\left(\sum_{j=0}^{\infty} \lambda^j u_j\right)\right]_{\lambda=0}$, $i = 0, 1, 2, ...$

Main Result

Theorem 1: Consider Bernoulli differential equation

$$u'(t) + p(t)u(t) = Q(t)u^n(t) \qquad (1),$$ with initial condition $u(0) = c$

where $p(t) = 0$ and $Q(t) = t^m$, $m \ge 0$

Then solution of the above equation is convergence series of

$$u(t)=\sum_{n=0}^{\infty}u_n(t)=u_0(t)+u_1(t)+u_2(t)+\cdots \qquad \text{where} \qquad u_0(t)=c$$

$$u_{k+1}(t)=\mathbb{S}^{-1}\left[\frac{1}{\upsilon^{\alpha}}\mathbb{S}\left[t^m A_k\right]\right]$$ where A_k is Adomian polynomial.

Proof: If $p(t)=0$ and $Q(t)=t^m,\ m\geq 0$ then (1) becomes

$$u'(t)=t^m\cdot u^n(t) \qquad (2)$$

here $u^n(t)$ is nonlinear term now by Sadik transform to (2) we get

$$\mathbb{S}(\upsilon^{\alpha},\beta)=\frac{1}{\upsilon^{\alpha+\beta}}c+\frac{1}{\upsilon^{\alpha}}\mathbb{S}\left(t^m u^n(t)\right) \qquad (3)$$

if we apply inverse Sadik transform we get the solution $u(t)$ but for non–linear part we apply Adomian polynomial method for which we represent solution $u(t)$ by infinite series

$$u(t)=\sum_{i=0}^{\infty}u_i(t) \text{ and } u^n(t)=\sum_{i=0}^{\infty}A_i \qquad (4)$$

$$\mathbb{S}\left(\sum_{i=0}^{\infty}u_i(t)\right)=\frac{1}{\upsilon^{\alpha+\beta}}c+\frac{1}{\upsilon^{\alpha}}S\left(t^m\sum_{i=0}^{\infty}A_i\right) \qquad (5)$$

$$\sum_{i=0}^{\infty}u_i(t)=c+\mathbb{S}^{-1}\left(\frac{1}{\upsilon^{\alpha}}\mathbb{S}\left(t^m\sum_{i=0}^{\infty}A_i\right)\right) \qquad (6)$$

$$\therefore u_0(t)=c \qquad u_{k+1}(t)=\mathbb{S}^{-1}\left(\frac{1}{\upsilon^{\alpha}}\mathbb{S}\left(t^m A_k\right)\right) \qquad (7)$$

calculating $u_1(t),\ u_2(t),\ u_3(t)\cdots$ from (7) we get solution as

$$u(t)=\sum_{n=0}^{\infty}u_n(t)=u_0(t)+u_1(t)+u_2(t)+\cdots$$

Example 1. Find solution of Bernoulli nonlinear Differential equation (2) for the different conditions

(1) $m=1,\ n=1,$ if $u(0)=e$

(2) $m=1,\ n=-1,$ if $u(0)=1$

(3) $m=0,\ n=2,$ if $u(0)=1$

(1) If $m=1,\ n=1,$ if $u(0)=e$ then (2) become

$$u'(t)=t.u(t) \qquad (8)$$

applying (7) to (8) where $u(t)=\sum_{i=0}^{\infty}A_i$ we get, $\qquad u_0(t)=e$

$$u_{k+1}(t)=\mathbb{S}^{-1}\left[\frac{1}{\upsilon^{\alpha}}\mathbb{S}\left[t\cdot A_k\right]\right] \qquad (9)$$

using (3) we can find Adomian polynomials as, $A_0=\frac{1}{0!}\left[u_0\right]=u_0=e$

$$\therefore u_1(t)=\mathbb{S}^{-1}\left[\frac{1}{\upsilon^{\alpha}}\mathbb{S}\left[t.A_0\right]\right]=e\mathbb{S}^{-1}\left[\frac{1}{\upsilon^{3\alpha+\beta}}\right]=e\frac{t^2}{2!}$$

$$A_1=\frac{1}{1!}\left[\frac{d}{d\lambda}N\left(u_0+\lambda u_1\right)\right]_{\lambda=0}=\frac{d}{d\lambda}\left(u_0+\lambda u_1\right)$$

$$\therefore u_2(t)=\mathbb{S}^{-1}\left[\frac{1}{\upsilon^{\alpha}}\mathbb{S}\left[t.A_1\right]\right]==\frac{e}{2!}\mathbb{S}^{-1}\left[\frac{1}{\upsilon^{\alpha}}\mathbb{S}\left[t^3\right]\right]=\frac{e}{2!}\mathbb{S}^{-1}\left[\frac{1}{\upsilon^{\alpha}}.\frac{3!}{\upsilon^{4\alpha+\beta}}\right]=\frac{e}{2!}\left(\frac{t^2}{2}\right)^2$$

$$A_2=\frac{1}{2!}\left[\frac{d^2}{d\lambda^2}N\left(u_0+\lambda u_1+\lambda^2 u_2\right)\right]_{\lambda=0}=\frac{1}{2!}\left[\frac{d^2}{d\lambda^2}\left(u_0+\lambda u_1+\lambda^2 u_2\right)\right]_{\lambda=0}$$

$$u_3(t)=\mathbb{S}^{-1}\left[\frac{1}{\upsilon^{\alpha}}\mathbb{S}\left[t.A_2\right]\right]=\mathbb{S}^{-1}\left[\frac{1}{\upsilon^{\alpha}}\mathbb{S}\left[t.e\frac{t^4}{8}\right]\right]=\frac{e}{8}\mathbb{S}^{-1}\left[\frac{1}{\upsilon^{\alpha}}\mathbb{S}\left[t^5\right]\right]=\frac{e}{3!}\left(\frac{t^2}{2}\right)^3$$

similarly $u_n(t)=\frac{e}{n!}\left(\frac{t^2}{2}\right)^n$

then solution is convergence series $u(t)=\sum_{n=0}^{\infty}u_n(t)=u_0(t)+u_1(t)+u_2(t)+\cdots$

$$u(t)=e+e\frac{t^2}{2!}+\frac{e}{2!}\left(\frac{t^2}{2}\right)^2+\frac{e}{3!}\left(\frac{t^2}{2}\right)^3+\cdots+\frac{e}{n!}\left(\frac{t^2}{2}\right)^n+\cdots$$

$$u(t)=e.e^{\frac{t^2}{2}}=e^{\frac{t^2}{2}+1} \qquad (10)$$

(2) If $m=1,\ n=-1,$ if $u(0)=1$ then (9) become

$$u'(t)=t.\left[u(t)\ \right]^{-1} \qquad (11)$$

applying (7) to (8) where $\left[u(t)\right]^{-1}=\sum_{i=0}^{\infty}A_i$ we get $u_0(t)=1\to A_0=1$

$$u_{k+1}(t)=\mathbb{S}^{-1}\left[\frac{1}{\upsilon^{\alpha}}\mathbb{S}\left[t.A_k\right]\right] \qquad (12)$$

$$\therefore u_1(t)=\mathbb{S}^{-1}\left[\frac{1}{\upsilon^{\alpha}}\mathbb{S}\left[t.A_0\right]\right]=\mathbb{S}^{-1}\left[\frac{1}{\upsilon^{\alpha}}\mathbb{S}(t)\right]=\frac{t^2}{2!}$$

$$A_1=\frac{1}{1!}\left[\frac{d}{d\lambda}N\left(u_0+\lambda u_1\right)\right]_{\lambda=0}=\frac{d}{d\lambda}\left(u_0+\lambda u_1\right)^{-1}=(-1)\left(u_0\right)^{-2}\left(u_1\right)$$

$$\therefore u_2(t)=\mathbb{S}^{-1}\left[\frac{1}{\upsilon^\alpha}\mathbb{S}\left[t.A_1\right]\right]=-\frac{1}{2}\mathbb{S}^{-1}\left[\frac{1}{\upsilon^\alpha}.\frac{3!}{\upsilon^{4\alpha+\beta}}\right]=-\frac{t^4}{8}$$

$$A_2=\frac{1}{2!}\left[\frac{d^2}{d\lambda^2}N\left(u_0+\lambda u_1+\lambda^2 u_2\right)\right]_{\lambda=0}=-\frac{1}{2}\left[-2\frac{t^4}{4}-\frac{t^4}{4}\right]=\frac{3}{8}t^4$$

$$\therefore u_3(t)=\mathbb{S}^{-1}\left[\frac{1}{\upsilon^\alpha}\mathbb{S}\left[t.A_2\right]\right]=\frac{3}{8}\mathbb{S}^{-1}\left[\frac{1}{\upsilon^\alpha}.\frac{5!}{\upsilon^{6\alpha+\beta}}\right]=\frac{t^6}{16}$$

similarly $\quad u_4(t)=-\frac{5t^8}{128},\ u_5(t)=\frac{7t^{10}}{256},\ldots\ldots\ldots$

then solution is convergence series $\quad u(t)=\sum_{n=0}^{\infty}u_n(t)=u_0(t)+u_1(t)+u_2(t)+\cdots$

$$\therefore u(t)=\sqrt{1+t^2} \tag{13}$$

(3) If $m=0,\ n=2,$ if $u(0)=1$ Bernoulli nonlinear differential equation (2) will be

$$u'(t)=u^2(t) \tag{14}$$

applying (7) to (14) where $\left[u(t)\right]^2=\sum_{i=0}^{\infty}A_i$ we get $\quad u_0(t)=1\rightarrow\ A_0=1$

$$u_{k+1}(t)=\mathbb{S}^{-1}\left[\frac{1}{\upsilon^\alpha}\mathbb{S}\left[A_k\right]\right] \tag{15}$$

$$u_1(t)=\mathbb{S}^{-1}\left[\frac{1}{\upsilon^\alpha}\mathbb{S}\left[A_0\right]\right]=\mathbb{S}^{-1}\left[\frac{1}{\upsilon^\alpha}\mathbb{S}[1]\right]=\mathbb{S}^{-1}\left[\frac{1}{\upsilon^\alpha}.\frac{1}{\upsilon^{\alpha+\beta}}\right]=t$$

$$A_1=\frac{1}{1!}\left[\frac{d}{d\lambda}N\left(u_0+\lambda u_1\right)\right]_{\lambda=0}=2u_0u_1=2t$$

$$\therefore u_2(t)=\mathbb{S}^{-1}\left[\frac{1}{\upsilon^\alpha}\mathbb{S}\left[A_1\right]\right]=\mathbb{S}^{-1}\left[\frac{1}{\upsilon^\alpha}.\frac{2}{\upsilon^{2\alpha+\beta}}\right]=t^2$$

$$A_2=\frac{1}{2!}\left[\frac{d^2}{d\lambda^2}\left(u_0+\lambda u_1+\lambda^2 u_2\right)^2\right]_{\lambda=0}=\left[\left(2u_0u_2\right)+\left(u_1\right)^2\right]=3t^2$$

$$\therefore u_3(t)=\mathbb{S}^{-1}\left[\frac{1}{\upsilon^\alpha}\mathbb{S}\left[A_2\right]\right]=\mathbb{S}^{-1}\left[\frac{1}{\upsilon^\alpha}\mathbb{S}\left[3t^2\right]\right]=t^3$$

Similarly $\quad u_4(t)=t^4,\ u_5(t)=t^5\cdots$

then solution is convergence series $\quad u(t)=\sum_{n=0}^{\infty}u_n(t)=u_0(t)+u_1(t)+u_2(t)+\cdots$

$$\therefore u(t)=1+t+t^2+t^3+\ldots\ldots=\frac{1}{1-t} \tag{16}$$

Theorem: 3.2. Consider Bernoulli differential equation (1)

$u'(t)+p(t)u(t)=Q(t)u^n(t)$ with initial condition $u(0)=c$

where $p(t)=t^s$ and $Q(t)=t^m$ $s, m \geq 0$ then show that solution of the above equation is convergence series of $u(t)=\sum_{n=0}^{\infty} u_n(t)=u_0(t)+u_1(t)+u_2(t)+\cdots$

where $u_0(t)=c$, $u_{k+1}(t)=\mathbb{S}^{-1}\left[\frac{1}{\upsilon^{\alpha}}\left(\mathbb{S}\left(t^m A_k\right)-\mathbb{S}\left(t^s u_k(t)\right)\right)\right]$

Proof: If where $p(t)=t^S$ and $Q(t)=t^m$ $s, m \geq 0$ then (1) becomes

$$u'(t)=t^m u^n(t)-t^s u(t) \tag{17}$$

now applying Sadik transform

$$\mathbb{S}\left(\upsilon^{\alpha},\beta\right)=\frac{1}{\upsilon^{\alpha+\beta}}c+\frac{1}{\upsilon^{\alpha}}\left(\mathbb{S}\left(t^m u^n(t)\right)-\mathbb{S}\left(t^s u(t)\right)\right) \tag{18}$$

if we apply inverse Sadik transform we get the solution $u(t)$ but for non–linear part we apply Adomian polynomial method for which we represent solution $u(t)$ by infinite series $u(t)=\sum_{i=0}^{\infty} u_i(t)$ $u^n(t)=\sum_{i=0}^{\infty} A_i$

$$\mathbb{S}\left[\sum_{i=0}^{\infty} u_i(t)\right]=\frac{1}{\upsilon^{\alpha+\beta}}c+\frac{1}{\upsilon^{\alpha}}\left(\mathbb{S}\left(t^m\sum_{i=0}^{\infty} A_i\right)-\mathbb{S}\left(t^p\sum_{i=0}^{\infty} u_i(t)\right)\right)$$

inverse Sadik transform gives

$$\sum_{i=0}^{\infty} u_i(t)=c+\mathbb{S}^{-1}\left[\frac{1}{\upsilon^{\alpha}}\left(\mathbb{S}\left(t^m\sum_{i=0}^{\infty} A_i\right)-\mathbb{S}\left(t^p\sum_{i=0}^{\infty} u_i(t)\right)\right)\right]$$

comparing both sides for i=0, and i=k+1

$$u_0(t)=c,$$

$$u_{k+1}(t)=\mathbb{S}^{-1}\left[\frac{1}{\upsilon^{\alpha}}\left(\mathbb{S}\left(t^m A_k\right)-\mathbb{S}\left(t^s u_k(t)\right)\right)\right] \tag{19}$$

Calculating $u_1(t), u_2(t), u_3(t)\cdots$ from (19) we get solution by convergence series

$$u(t)=\sum_{n=0}^{\infty} u_n(t)=u_0(t)+u_1(t)+u_2(t)+\cdots$$

Example 2. Consider nonlinear differential equation

$$u'(t)+u(t)=t\left(u(t)\right)^2 \tag{20}$$

with initial condition $u(0)=1$.

Here $p=0,\ m=1$ $\therefore p(t)=1$ and $Q(t)=t$ and $\left[u(t)\right]^2=\sum_{i=0}^{\infty} A_i$

applying Sadik transform from (19) we get,

$$u_0(t) = A_0 = 1$$

$$u_{k+1}(t) = \mathbb{S}^{-1}\left[\frac{1}{\upsilon^{\alpha}}\left(\mathbb{S}(t.A_k) - \mathbb{S}(u_k(t))\right)\right] \qquad (21)$$

$$\therefore u_1(t) = \mathbb{S}^{-1}\left[\frac{1}{\upsilon^{\alpha}}\left(\mathbb{S}(t.A_0) - \mathbb{S}(u_0(t))\right)\right] = \frac{t^2}{2} - t$$

$$A_1 = \frac{1}{1!}\left[\frac{d}{d\lambda}N(u_0 + \lambda u_1)\right]_{\lambda=0} = t^2 - 2t$$

$$\therefore u_2(t) = \mathbb{S}^{-1}\left[\frac{1}{\upsilon^{\alpha}}\left(\mathbb{S}(t.A_1) - \mathbb{S}(u_1(t))\right)\right] = \frac{6t^4}{4!} - \frac{5t^3}{3!} + \frac{t^2}{2!} = \frac{t^4}{4} - \frac{5t^3}{6} + \frac{t^2}{2}$$

$$A_2 = \frac{1}{2!}\left[\frac{d^2}{d\lambda^2}\left(u_0 + \lambda u_1 + \lambda^2 u_2\right)^2\right]_{\lambda=0} = \frac{t^4}{2} - \frac{5t^3}{3} + t^2$$

$$\therefore u_3(t) = \mathbb{S}^{-1}\left[\left(\frac{60}{\upsilon^{7\alpha+\beta}} - \frac{46}{\upsilon^{6\alpha+\beta}} + \frac{11}{\upsilon^{5\alpha+\beta}} - \frac{1}{\upsilon^{4\alpha+\beta}}\right)\right] = \frac{t^6}{12} - \frac{46}{5!}t^5 + \frac{11}{4!}t^4 - \frac{t^3}{3!}$$

we get solution by convergence series

$$u(t) = \sum_{n=0}^{\infty} u_n(t) = u_0(t) + u_1(t) + u_2(t) + \cdots$$

$$\therefore u(t) = \frac{1}{1+t} \qquad (22)$$

Conclusions

In this paper we have solved Bernoulli Differential equation for special conditions by Sadik Transform along with ADM. Using different values of α and β we can find solution of $u'(t) = t^m u^n(t)$ (2) by Laplace Transform, Sumudu Transform, Elzaki Transform, Kamal Transform, Abood Transform, Tarig Transform and Mohand Transform etc. combined with ADM, For example $\alpha = -1,\ \beta = -1,\ E\{f(t)\} = T(\upsilon)$ then (7) becomes expression for Elzaki Transform combined with ADM, $\therefore$ $u_0(t) = c$ & $u_{k+1}(t) = E^{-1}\left[\upsilon E\left[t^m A_k\right]\right]$ by applying formulae of Elzaki and ADM then inverse Elzaki formulae we get same solution.

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