Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 21, Number 3 (2025), pp. 351-367
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http://www.ripublication.com/gjpam.htm

Lacunary Statistical Convergence in Partial Metric Spaces

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Abstract

In the present paper, notions of lacunary statistically p-convergence and lacunary statistically p-boundedness are introduced in a partial metric space (X, p) and a relation between these two concepts is established. Apart from this, it is demonstrated that a p-bounded and lacunary statistically p-convergent sequence is lacunary strongly p-Cesàro summable. Finally, we explored the inclusion relation between lacunary statistical p-convergent and statistical p-convergent sequences.

Keywords: Lacunary statistical convergence, partial metric space, lacunary statistical boundedness, lacunary density.

MSC: 46A45, 40A05, 40A35.

1. INTRODUCTION

The present work mainly concerns with three concepts : statistical convergence, lacunary statistical convergence and partial metric space. Most of the work in the field of sequence space is dominated by the sequence of scalars (real or complex). Through this study, we contribute to sequence space by adding sequences from an arbitrary non-empty set X, via partial metric. To go through, let us first recall some basic tools.

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In 1951, Statistical convergence of real sequences was introduced in short by Fast [8]. Later on this concept was studied as "convergence in density" by Buck [4] in 1953. It is also a part of monograph by Zygmund [31] and referred as almost convergence. Steinhaus [27] and Schoenberg [26] introduced and studied this concept independently in connection with summability of sequences. Later on this concept of statistical convergence and its various extensions have been explored by many more mathematicians and now it is so vigour and broad that find its applications in different areas of mathematics, such as measure theory, trigonometric series, Fourier series and others.

Statistical convergence has its main pillar as natural density which is defined as:

Definition 1.1. *The natural density of a subset K of $\mathbb{N}$ is denoted by $\delta(K)$ and defined as*

$$\delta(K) = \lim_{n\to\infty} \frac{1}{n}\mathrm{card}(\{k \in K : k \leq n\}),$$

provided the limit exists. It is easily verified that $\delta(K) = 0$ for finite subset K of $\mathbb{N}$ and $\delta(K) + \delta(\mathbb{N} - K) = 1$ for every $K \subseteq \mathbb{N}$. For a detailed account of natural density, one may peep into Niven and Zuckerman [21].

Definition 1.2. *([24]) A real valued sequence $\langle x_k\rangle$ is statistically convergent to $L \in \mathbb{R}$ if for each $\varepsilon > 0$, $\delta(\{|x_k - L| \geq \varepsilon\}) = 0$, i.e., $\lim_{n\to\infty} \frac{1}{n}\mathrm{card}(\{k \leq n : |x_k - L| > \varepsilon\}) = 0$. Here L is referred as statistical limit of $\langle x_k\rangle$. We write $x_k \to L(S)$ and by $S(c)$ we denote the set of all statistically convergent real sequences.*

With the passage of time, various generalization of this notion, that is, $\lambda-$statistical convergence, $A-$statistical convergence, $I-$convergence, rough statistical convergence, $\mu-$statistical convergence, lacunary statistical convergence, deferred statistical convergence, convergence of order α etc. have been appeared and also these generalizations have been studied for vector valued sequences by many more mathematicians. One may refer to [1, 3, 5, 6, 7, 10, 11, 12, 14, 17, 18, 19, 23, 28, 29, 30].

Before proceeding for lacunary statistical convergence, we recall lacunary sequence and lacunary density.

Following Freedman et al. [9], a lacunary sequence $\theta = \langle k_r\rangle_{r=0}^{\infty}$ is an increasing sequence such that $k_r - k_{r-1} \to \infty$, where $k_0 = 0$, $k_r \geq 0$. Here we notate $I_r = (k_{r-1}, k_r]$, $h_r = k_r - k_{r-1}$ and $q_r = \dfrac{k_r}{k_{r-1}}$.

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There is a strong relation between the space $|\sigma_1|$ of strongly Cesàro summable sequences where

$$|\sigma_1| = \left\{ \langle x_k \rangle : \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} |x_k - L| = 0 \; for \; some \; L \right\}$$

and the space N^θ, where

$$N^\theta = \left\{ \langle x_k \rangle : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} |x_k - L| = 0 \; for \; some \; L \right\}.$$

Fridy and Orhan [12] in 1993 studied a new variant of statistical convergence, named as lacunary statistical convergence that is in the same relation with statistical convergence as N^θ with $|\sigma_1|$.

Definition 1.3. *([12]) A real valued sequence $\langle x_k \rangle$ is lacunary statistical convergent to L or we can say $x_k \to L(S^\theta)$ if for every $\varepsilon > 0$, $\lim_{r\to\infty} \frac{1}{h_r} \text{card}(\{k \in I_r : |x_k - L| \geq \varepsilon\}) = 0$. By $S^\theta(c)$ we notate the class of all lacunary statistical convergent sequences of reals, i.e.,*

$$S^\theta(c) = \{\langle x_k \rangle : x_k \to L(S^\theta) \; for \; some \; L\}.$$

As statistical convergence has its deep root in the notion of natural density, similar is the relation of lacunary statistical convergence with lacunary density (or $\theta-$density).

Definition 1.4. *([3]) For a given lacunary sequence $\theta = \langle k_r \rangle$, the lacunary density (or $\theta-$density) of $K \subseteq \mathbb{N}$ is defined as $\delta^\theta(K) = \lim_{r\to\infty} \frac{1}{h_r} card(\{k \in I_r : k \in K\})$.*

Definition 1.5. *For a given lacunary sequence $\theta = \langle k_r \rangle$, if the set of indicies k 's for which $\langle x_k \rangle$ does not satisfy property P has zero lacunary density, then we say $\langle x_k \rangle$ satisfies P for "almost all k with respect to θ" abbreviated as "a.a. k w.r.t. θ"*

Lacunary statistical convergence, now may be redefined as:

Definition 1.6. *A sequence $\langle x_k \rangle$ of reals is lacunary statistical convergent to $L \in \mathbb{R}$ if for $\varepsilon > 0$,*

$$\delta^\theta(\{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\}) = 0,$$
$$i.e., \; |x_k - L| < \varepsilon \quad \text{a.a. } k \text{ w.r.t } \theta.$$

The credit of introducing the idea of partial metric space goes to Matthews [16] in 1994. Initially the concept of partial metric space was used in the field of computer science, but now it has been extensively used in biological sciences, information science and fixed point theory etc.

Definition 1.7. *([16]) Let $X \neq \emptyset$. A function $p : X \times X \to \mathbb{R}$ satisfying the following*

(p_1) $0 \leq p(x,x) \leq p(x,y)$

(p_2) $p(x,x) = p(x,y) = p(y,y) \iff x = y$

(p_3) $p(x,y) = p(y,x)$

(p_4) $p(x,y) \leq p(x,z) + p(z,y) - p(z,z)$ *for all* $x,y,z \in X$,

is said to be a partial metric on X and (X,p) is called a partial metric space.

It is clear from axiom (p_1) that $|p(x_k,x) - p(x,x)|$ and $p(x_k,x) - p(x,x)$ are the same thing, for any sequence $\langle x_k \rangle$ in X and $x \in X$.

In comparison to a metric on X, we can say a partial metric p is precisely metric $p : X \times X \to \mathbb{R}$ such that $\forall\ x \in X,\ p(x,x) = 0$. That is, in the definition of partial metric space, only one side axiom of metric is preserved, i.e., $\forall\ x,y \in X,\ p(x,y) = 0 \Rightarrow x = y$ and other half that is, $x = y \Rightarrow p(x,y) = 0$ need not hold good. For a detailed description of partial metric space, one may refer [16, 20, 25].

Nuray [22], Bayram et al. [2] and Kumar et al. [15] stepped into partial metric space via statistical convergence and introduced notion of statistical convergence in partial metric space. We call this notion as statistical p-convergence.

Definition 1.8. *A sequence $\langle x_k \rangle$ in partial metric (X,p) is said to be statistically p-convergent to some $a \in X$ if for given $\varepsilon > 0$, $\delta(\{k \in \mathbb{N} : p(x_k,a) \geq p(a,a)+\varepsilon\}) = 0$ and we write it as $x_k \xrightarrow{p} a(S)$. By $S(c^p)$, we notate the class of all statistically p-convergent sequence from (X,p).*

Definition 1.9. *Let $\langle x_k \rangle$ be sequence in (X,p) and $a \in X$. If for given $\varepsilon > 0$, $\exists$ a positive integer k_0 such that $p(x_k,a) \leq p(a,a) + \varepsilon$ for all $k \geq k_0$, then we say $\langle x_k \rangle$ is p-convergent to a. We write $x_k \xrightarrow{p} a$ and c^p for the class of all p-convergent sequences.*

Definition 1.10. *A sequence $\langle x_k \rangle$ in partial metric space (X,p) is said to be p-bounded if $\exists$ some $a \in X$ and $M > 0$ such that $p(x_k,a) < p(a,a) + M\ \forall\ k \geq 1$. We write b^p as the class of all p-bounded sequences.*

In the present paper, we are going to explore lacunary statistical convergence in partial metric space (X,p) by introducing the notion of lacunary statistical p-convergence. Necessary and sufficient condition on lacunary sequence θ for the equality between the space $S(c^p)$ and $S^\theta(c^p)$ have been established. It is also investigated that a p-bounded and lacunary statistically p-convergent sequence is lacunary strongly p-Cesàro summable.

2. LACUNARY STATISTICAL p-CONVERGENCE AND LACUNARY STATISTICAL p-BOUNDEDNESS

In this section, we introduce the concepts of lacunary statistically p-convergence and lacunary statistically p-boundedness and investigate various properties of lacunary statistically p-convergent sequences. It is also observed that every lacunary statistically p-convergent sequence has a lacunary statistically dense, p-convergent subsequence.

Definition 2.1. *A sequence $\langle x_k \rangle$ in partial metric space (X, p) is said to be lacunary statistically p-convergent to $a \in X$ if for $\varepsilon > 0$,*

$$\lim_{r\to\infty} \frac{1}{h_r} \operatorname{card}(\{k \in I_r : p(x_k, a) \geq p(a, a) + \varepsilon\}) = 0.$$

We write $x_k \xrightarrow{p} a(S^\theta)$ and $S^\theta(c^p) = \{\langle x_k \rangle : x_k \in X,\ x_k \xrightarrow{p} a(S^\theta)\}$ for some $a \in X$.

Theorem 2.2. *$c^p \subset S^\theta(c^p)$, inclusion is proper.*

Proof. Let $\langle x_k \rangle$ be a sequence in (X, p) which is p-convergent to $a \in X$. Then for given $\varepsilon > 0$, $\exists\ k_0 \in \mathbb{N}$ we have $p(x_k, a) < p(a, a) + \varepsilon$ for all $k \geq k_0$. As lacunary density of a finite set is zero, so the result holds. For proper inclusion, consider the following example:

Let $X = \mathbb{R}$ and p be the partial metric defined by $p(\xi, \eta) = |\xi - \eta|$; $\xi, \eta \in \mathbb{R}$. Take sequence

$$x_i = \begin{cases} k_{r-1} + 1 & \text{for } i = k_{r-1} + 1, \\ 0 & \text{otherwise.} \end{cases} \qquad \text{on } I_r,\ r = 1, 2, 3 \ldots$$

Let, if possible, $\langle x_i \rangle$ is p-convergent to some $a \in X$. Then for given $\varepsilon > 0$, $\exists\ k_0 \in \mathbb{N}$ we have $p(x_i, a) < p(a, a) + \varepsilon$ for all $i \geq k_0$, i.e., $|k_{r-1} + 1 - a| < \varepsilon\ \forall\ k_{r-1} + 1 \geq k_0\ \forall\ r = 1, 2, 3, \ldots$
a contradiction as $k_r \to \infty$. However, $\langle x_i \rangle$ is lacunary statistically p-convergent to $0 \in X$, because for $\varepsilon > 0$, $\lim_{r\to\infty} \frac{1}{h_r} \operatorname{card}(\{i \in I_r : p(x_i, 0) \geq p(0, 0) + \varepsilon\}) = \lim_{r\to\infty} \frac{1}{h_r} = 0.$
□

Theorem 2.3 ([13, Proposition 2.8.])**.** *Let f be a modulus function (unbounded). Then $S^f_{p,q} - \lim x_k = L$ iff $\exists\ K \subseteq \mathbb{N}$ such that $\delta^f_{p,q}(K) = 0$ and $\lim_{k\in\mathbb{N}-K} x_k = L$.*

For a lacunary sequence $\theta = \langle k_r \rangle$, taking $f(x) = x,\ p(r) = k_{r-1}$ and $q(r) = k_r,\ r = 1, 2, 3, \ldots$ we have,

Corollary 2.4. *A sequence $\langle x_k \rangle$ of reals is lacunary statistical convergent to L iff $\exists\ K \subseteq \mathbb{N}$ such that $\delta^\theta(K) = 0$ and $\lim_{k \in \mathbb{N}-K} x_k = L$.*

Theorem 2.5. *Let $\langle x_k \rangle$ be a sequence in a partial metric space (X, p). Then $\langle x_k \rangle$ is a lacunary statistical p-convergent to $a \in X$ if and only if there exists $K \subseteq \mathbb{N}$ such that $\delta^\theta(K) = 0$ and $\langle x_k \rangle_{k \in \mathbb{N}-K} \xrightarrow{p} a$.*

Proof. Let $x_k \xrightarrow{p} a(S^\theta)$. Then for each $\varepsilon > 0$,

$$\delta^\theta(\{k \in \mathbb{N} : |y_k| \geq \epsilon\}) = 0,$$

where $y_k = p(x_k, a) - p(a, a)$. Thus $\langle y_k \rangle$ is a lacunary statistical convergent sequence of reals converging to 0, so by Corollary 2.4, $\exists\ K \subseteq \mathbb{N}$ such that $\delta^\theta(K) = 0$ and $\lim_{k \in \mathbb{N}-K} y_k = 0$. This yields $\lim_{k \in \mathbb{N}-K} p(x_k, a) = p(a, a)$

Conversely, it is given that $\langle y_k \rangle_{k \in \mathbb{N}-K} \xrightarrow{p} a$. So for $\varepsilon > 0$, $\exists\ k_0 \in \mathbb{N}$ such that $p(x_k, a) < p(a, a) + \varepsilon$ for all $k \in \mathbb{N} - K$ with $k \geq k_0$. From this we conclude

$$\{k : p(x_k, a) \geq p(a, a) + \varepsilon\} \subseteq K \cup \{1, 2, \ldots, k_0 - 1\}.$$

As every finite set has zero lacunary density and $\delta^\theta(K) = 0$, so the result follows. $\square$

Definition 2.6. *A sequence $\langle x_k \rangle$ in (X, p) is said to be lacunary statistically p-bounded if $\exists$ some $a \in X$ and $M > 0$ such that $\lim_{r \to \infty} \frac{1}{h_r} \text{card}(\{k \in I_r : p(x_k, a) > p(a, a) + M\}) = 0$.*

The class of all lacunary statistically p-bounded sequences is notated by $S^\theta(b^p)$.

Definition 2.7. *The diameter $d(X)$ of a partial metric space (X, p) is defined as $d(X) = sup_{x,y \in X}\{p(x, y) - p(x, x)\}$. Moreover if $d(X) < \infty$, then we termed X as a bounded partial metric space.*

Theorem 2.8. *Lacunary statistically p-convergence implies lacunary statistically p-boundedness. Reverse implication need not hold good.*

Proof. Let $\langle x_k \rangle$ be a lacunary statistically p-convergent to some $a \in X$. Then for given $\varepsilon > 0$, $\lim_{r \to \infty} \frac{1}{h_r} \text{card}(\{k \in I_r : p(x_k, a) > p(a, a) + \varepsilon\}) = 0$. Now for sufficiently large M, we may assert that, $\text{card}(\{k \in I_r : p(x_k, a) > p(a, a) + M\}) \leq \text{card}(\{k \in I_r : p(x_k, a) > p(a, a) + \varepsilon\})$ and hence the result follows.

For converse part, consider the following example:
Let $\theta = (2^r)$, $X = \mathbb{R}$ and p be the partial metric defined by $p(\xi, \eta) = |\xi - \eta|$; $\xi, \eta \in \mathbb{R}$.

Consider a sequence $\langle x_k \rangle$ in X as

$$x_k = \begin{cases} -1 & \text{if k is odd} \\ 1 & \text{if k is even} \end{cases} \quad for\ k \in \mathbb{N}\ .$$

Now $\lim_{r\to\infty} \frac{1}{h_r} \text{card}(\{k \in I_r : p(x_k, 1) > p(1,1) + \varepsilon\}) = \frac{1}{2}$. Thus $\langle x_k \rangle$ is not lacunary statistically p-convergent to 1. Similar can be proved for -1. As every bounded sequence is lacunary statistically p-bounded, hence the result follows. □

Remark 2.9. *Subsequence of a lacunary statistically p-convergent sequence need not be lacunary statistically p-convergent. For this consider the following example:*

Let $X = \mathbb{R}$ be the partial metric space equipped with partial metric p defined as $p(\xi, \eta) = |\xi - \eta|$; $\xi, \eta \in \mathbb{R}$.

$$x_i = \begin{cases} k_{r-1} + 1 & for\ i = k_{r-1} + 1, \\ 0 & otherwise \end{cases} \quad on\ I_r,\ r = 1, 2, 3 \ldots \quad .$$

Then $\langle x_i \rangle$ is lacunary $p-$statistically convergent to 0. Now $\langle k_0 + 1, k_1 + 1, k_2 + 1, \ldots \rangle$ is a subsequence of $\langle x_i \rangle$ which is not lacunary statistically p-convergent.

However, the following theorem characterizes the lacunary statistically p-convergent sequence in term of lacunary statistically dense lacunary statistically p-convergent subsequences.

Theorem 2.10. *A sequence $\langle x_k \rangle$ is lacunary statistically p-convergent iff every lacunary statistically dense subsequence of $\langle x_k \rangle$ is lacunary statistically p-convergent.*

Proof. Let $\langle x_k \rangle$ is lacunary statistically p-convergent to a. So for given $\varepsilon > 0$, we have

$$\lim_{r\to\infty} \frac{1}{h_r} \text{card}(\{k \in I_r : p(x_k, a) > p(a,a) + \varepsilon\}) = 0,\ i.e.,\ \delta^\theta(A) = 0$$

where $A = \{k \in \mathbb{N} : p(x_k, a) > p(a,a) + \varepsilon\}$ and $\langle x_{k_n} \rangle_{n \in \mathbb{N}}$ is lacunary statistically dense subsequence of $\langle x_k \rangle$ which is not lacunary $p-$statistically convergent, i.e.,

$$\liminf_{r\to\infty} \frac{1}{h_r} \text{card}(\{k_n \in I_r : p(x_{k_n}, a) > p(a,a) + \varepsilon\}) = d,\ where\ d \in (0,1).$$

Now $\text{card}(\{k \in I_r : p(x_k, a) > p(a,a) + \varepsilon\}) \geq \text{card}(\{k_n \in I_r : p(x_{k_n}, a) > p(a,a) + \varepsilon\})$.
So $\liminf_{r\to\infty} \frac{1}{h_r} card(\{k \in I_r : p(x_k, a) > p(a,a) + \varepsilon\}) \geq d \neq 0$, a contradiction to given.
Converse part follows from the fact that every sequence is a lacunary statistical dense subsequence of itself. □

Theorem 2.11. *$b^p \subset S^\theta(b^p)$, inclusion is proper.*

Proof. Let $\langle x_k \rangle$ be a p-bounded sequence in (X, p). Then there exists $a \in X$ and $M > 0$ such that $p(x_k, a) < p(a, a) + M\ \forall\ k \geq 1$. As every empty set has zero lacunary density, so $\langle x_k \rangle$ is lacunary statistically p-bounded. For proper inclusion, consider the example cited in Theorem 2.2. The result shows that the class of lacunary statistically p-bounded sequences is much wider than the class of p-bounded sequences. □

Remark 2.12. *As $c^p \subset b^p$ and $b^p \subset S^\theta(b^p)$ so we have $c^p \subset S^\theta(b^p)$.*

Theorem 2.13. *A sequence $\langle x_k \rangle$ in (X, p) is lacunary statistically p-convergent to some $a \in X$ iff $\exists$ a sequence $\langle y_k \rangle$ p-convergent to a such that $y_k = x_k$ a.a. k w.r.t. θ.*

Proof. Let $\langle x_k \rangle$ be a lacunary statistically p-convergent to some $a \in X$. So, for each $\varepsilon > 0$, we get $\delta^\theta(K) = 0$ where $K = \{k \in \mathbb{N} : p(x_k, a) > p(a, a) + \varepsilon\}$

$$y_k = \begin{cases} x_k & \text{for } k \in \mathbb{N} - K \\ a & \text{for } k \in K. \end{cases}$$

Now $\{k \in \mathbb{N} : y_k \neq x_k\} \subseteq K$ and so $y_k = x_k$ a.a. k w.r.t. θ. Also

$$p(y_k, a) = \begin{cases} p(x_k, a) & \text{for } k \in \mathbb{N} - K \\ p(a, a) & \text{for } k \in K \end{cases}$$

implies $p(y_k, a) < p(a, a) + \varepsilon$ for all $k \geq 1$. Thus $\langle y_k \rangle$ is p-convergent sequence and $y_k = x_k$ a.a. k w.r.t. θ.

Conversely, let $\exists\ k_0 \in \mathbb{N}$ such that $p(y_k, a) < p(a, a) + \varepsilon\ \forall\ k \geq k_0$. The result follows from the inclusion relation $\{k : p(x_k, a) \geq p(a, a) + \varepsilon\} \subseteq K \cup \{1, 2, 3, \ldots, k_0 - 1\}$. □

Following on the similar lines, we have

Theorem 2.14. *A sequence $\langle x_k \rangle$ is lacunary statistically p-bounded iff $\exists$ a p-bounded sequence $\langle y_k \rangle$ such that $y_k = x_k$ a.a. k w.r.t. θ.*

Remark 2.15. *Example in Remark 2.9 asserts that a subsequence of a lacunary statistically p-bounded sequence need not be lacunary statistically p-bounded.*

Using the same technique as in Theorem 2.10, we give a characterization of the lacunary statistically p-boundedness in terms of the lacunary statistically dense lacunary statistically p-bounded subsequences of it, in terms of following.

Theorem 2.16. *A sequence $\langle x_k \rangle$ is lacunary statistically p-bounded iff every lacunary statistically dense subsequence of $\langle x_k \rangle$ is lacunary statistically p-bounded.*

Theorem 2.17. *A partial metric space (X, p) is bounded iff the set of all p-bounded sequences coincides with the set of all lacunary statistically p-bounded sequences.*

Proof. As X is bounded, so let $M = \sup_{x,y\in X} |p(x, y) - p(x, x)|$. Then for all $y \in X$ (and arbitrary x but fixed), we have $|p(x, y) - p(x, x)| \leq M$. Then for any sequence $\langle x_k \rangle$ in X, we have $|p(x_k, x) - p(x, x)| \leq M \ \forall\ k \geq 1$ and hence $\langle x_k \rangle$ is p-bounded. This implies that $\langle x_k \rangle$ is lacunary statistically p-bounded, in view of Theorem 2.11.

Conversely, suppose if possible, $\sup_{x,y\in X} |p(x, y) - p(x, x)| = \infty$. Then $\exists\, x_0 \in X$ such that $\sup_{y\in X} |p(x_0, y) - p(x_0, x_0)| = \infty$. This implies that for each $n \in \mathbb{N}$, $\exists\ x_n \in X$ such that
$p(x_0, x_n) - p(x_0, x_0) > n$ i.e., $p(x_0, x_n) > n + p(x_0, x_0)$ for all $n \in \mathbb{N}$. Take

$$y_i = \begin{cases} x_i & \text{if } i = k_{r-1} + 1,\ for\ r = 1, 2, 3, ... \\ x_0 & \text{otherwise.} \end{cases} \qquad on\ I_r$$

Now $\{i \in I_r : p(y_i, x_0) > p(x_0, x_0) + M\} \subseteq \{k_{r-1} + 1\}$ implies that

$$0 \leq \lim_{r\to\infty} \frac{1}{h_r} \text{card}(\{i \in I_r : p(y_i, x_0) > p(x_0, x_0) + M\}) = \lim_{r\to\infty} \frac{1}{h_r} = 0.$$

This implies $\langle y_i \rangle$ is lacunary statistically p-bounded. However $\langle y_i \rangle$ is not p-bounded, as

$$p(y_{k_{r-1}+1}, x_0) = p(x_0, y_{k_{r-1}+1}) = p(x_{k_{r-1}+1}, x_0) > k_{r-1}+1+p(x_0, x_0) \text{ for all } r = 1, 2, 3, ...$$

Thus $\langle y_i \rangle$ is lacunary statistically p-bounded sequence in partial metric space (X, p) which is not p-bounded, a contradiction to given. □

3. LACUNARY STRONGLY p-CESÀRO SUMMABLE SPACES

We start this section by introducing the notion of lacunary strongly p-Cesàro summability and it is established that in case of p-bounded sequence, the lacunary strongly p-Cesàro summability and lacunary statistically p-convergence are one and same thing. We also determine that $1 < \liminf_r q_r < \limsup_r q_r < \infty$ is necessary and sufficient condition for the coincidence of the spaces $S^\theta(c^p)$ and $S(c^p)$, by following the same approach as in [9]. Apart from this, results have been established for inclusion between two arbitrary lacunary methods of $p-$statistical convergence.

Definition 3.1. *Let $\langle x_k \rangle$ be a sequence in partial metric space (X, p). The sequence $\langle x_k \rangle$ is lacunary strongly p-Cesàro summable to $a \in X$ if* $\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} |p(x_k, a) - p(a, a)| = 0$.

We use the notation $N^\theta(c^p)$ for the set of all lacunary strongly p-Cesàro summable sequences, i.e.,

$$N^\theta(c^p) = \left\{ \langle x_k \rangle : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} |p(x_k,a) - p(a,a)| = 0 \; for\ some\ a \in X \right\}$$

and we write $x_k \xrightarrow{a} (N^\theta)$.

Theorem 3.2. *For a given lacunary sequence $\theta = \langle k_r \rangle$, we have the following*

(1) $x_k \xrightarrow{p} a(N^\theta)$ implies $x_k \xrightarrow{p} a(S^\theta)$ i.e., $N^\theta(c^p) \subset S^\theta(c^p)$. Every lacunary strongly p-Cesàro summable sequence is lacunary statistically p-convergent to same limit.

(2) Inclusion in part (1) is proper.

(3) Every p-bounded and lacunary statistically p-convergent sequence is lacunary strongly p-Cesàro summable sequence, i.e., $S^\theta(c^p) \cap b^p = N^\theta(c^p) \cap b^p$.

Proof.

(1) For $\varepsilon > 0$, we have

$$\begin{aligned}\sum_{k\in I_r} |p(x_k,a) - p(a,a)| &\geq \sum_{\substack{k\in I_r \\ |p(x_k,a)-p(a,a)|>\varepsilon}} |p(x_k,a) - p(a,a)| \\ &\geq \varepsilon \operatorname{card}(\{k \in I_r : |p(x_k,a) - p(a,a)| \geq \varepsilon\}\end{aligned}$$

and hence the result follows.

(2) For proper inclusion, consider the following example. Let $X = \mathbb{R}$ and p be the partial metric defined by $p(\xi,\eta) = |\xi - \eta|$; $\xi,\eta \in \mathbb{R}$. Construct a sequence as follows:
Take x_k to be $1, 2, 3, ..., [\sqrt{h_r}]$ at the first $[\sqrt{h_r}]$ integers on I_r and $x_k = a$ otherwise for all, $r = 1, 2, 3, ...$ where $a \in [0,1) - \{\frac{1}{2}\}$ and $[\cdot]$ denote the greatest integer function. Then for every $\varepsilon > 0$, $\frac{1}{h_r}\operatorname{card}(\{k \in I_r : |p(x_k,a) - p(a,a)| \geq \varepsilon\}) \leq \frac{1}{h_r}[\sqrt{h_r}] \to 0$ as $r \to \infty$ and so $\langle x_k \rangle \in S^\theta(c^p)$. On the other hand,

$$\begin{aligned}\frac{1}{h_r}\sum_{k\in I_r} |p(x_k,a) - p(a,a)| &= \frac{1}{h_r}\sum_{k\in I_r} |x_k - a| \\ &= \frac{1}{h_r}\left[|1-a| + |2-a| + ... + |[\sqrt{h_r}] - a|\right] \\ &= \frac{1}{h_r}\left(\frac{[\sqrt{h_r}]([\sqrt{h_r}]+1)}{2} - [\sqrt{h_r}]a\right) \to \frac{1}{2} - 0 \; as\ r \to \infty.\end{aligned}$$

This implies that $\langle x_k \rangle \notin N^\theta(c^p)$.

(3) Let $\langle x_k \rangle$ be a p-bounded and lacunary statistically p-convergent to $a \in X$. Then

$$\frac{1}{h_r}\sum_{k\in I_r}|p(x_k,a)-p(a,a)|$$
$$=\frac{1}{h_r}\left(\sum_{\substack{k\in I_r\\|p(x_k,a)-p(a,a)|<\varepsilon}}|p(x_k,a)-p(a,a)|+\sum_{\substack{k\in I_r\\|p(x_k,a)-p(a,a)|\geq\varepsilon}}|p(x_k,a)-p(a,a)|\right)$$

As $\langle x_k \rangle$ is p-bounded sequence, so $\exists$ some $b \in X$ and $M > 0$ such that

$$p(x_k,b) < p(b,b) + M \ \forall \ k \geq 1.$$

Now $p(x_k,a) \leq p(x_k,b) + p(b,a) - p(b,b) \leq p(b,b) + M + p(a,b) - p(b,b)$,

$$i.e., p(x_k,a) \leq T + p(a,a) \ where \ T = M + p(a,b) - p(a,a),$$

$$i.e., p(x_k,a) - p(a,a) \leq T \ \forall \ k \geq 1.$$

This implies

$$\frac{1}{h_r}\sum_{k\in I_r}|p(x_k,a)-p(a,a)|$$
$$\leq \frac{1}{h_r}\left(\sum_{\substack{k\in I_r\\|p(x_k,a)-p(a,a)|<\varepsilon}}|p(x_k,a)-p(a,a)|+T\,\mathrm{card}(\{k\in I_r:|p(x_k,a)-p(a,a)|\geq\epsilon\})\right)$$
$$\leq \frac{\varepsilon}{h_r}h_r+\frac{T}{h_r}\mathrm{card}(\{k\in I_r:|p(x_k,a)-p(a,a)|\geq\varepsilon\})$$

and hence the result follows. □

Theorem 3.3. *Let* $\theta = \langle k_r \rangle$ *be any lacunary sequence. Then* $S(c^p) \subset S^\theta(c^p)$ *if and only if* $\liminf_r q_r > 1$.

Proof. Let $\liminf_r q_r > 1$. Then $\exists\ \delta > 0$ such that $q_r > 1 + \delta$ for all $r \geq 1$. As $q_r = \dfrac{k_r}{k_{r-1}}$, so we have $\dfrac{h_r}{k_{r-1}} \geq \delta$. This implies $\dfrac{k_{r-1}}{h_r} \leq \dfrac{1}{\delta}$. On adding 1 to both sides, we get $\dfrac{k_r}{h_r} \leq \dfrac{1+\delta}{\delta}$.

Let $x_k \xrightarrow{p} a(S)$, then for every $\varepsilon > 0$ and sufficiently large r, we get

$$\begin{aligned}\frac{1}{k_r}\mathrm{card}(\{k<k_r:p(x_k,a)>p(a,a)+\varepsilon\}) &\geq \frac{1}{k_r}\mathrm{card}(\{k\in I_r:p(x_k,a)>p(a,a)+\varepsilon\})\\&\geq \frac{h_r}{k_r}\frac{1}{k_r}\mathrm{card}(\{k\in I_r:p(x_k,a)>p(a,a)+\varepsilon\})\\&\geq \frac{\delta}{1+\delta}\frac{1}{k_r}\mathrm{card}(\{k\in I_r:p(x_k,a)>p(a,a)+\varepsilon\})\end{aligned}$$

and hence the result.

Conversely, suppose $\liminf_r q_r = 1$. Since $\theta = \langle k_r \rangle$ is lacunary, so following [9], we can select a subsequence $\langle k_{r_j} \rangle$ of $\langle k_r \rangle$ satisfying $\dfrac{k_{r_j}}{k_{r_j-1}} < 1 + \dfrac{1}{j}$ where $r_j > r_{j-1} + 2$. Let $X = \mathbb{R}$ be a partial metric space with partial metric defined as $p(\xi, \eta) = |\xi - \eta|$; $\xi, \eta \in \mathbb{R}$. Define

$$x_k = \begin{cases} 1 & \text{if } k \in I_{r_j}, \\ 0 & \text{otherwise.} \end{cases} \qquad for\ j = 1, 2, 3, ...$$

Then $\lim\limits_{j\to\infty} \dfrac{1}{h_{r_j}} \mathrm{card}(\{k \in I_{r_j} : p(x_k, 0) > p(0,0) + \varepsilon\}) = \lim\limits_{j\to\infty} \dfrac{h_{r_j}}{h_{r_j}} = 1$ and for $r \neq r_j$ $\lim\limits_{r\to\infty} \dfrac{1}{h_r} \mathrm{card}(\{k \in I_r : p(x_k, 0) > p(0, 0 + \varepsilon\}) = \lim\limits_{r\to\infty} \dfrac{0}{h_r} = 0$, hence $\langle x_k \rangle \notin S^\theta(c^p)$. However for any sufficiently large integer t, we can find the unique j such that $k_{r_j-1} < t \leq k_{r_{j+1}-1}$ and

$$\begin{aligned} \frac{1}{t} card(\{k < t : p(x_k, 0) > p(0,0) + \varepsilon\}) &= \frac{1}{t} \mathrm{card}(\{k < t : |x_k| > \varepsilon\}) \\ &\leq \frac{k_{r_{j-1}} + h_{r_j}}{k_{r_j-1}} \\ &\leq \frac{k_{r_{j-1}}}{k_{r_j-1}} + \frac{h_{r_j}}{k_{r_j-1}} \\ &< \frac{1}{j} + \frac{1}{j} \to 0\ as\ t \to \infty. \end{aligned}$$

This implies that $\langle x_k \rangle \in S(c^p)$. □

Theorem 3.4. *For any lacunary sequence* $\theta = \langle k_r \rangle$, $S^\theta(c^p) \subset S(c^p)$ *iff* $\limsup_r q_r < \infty$.

Proof. Let $\limsup_r q_r < \infty$. Then there exists $M > 0$ such that $q_r < M$ for all $r \geq 1$. Suppose that $\langle x_k \rangle \in S^\theta(c^p)$. Then there exist $a \in X$ and $\varepsilon > 0$ such that

$$\lim_{r\to\infty} \frac{1}{h_r} \mathrm{card}(\{k \in I_r : p(x_k, a) \geq p(a,a) + \varepsilon\}) = 0,\ i.e.,\ \lim_{r\to\infty} \frac{M_r}{h_r} = 0$$

where $M_r = \mathrm{card}(\{k \in I_r : p(x_k, a) \geq p(a,a) + \varepsilon\})$. So for given $\varepsilon > 0$, $\exists\ r_0 \in \mathbb{N}$ such that $\dfrac{M_r}{h_r} < \varepsilon$ for all $r > r_0$. Let $T = \sup\{M_r : 1 \leq r \leq r_0\}$ and n be any integer

$k_{r-1} < n \leq k_r$. Then we can write,

$$\begin{aligned}
&\frac{1}{n} card\left(\{k \leq n : p(x_k, a) \geq p(a,a) + \varepsilon\}\right) \\
&\quad < \frac{1}{k_{r-1}} \text{card}\left(\{k \leq k_r : p(x_k, a) \geq p(a,a) + \varepsilon\}\right) \\
&\quad = \frac{1}{k_{r-1}} \left\{M_1 + M_2 + ... + M_{r_0} + M_{r_0+1} + ... + M_r\right\} \\
&\quad \leq \frac{r_0 T}{k_{r-1}} + \frac{1}{k_{r-1}} \{M_{r_0+1} + M_{r_0+2} + ... + M_r\} \\
&\quad = \frac{r_0 T}{k_{r-1}} + \frac{1}{k_{r-1}} \left\{h_{r_0+1} \frac{M_{r_0+1}}{h_{r_0+1}} + h_{r_0+2} \frac{M_{r_0+2}}{h_{r_0+2}} + ... + h_r \frac{M_r}{h_r}\right\} \\
&\quad \leq \frac{r_0 T}{k_{r-1}} + \frac{1}{k_{r-1}} \left(\underset{r > r_0}{sup} \frac{M_r}{h_r}\right) \{h_{r_0+1} + h_{r_0+2} + ... + h_r\} \\
&\quad \leq \frac{r_0 T}{k_{r-1}} + \frac{1}{k_{r-1}} \varepsilon \left(k_r - k_{r_0}\right) \\
&\quad = \frac{r_0 T}{k_{r-1}} + \varepsilon \left(\frac{k_r}{k_{r-1}} - \frac{k_{r_0}}{k_{r-1}}\right) \\
&\quad \leq \frac{r_0 T}{k_{r-1}} + \varepsilon . M
\end{aligned}$$

and hence the result follows immediately.

Conversely, suppose that $\limsup_r q_r = \infty$. We can select a subsequence $\langle k_{r_j}\rangle$ of the lacunary sequence $\theta = \langle k_r \rangle$ such that $q_{r_j} > j$. Consider the partial metric space (X, p) where $X = \mathbb{R}$ and $p(\xi, \eta) = |\xi - \eta|$; $\xi, \eta \in \mathbb{R}$. Define

$$x_i = \begin{cases} 1 & \text{if } k_{r_j - 1} < i \leq 2k_{r_j - 1}, \\ 0 & \text{otherwise.} \end{cases} \qquad for\ j = 1, 2, 3, ...$$

Then $\tau_{r_j} = \frac{1}{h_{r_j}} \sum_{i \in I_{r_j}} |p(x_i, 0) - p(0,0)| = \frac{k_{r_j - 1}}{h_{r_j}} = \frac{k_{r_j - 1}}{k_{r_j} - k_{r_j - 1}} < \frac{1}{j}$ and if $r \neq r_j$, $\tau_r = 0$. This implies $\langle x_i \rangle \in N^\theta(c^p)$ and hence in view of Theorem 3.2, $\langle x_i \rangle \in S^\theta(c^p)$. Now

$$\begin{aligned}
&\frac{1}{2k_{r_j-1}} \text{card}\left(\left\{k \leq 2k_{r_j - 1} : |p(x_i, 0) - p(0,0) > \varepsilon\right\}\right) \\
&\qquad = \frac{1}{2k_{r_j-1}} \text{card}\left(\left\{k \leq 2k_{r_j - 1} : |x_i| > \varepsilon\right\}\right) \\
&\qquad \geq \frac{1}{2k_{r_j-1}} k_{r_j - 1} = \frac{1}{2} \ and\ so\ \langle x_i \rangle \notin S(c^p). \quad \square
\end{aligned}$$

Corollary 3.5. *For a lacunary sequence $\theta = \langle k_r \rangle$, we have $S^\theta(c^p) = S(c^p)$ if and only*

if
$1 < \liminf q_r < \limsup_r q_r < \infty$.

Definition 3.6. *A lacunary sequence $\theta^{'} = \langle k_r^{'} \rangle$ is said to be lacunary refinement of lacunary sequence $\theta = \langle k_r \rangle$ if $\langle k_r^{'} \rangle \supseteq \langle k_r \rangle$.*

Theorem 3.7. *For a lacunary refinement $\theta^{'}$ of θ, we have $x \notin N^{\theta}(c^p) \Rightarrow x \notin N^{\theta^{'}}(c^p)$.*

Proof. Let $\langle x_k \rangle \notin N^{\theta}(c^p)$. Then for any real number a, $\lim\limits_{n\to\infty} \dfrac{1}{h_r} \sum_{k \in I_r} |p(x_k, a) - p(a,a)| \neq 0$. Then there exists $\varepsilon > 0$ and a subsequence $\langle k_{r_j} \rangle$ of $\langle k_r \rangle$ such that $\dfrac{1}{h_{r_j}} \sum_{k \in I_{r_j}} |p(x_k, a) - p(a,a)| \geq \varepsilon$. Writing $I_{r_j} = I_{s+1}^{'} \cup I_{s+2}^{'} \cup ... \cup I_{s+p}^{'}$, where $k_{r_j - 1} = k_s^{'} < k_{s+1}^{'} < ... < k_{s+p}^{'} = k_{r_j}$ we have

$$\frac{\sum_{I_{s+1}^{'}} |p(x_i,a) - p(a,a)| + \sum_{I_{s+2}^{'}} |p(x_i,a) - p(a,a)| + ... + \sum_{I_{s+p}^{'}} |p(x_i,a) - p(a,a)|}{h_{s+1}^{'} + h_{s+2}^{'} + ... + h_{s+p}^{'}} \geq \varepsilon$$

This implies for some j, we have $\dfrac{1}{h_{s+j}^{'}} \sum_{I_{s+j}^{'}} |p(x_i,a) - p(a,a)| \geq \varepsilon$ and as a result $\langle x_i \rangle \notin N^{\theta^{'}}(c^p)$.

The following theorems are generalizations of corresponding theorems ([11], [14]) for real valued sequences to X-values sequences.

Theorem 3.8. $x_k \xrightarrow{p} a \left(S^{\theta^{'}}\right)$ *implies* $x_k \xrightarrow{p} a \left(S^{\theta}\right)$.

Theorem 3.9. *For a lacunary refinement $\theta^{'} = \langle k_r^{'} \rangle$ of the lacunary sequence $\theta = \langle k_r \rangle$, let $I_r = (k_{r-1}, k_r]$ and $I_r^{'} = \left(k_{r-1}^{'}, k_r^{'}\right]$, $r = 1, 2, 3, ...$*

(1) If there exists $\delta > 0$ such that $\dfrac{|I_j^{'}|}{|I_i|} \geq \delta$ for every $I_j^{'} \subset I_i$, then $S^{\theta}(c^p) \subseteq S^{\theta^{'}}(c^p)$.

(2) Let $I_ij = I_i \cap I_j^{'}$; $i, j = 1, 2, 3, ...$ If $\exists\ \delta > 0$ such that $\dfrac{|I_{ij}|}{|I_i|} \geq \delta$ for every $i, j = 1, 2, 3, ...$ provided $I_{ij} \neq \emptyset$, then $S^{\theta}(c^p) \subseteq S^{\theta^{'}}(c^p)$.

(3) If $\exists\ \delta > 0$, $\dfrac{|I_{ij}|}{|I_j^{'}|} \geq \delta$ for every $i, j = 1, 2, 3, ...$ provided $I_{ij} \neq \emptyset$, then $S^{\theta^{'}}(c^p) \subseteq S^{\theta}(c^p)$.

(4) If $\exists\ \delta > 0$, $\dfrac{|I_{ij}|}{|I_i| + |I_j^{'}|} \geq \delta$ for every $i, j = 1, 2, 3, ...$ provided $I_{ij} \neq \emptyset$, then $S^{\theta}(c^p) = S^{\theta^{'}}(c^p)$.

ACKNOWLEDGEMENT

The authors would like to thank the referees for their helpful comments and suggestions, which improved the presentation of this paper.

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