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On a Cohomological Characterization of a Free Profinite Product of Two Profinite Groups with Commuting Subgroups

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Abstract

This is an investigation on the structure of free pro-$\mathcal{C}$ products of pro-$\mathcal{C}$ groups with commuting subgroups where $\mathcal{C}$ is the class of all finite solvable groups. First, we give necessary and sufficient condition such that a free pro-$\mathcal{C}$ product of pro-$\mathcal{C}$ groups with commuting subgroups, is $\mathcal{C}$-residual. Then some conditions under which, a given free pro-$\mathcal{C}$ product of pro-$\mathcal{C}$ groups with commuting subgroups can be written as a free pro-$\mathcal{C}$ product with amalgamation of two free pro-$\mathcal{C}$ products with amalgamation (what we call a double pro-$\mathcal{C}$ amalgamation), are derived. Further, we characterize, using cohomological conditions, when a pro-$\mathcal{C}$ group is the free pro-$\mathcal{C}$ product with commuting subgroups of some of its subgroups.

Keywords: pro-$\mathcal{C}$ group, pro-$\mathcal{C}$ topology, free pro-$\mathcal{C}$ product of pro-$\mathcal{C}$ groups with amalgamation, free pro-$\mathcal{C}$ product of pro-$\mathcal{C}$ groups with commuting subgroups, group of continuous derivations.

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1. INTRODUCTION

The study of free constructions of abstracts groups began in 1882 when Walter Von Dyck pointed out in his paper *Gruppentheoretische Studien* the concept of free groups. The algebraic study of this notion began in 1924 by Jakop Nielsen, who gave the name of free group and established many of its fundamental properties, see [14]. Since 1927, free products of groups with amalgamated subgroups, were introduced by Schreier and generalized since 1948 by Hanna Neumann. In [9], free constructions (free products, amalgamated free products, HNN-extensions and free products with commuting subgroups) of abstracts groups were clearly defined. In [8], Loginova pointed out a link between free products with amalgamation and free products with commuting subgroups by establishing that every free product with commuting subgroups can be written as amalgamated free product of two amalgamated free products, what we call a double amalgamation of abstracts groups.

Profinite groups are known since 1965 when J.P. Serre introduced them in his book titled *Cohomologie galoisienne* [21]. A profinite group G is the inverse limit of a projective system of finite groups, i.e., $G = \varprojlim_{i\in I} G_i$, where $(G_i)_{i\in I}$ is a projective system of finite (abstract) groups and I is a directed set. A profinite group G is isomorphic to a closed subgroup of a direct product of finite groups. A profinite group is a topological, compact, Haussdorff and totally disconnected group. A concrete example of a profinite group is the profinite completion of an abstract group. Given G an abstract group, the profinite completion $\widehat{G}$ of group G is the inverse limit of the projective system $(G/N)_{N\in\mathcal{N}}$ of the (finite) quotient groups G/N, where $\mathcal{N}$ is the collection of all normal subgroups of finite index of G, i.e., $\widehat{G} = \varprojlim_{N\in\mathcal{N}} G/N$. Many authors have studied profinite groups in different directions [3, 15, 20, 17, 19, 6, 5]. Luis Ribes and Pavel Zalesskii in [20] have introduced free constructions of profinite groups. They defined free profinite products, amalgamated free profinite products and profinite HNN-extensions of profinite groups. They studied the particular case of proper amalgamated free profinite products and proper profinite HNN-extensions of profinite groups. They gave examples of amalgamated free profinite product which are not proper and proved some conditions for their properness [16, 15]. Similarly, G. Mantika and D. Tieudjo defined free profinite product of profinite groups with commuting subgroups and they studied their properness. See [10]. Let G_1 and G_2 be two profinite groups, let H be a closed subgroup of G_1, K a closed subgroup of G_2 and A a closed common subgroup of G_1 and G_2. We denote by $G_1 \underset{A}{*} G_2$, $G_1 \underset{A}{\amalg} G_2$, $G_1 \underset{[H,K]}{*} G_2$ and $G_1 \underset{[H,K]}{\amalg} G_2$, the abstract amalgamation, the profinite amalgamation, the abstract free product with commuting subgroups and the free profinite product with commuting subgroups respectively.

Today, profinite groups have been generalized to pro$-\mathcal{C}$ groups, where $\mathcal{C}$ is a class of finite groups. A pro-$\mathcal{C}$ group G is the inverse limit of a projective system of groups of $\mathcal{C}$. When $\mathcal{C}$ is the class of all finite groups, all finite p-groups, all finite solvable groups and all finite nilpotent groups, then we say profinite groups, pro-p groups, pro-solvable groups and pro-nilpotent groups respectively. When G is an abstract group, its profinite completion with respect to $\mathcal{C}$ is called the pro-$\mathcal{C}$ completion of group G and denoted by $\widehat{G}^{\mathcal{C}}$. Also, $\widehat{G}^{\mathcal{C}}$ is a concrete example of a pro-$\mathcal{C}$ group. Therefore, free pro-$\mathcal{C}$ products of pro-$\mathcal{C}$ groups with amalgamation are defined. See [20]. The topology on an abstract group G given by the fundamental system of neighborhoods of the identity consisting of the collection of all its subgroups belonging in $\mathcal{C}$, is called a *pro-$\mathcal{C}$ topology* on the group G. With this topology, G becomes a topological group. A subset S of a group G *is closed* in the pro-$\mathcal{C}$ topology of G if for any element $g \in G \setminus S$, there exists a normal subgroup K of finite index in G with $G/K \in \mathcal{C}$ such that $g \notin SK$. When the trivial group is closed in the pro-$\mathcal{C}$ topology of a group G, then we say that the group G is *$\mathcal{C}$-residual*. Equivalently, G is $\mathcal{C}$-residual if for any $g \neq 1_G$ there exists a normal subgroup K in G such that $G/K \in \mathcal{C}$ and $g \notin K$. This means that, for every $g \neq 1_G$, there exists a homomorphism φ from G onto a group of $\mathcal{C}$ such that $\varphi(g) \neq 1_G$. A subgroup H of a group G is *$\mathcal{C}$-separable* if it is closed in the pro-$\mathcal{C}$ topology of G. Equivalently, a subgroup H of a group G is *$\mathcal{C}$-separable* if for any $a \in G\backslash H$, there exists a homomorphism φ from G onto a group of $\mathcal{C}$ such that $\varphi(a) \notin \varphi(H)$. D. Tieudjo in [22] recalled root-class residuality of free groups and free products of root-class residual groups. He proved some sufficient conditions for root-class residuality of generalized residual groups. Loginova in [8] proved necessary and sufficient condition such that a free product with commuting subgroups of residually finite p-groups, is again residually finite p-group. In this paper, we study the case where $\mathcal{C}$ is the class of all finite solvable groups. We prove:

Theorem 1.1
Let $\mathcal{C}$ be the class of all finite solvable groups. Let G_1 and G_2 be two $\mathcal{C}$-residual groups and let H and K be nonidentity subgroups of G_1 and G_2 respectively. Assume that H is central in G_1, K is abelian and commute with G_1. Then, $G = G_1 \underset{[H,K]}{*} G_2$, the free product of G_1 and G_2 with commuting subgroups H and K, is $\mathcal{C}$-residual if and only if the subgroups H and K are $\mathcal{C}$-separable in G_1 and G_2 respectively.

Also, when studiying the residual finiteness of free products of abstract groups with commuting subgroups, Loginova in [8] established that this construction can be written as double amalgamation. That is, given G_1 and G_2 two abstract groups with respective subgroups H and K, the following situation holds:

$$G_1 \underset{[H,K]}{*} G_2 = (G_1 \underset{H}{*} (H \times K)) \underset{H \times K}{*} ((H \times K) \underset{K}{*} G_2).$$

For profinite groups or pro$-\mathcal{C}$ groups in general, this is not always true. Let $\mathcal{C}$ be the class of all finite solvable groups. Then, $\mathcal{C}$ is subgroup closed and is also closed under taking quotients, under forming finite direct products, under extensions, and for any group G with normal subgroups H and K such that $G/H, G/K \in \mathcal{C}$, then $G/H \cap K \in \mathcal{C}$. See [20, 22]. $\mathcal{C}$ is an example of root-class. See [22]. In this paper, under some conditions, we write the free pro-$\mathcal{C}$ product with commuting subgroups of pro-$\mathcal{C}$ groups as a pro$-\mathcal{C}$ product with amalgamation of two pro$-\mathcal{C}$ products with amalgamation. That is:

Theorem 1.2
Let $\mathcal{C}$ be the class of all finite solvable groups. Let G_1 and G_2 be two pro-$\mathcal{C}$ groups with respective closed subgroups H and K. If H is central in G_1, K is abelian and commute with G_1, and H and K are $\mathcal{C}$-separable and satisfy $\widehat{HK}^{\mathcal{C}} = HK$, then we have:

$$G_1 \underset{[H,K]}{\amalg} G_2 = (G_1 \underset{H}{\amalg} (H \times K)) \underset{H \times K}{\amalg} ((H \times K) \underset{K}{\amalg} G_2).$$

Some free constructions of groups (abstract or topological case) were also characterized with cohomology tools [13, 17, 18]. Let $\mathcal{C}$ be the class of all finite solvable groups. The pro-$\mathcal{C}$ completion of $\mathbb{Z}$, the ring of integers, is the free pro-$\mathcal{C}$ group on a single generator noted by $\mathbb{Z}_{\hat{\mathcal{C}}}$. It has an obvious structure of a compact, Haussdorff ring. See [7]. Let R be a commutative ring and let G be a profinite group. The abstract group algebra (or group ring) $[RG]$ consists of all formal sums $\sum_{g_i \in G} a_i g_i$ ($a_i \in R$, where a_i is zero for all but a finite number of indices), with natural addition defined by $(\sum a_i g_i) + (\sum b_i g_i) = \sum (a_i + b_i) g_i$ and multiplication defined by $(\sum a_i g_i)(\sum b_j g_j) = \sum c_k g_k$ where $c_k = \sum_{g_i g_j = g_k} a_i b_j$. Let G be a pro-$\mathcal{C}$ group. The complete group algebra of G is defined by $[[\mathbb{Z}_{\hat{\mathcal{C}}} G]] = \varprojlim_U [\mathbb{Z}_{\hat{\mathcal{C}}} G/U]$, where U runs through the open normal subgroups of G. $[[\mathbb{Z}_{\hat{\mathcal{C}}} G]]$ is a profinite ring. Throughout this paper, DMod$([[\mathbb{Z}_{\hat{\mathcal{C}}} G]])$ denotes the category of discrete $[[\mathbb{Z}_{\hat{\mathcal{C}}} G]]$-modules. Let now M be a closed subgroup of G. For $A \in$ DMod$([[\mathbb{Z}_{\hat{\mathcal{C}}} G]])$, define

$$Der_M(G, A) = \{d : G \longrightarrow A |\ d(xy) = xd(y) + d(x), \forall x, y \in G,\ d|_M = 0\},$$

the group of all continuous derivations from G to A vanishing on M. L. Ribes and P. Zalesskii characterized cohomologically free pro-$\mathcal{C}$ products of pro-$\mathcal{C}$ groups with amalgamation, where $\mathcal{C}$ is an extension closed variety of finite solvable groups. See

[20, Theorem 9.3.1]. In this paper, following L. Ribes and P. Zalesskii, we obtain an analogous criterion in terms of derivations, when a pro-$\mathcal{C}$ group G is a free pro-$\mathcal{C}$ product with commuting subgroups of its subgroups. We prove:

Theorem 1.3
Let $\mathcal{C}$ be the class of all finite solvable groups. Let G be a pro-$\mathcal{C}$ group. Let G_1 and G_2 be closed subgroups of G. If H and K are respective subgroups of G_1 and G_2 such that H is central in G_1, K is abelian and commute with G_1, and H and K are $\mathcal{C}$-separable and satisfy $\widehat{HK}^{\mathcal{C}} = HK$, then the following conditions are equivalent:

1. *$G = G_1 \underset{[H,K]}{\amalg} G_2$ (free pro-$\mathcal{C}$ product with commuting subgroups);*
2. *The natural homomorphism*
   $\psi_G : \mathit{Der}_{H\times K}(G, A) \longrightarrow \mathit{Der}_{H\times K}(G_1 \amalg K, A) \times \mathit{Der}_{H\times K}(G_2 \amalg H, A)$,
   $(f \longmapsto (f|_{G_1\amalg K}, f|_{G_2\amalg H})$
   is an isomorphism for all $[[\mathbb{Z}_{\hat{\mathcal{C}}}G]]-$modules $A \in \mathcal{C}$.

2. PRELIMINARIES NOTIONS AND RESULTS

In this section, we recall definitions and properties of some notions we will use. One can refer to [2, 4, 10, 20, 23] for more details.
In all what follows below, $\mathcal{C}$ is the class of all finite solvable groups.

2.1. Basic notions

Definition 2.1
Let G be an abstract group. A profinite topology on G is a group topology on G such that the subgroups of finite index of G is a basis of neighborhoods of the identity element of G.
If this basis is consisting of subgroups N of finite index of G such that every quotient G/N belongs to the class $\mathcal{C}$, the above topology is the pro-$\mathcal{C}$ topology of G.

Definition 2.2
Let G be a pro-$\mathcal{C}$ group. A closed subgroup H of G is called the retract semidirect factor of G if there exists a closed normal subgroup K of G satisfying: $G = HK$ and $H \cap K = \{1\}$. K is then said to be a normal complement of H.
Equivalently, a closed subgroup H is the retract semidirect factor of G if there exists a continuous homomorphism $\nu : G \to H$ such that $\nu \circ s = id_H$ where $s : H \to G$ is the canonical homomorphism.

It is obvious that:

Remark 2.1
Let G be a pro-$\mathcal{C}$ group and let H be a closed subgroup of G with a normal complement K. G can be written as a semidirect product $K \rtimes H$.

2.2. Free abstract and pro-$\mathcal{C}$ products of abstract and pro-$\mathcal{C}$ groups with amalgamations

Definition 2.3 (Pushout: Free abstract product of abstract groups with amalgamation)
Let G_1, G_2 and H be abstract groups and let $\sigma : H \to G_1$ and $\tau : H \to G_2$ be monomorphisms. A free product of G_1 and G_2 with amalgamated subgroup H is a group G together with homomorphisms $\varphi_1 : G_1 \to G$ and $\varphi_2 : G_2 \to G$ such that $\varphi_1\sigma = \varphi_2\tau$, satisfying the following universal property: for any pair of homomorphisms $\psi_1 : G_1 \to K$ and $\psi_2 : G_2 \to K$ into a group K with $\psi_1\sigma = \psi_2\tau$, there exists a unique homomorphism $\psi : G \to K$ such that $\psi\varphi_1 = \psi_1$ and $\psi\varphi_2 = \psi_2$.
We denote by $G_1 \underset{H}{\star} G_2$ the free product of groups G_1 and G_2 with amalgamated subgroup H.

Definition 2.4
Let $G_1 \underset{H}{\star} G_2$ be the free product of groups G_1 and G_2 with amalgamated subgroup H.

1. *Let R and S be normal subgroups of finite index in groups G_1 and G_2 respectively. The subgroups R and S are (H, H)-compatible if $R \cap H = S \cap H$.*

2. *A family $(R_i)_{i\in I}$ of subgroups of a group G is called a filtration if $\underset{i\in I}{\cap} R_i = \{1\}$. And the family $(R_i)_{i\in I}$ is called a H-filtration if it is a filtration, and in addition we have $\underset{i\in I}{\cap} HR_i = H$.*

Definition 2.5 (Free pro$-\mathcal{C}$ product of pro-$\mathcal{C}$ groups with amalgamation)
Let G_1, G_2 and H be pro$-\mathcal{C}$ groups and let $f_1 : H \to G_1$ and $f_2 : H \to G_2$ be continuous monomorphisms. A free pro$-\mathcal{C}$ product of G_1 and G_2 with amalgamated subgroup H is a pro$-\mathcal{C}$ group G together with continuous homomorphisms $\varphi_1 : G_1 \to G$ and $\varphi_2 : G_2 \to G$ such that $\varphi_1 f_1 = \varphi_2 f_2$, satisfying the following universal property: for any pair of continuous homomorphisms $\psi_1 : G_1 \to K$ and $\psi_2 : G_2 \to K$ into a pro$-\mathcal{C}$ group K with $\psi_1 f_1 = \psi_2 f_2$, there exists a unique continuous homomorphism $\psi : G \to K$ such that $\psi\varphi_1 = \psi_1$ and $\psi\varphi_2 = \psi_2$.

2.3. Free abstract and pro-$\mathcal{C}$ products of abstract and pro-$\mathcal{C}$ groups with commuting subgroups

Definition 2.6 (Free abstract product of abstract groups with commuting subgroups) *Let G_1, G_2, H and K be abstract groups and let $\sigma : H \to G_1$ and $\tau : H \to G_2$ be monomorphisms. A free product of G_1 and G_2 with commuting subgroups H and K is a group G together with homomorphisms $\varphi_1 : G_1 \to G$ and $\varphi_2 : G_2 \to G$ such that $[\varphi_1(\sigma(H)); \varphi_2(\tau(K))] = 1$, satisfying the following universal property: for any pair of homomorphisms $\psi_1 : G_1 \to K$ and $\psi_2 : G_2 \to K$ into a group K with $[\psi_1(\sigma(H)); \psi_2(\tau(K))] = 1$, there exists a unique homomorphism $\psi : G \to K$ such that $\psi\varphi_1 = \psi_1$ and $\psi\varphi_2 = \psi_2$. We denote this group by $G = G_1 \underset{[H,K]}{*} G_2$.*

We prove the following:

Proposition 2.1
Let G_1 and G_2 be two pro$-\mathcal{C}$ groups with respective closed subgroups H and K. Then, the pro-$\mathcal{C}$ topology of $G = G_1 \underset{[H,K]}{*} G_2$ induces on G_1, G_2, H and K their pro-$\mathcal{C}$ topologies.

Proof. To prove that the pro-$\mathcal{C}$ topology of G induces on G_2 (for example, and the similar reason for G_1, H and K) its pro-$\mathcal{C}$ topology, it suffices to prove that G_2 is the retract semidirect factor of G. Indeed, assume that G_2 is a retract semidirect factor of G and let prove that the pro-$\mathcal{C}$ topology of G induces on G_2 its pro-$\mathcal{C}$ topology. So, let M_2 be a normal subgroup of G_2 of finite index such that $G_2/M_2 \in \mathcal{C}$. Since G_2 is a retract semidirect factor of G, there exists A, a normal subgroup of G such that $G = A \rtimes G_2$. Clearly $AM_2 \lhd_f G$ since

$$G_2/M_2 = G_2/(AM_2) \cap G_2 \simeq (AM_2)G_2/AM_2 = G/AM_2 \text{ where } (AM_2) \cap G_2 = M_2.$$

Therefore $G/AM_2 \in \mathcal{C}$, and it follows then that the pro-$\mathcal{C}$ topology of G induces on G_2 its pro-$\mathcal{C}$ topology.
Let now prove that G_2 is a retract semidirect factor of G. To do it, we will build a homomorphism $v : G \to G_2$ with $v \circ s = id_{G_2}$, where $s : G_2 \to G$ is the canonical homomorphism.
Since $G = G_1 \underset{[H,K]}{*} G_2$, so by the definition of free products, there is a (canonical) map $s : G_2 \to G$, including G_2 as a subgroup. By the universal property of free products, there exists a unique homomorphism $v : G \to G_2$ defined by the identity map $id : G_2 \to G_2$ and the trivial map $t : G_1 \to G_2$. Clearly, K commutes with the identity element, which is the image of H. This situation is illustrated by the following commutative diagram.

$$\begin{tikzcd}
H \arrow[d, "\sigma"'] & & K \arrow[d, "\tau"] \\
G_1 \arrow[r, "i_1"] \arrow[dr, "t"'] & G \arrow[d, dotted, "\exists! v"] & G_2 \arrow[l, "s"'] \arrow[dl, "id"] \\
& G_2 &
\end{tikzcd}$$

Therefore G_2 is a retract semidirect factor of G and the Proposition is proven.

□

Definition 2.7 (Free pro-$\mathcal{C}$ product of pro-$\mathcal{C}$ groups with commuting subgroups) *Let H be a closed subgroup of a pro-$\mathcal{C}$ group G_1 and let K be a closed subgroup of a pro-$\mathcal{C}$ group G_2. Let $\sigma : H \to G_1$ and $\tau : K \to G_2$ be the inclusion maps. The free pro-$\mathcal{C}$ product of the pro-$\mathcal{C}$ groups G_1 and G_2 with commuting subgroups H and K is the family $(G; \varphi_1; \varphi_2)$ where G is a pro-$\mathcal{C}$ group and $\varphi_1 : G_1 \to G$ and $\varphi_2 : G_2 \to G$ are continuous homomorphisms satisfying:*

(1) $[\varphi_1(\sigma(H)); \varphi_2(\tau(K))] = 1$ *and*

(2) If G' is a pro-$\mathcal{C}$ group with continuous homomorphims $\psi_1 : G_1 \to G'$ and $\psi_2 : G_2 \to G'$ such that $[\psi_1(\sigma(H)); \psi_2(\tau(K))] = 1$, then there exists a unique continuous homomorphism $\psi : G \to G'$ such that $\psi\varphi_1 = \psi_1$ and $\psi\varphi_2 = \psi_2$.

We denote by $G_1 \coprod\limits_{[H,K]} G_2$ the free pro-$\mathcal{C}$ product of pro-$\mathcal{C}$ groups G_1 and G_2 with commuting subgroups H and K, where G_1 and G_2 are two pro-$\mathcal{C}$ groups, H is a closed subgroup of the pro-$\mathcal{C}$ group G_1 and K is a closed subgroup of the pro-$\mathcal{C}$ group G_2. It is *proper* if the continuous homomorphisms $G_1 \to G_1 \coprod\limits_{[H,K]} G_2$ and $G_2 \to G_1 \coprod\limits_{[H,K]} G_2$ are one to one.

2.4. Complete group algebra and cohomology with coefficients in discrete modules

Let R be a profinite ring, i.e., R is an inverse limit of an inverse system of finite rings. A R-module is an abelian Hausdorff topological group M satisfying the analogous properties of abstract module over abstract ring. If M and N are two R-modules, we use

the notation $Hom_R(M, N)$ for the abelian group of all continuous $R-$homomorphisms from M to N as abelian profinite groups. We refer to a continuous $R-$homomorphism of $R-$modules as a morphism of R-modules. We will be interested in R-modules that are discrete. Discrete modules together with their morphisms form a category denotes by DMod(R). It is a subcategory of the category of all R-modules.
Let R be a commutative ring and let G be a profinite group. The abstract group algebra (or group ring) $[RG]$ consists of all formal sums $\sum_{g_i \in G} a_i g_i$ ($a_i \in R$, where a_i is zero for all but a finite number of indices), with natural addition defined by $(\sum a_i g_i) + (\sum b_i g_i) = \sum (a_i + b_i) g_i$ and multiplication defined by $(\sum a_i g_i)(\sum b_j g_j) = \sum c_k g_k$ where $c_k = \sum_{g_i g_j = g_k} a_i b_j$.
Now, let G be a pro-$\mathcal{C}$ group. In the context of pro$-\mathcal{C}$ groups, the analogue of the group ring is the concept of complete group algebra.

Definition 2.8
Let G be a pro-$\mathcal{C}$ group and R a profinite ring. The complete group algebra of G denoted by $[[RG]]$ is defined by

$$[[RG]] = \varprojlim_{U} [RG/U],$$

where U runs through the open normal subgroups of G.

Then, $[[RG]]$ is a profinite ring since we can express it as an inverse limit of finite rings, i.e.,

$$[[RG]] = \varprojlim [(R/I)(G/U)],$$

where I and U range over the open ideals of R and the open normal subgroups of G, respectively. See[20].
Recall also that every $[[RG]]$-module is a G-module (see Proposition 5.3.6 in [20]). DMod$([[\mathbb{Z}_{\hat{\mathcal{C}}}G]])$ denotes the category of discrete $[[\mathbb{Z}_{\hat{\mathcal{C}}}G]]$-modules.
Let G be a pro-$\mathcal{C}$ group and let A be a discrete G-module. Let $C^n(G, A)$ be the (abelian) group of all continuous functions $f : G^n \longrightarrow A$. Define a cochain complex

$$0 \longrightarrow C^0(G, A) \longrightarrow C^1(G, A) \longrightarrow ... \longrightarrow C^n(G, A) \overset{\partial^{n+1}}{\longrightarrow} C^{n+1}(G, A) \longrightarrow ...,$$

where ∂^{n+1} is defined as follow

$$(\partial^{n+1} f)(x_1, ..., x_{n+1}) = x_1 f(x_2, ..., x_{n+1})$$
$$+ \sum_{i=1}^{n} (-1)^i f(x_1, ..., x_i x_{i+1}, ..., x_{n+1}) + (-1)^{i+1} f(x_1, ..., x_n)$$

with $x_1, ..., x_{n+1} \in G$.

Definition 2.9
Let G be a pro-$\mathcal{C}$ group and let A be a discrete G-module. Then the $n-th$ cohomology group of G with coefficients in A is defined as the $n-th$ cohomology group of the above cochain complex, i.e.,

$$H^n(G,A) = ker(\partial^{n+1})/Im(\partial^n).$$

According to above definition, $H^1(G,A) = ker(\partial^2)/Im(\partial^1)$. The elements of $ker(\partial^2)$ are called $crossed\ homomorphisms$ or $derivations$ from G to A. So, a derivation $d : G \longrightarrow A$ is a continuous function such that $d(xy) = xd(y) + d(x)$, for all $x, y \in G$. We denote by $Der(G,A)$, the (abelian) group of derivations. The elements of $Im(\partial^1)$ are called $principal\ crossed\ homomorphisms$ or $inner\ derivations$. Each inner derivations $d_a : G \longrightarrow A$ is determined by an element $a \in A$ and is defined by $d_a(x) = xa - a$ ($x \in G$). See again [20].

3. PROOF OF THEOREM 1.1

The following property is the extension to $\mathcal{C}$-groups of ([1], Theorem 2.).

Lemma 3.1
Let $\mathcal{C}$ be the class of all finite solvable groups. Let $G = A \underset{H}{*} B$ be the free product of A and B, two groups of $\mathcal{C}$, with amalgamated subgroup H. If H is central in A or in B, then G is $\mathcal{C}$-residual.

Proof. Let A and B be groups of $\mathcal{C}$ with a common subgroup H. Suppose that H is central in A or in B. Let $G = A \underset{H}{*} B$ be the free product of A and B with amalgamated subgroup H. Using simultaneously ([11], Corollary 15.2, p. 532) and ([12], Theorem 4., p. 11), there is a finite group G_1 of $\mathcal{C}$ containing isomorphic copies A_1 and B_1 of A and B, respectively, with isomorphisms

$$\theta : A \longrightarrow A_1;\ \phi : B \longrightarrow B_1.$$

G_1 can be chosen such that the isomorphisms θ and ϕ coincide on H. See ([11], p. 532).
Since G is the free product of A and B with amalgamated subgroup H, it follows that θ and ϕ can be simultaneously extended to a homomorphism μ of G onto G_1. Consider K the kernel of μ. Since G_1 is finite, it follows that K is of finite index in G. Since μ is one-to-one when restricted to either A or B, it follows that

$$K \cap A = 1 = K \cap B.$$

In accordance to ([1], Theorem 2.), K is free. Consequently, using the fact that $\mathcal{C}$ can be seen as a root-class, K is $\mathcal{C}$-residual by ([22], Theorem 2.1). Finally, G is clearly $\mathcal{C}$-residual as a finite extension of a $\mathcal{C}$-residual group, and the Lemma is demonstrated. □

Proof of Theorem 1.1 Let $\mathcal{C}$ be the class of all finite solvable groups. Let G_1 and G_2 be two $\mathcal{C}$-residual groups and let H and K be nonidentity subgroups of G_1 and G_2 respectively such that H is central in G_1 and K is abelian and commute with G_1. Following Loginova in [8], we write G as a double amalgamation. That is,

$$G = (G_1 \underset{H}{*} (H \times K)) \underset{H\times K}{*} (G_2 \underset{K}{*} (H \times K)).$$

1. Assume that the subgroups H and K are not $\mathcal{C}$-separable. Let prove that G is not $\mathcal{C}$-residual. Since K is not $\mathcal{C}$-separable, let $a \in G_2 \setminus K$ and let η be an arbitrary homomorphism of G_2 onto a finite group in $\mathcal{C}$ such that $\eta(a) \in \eta(K)$. Let h be a nonidentity element of H. Then the element $w = [a, h]$ of the group $G_2 \underset{K}{*} (H \times K)$ differs from 1, since $w = a^{-1}h^{-1}ah$ is reduced in the free product with amalgamation $G_2 \underset{K}{*} (H \times K)$. Clearly, the image of this element under any homomorphism of $G_2 \underset{K}{*} (H \times K)$ onto a finite group in $\mathcal{C}$ equals 1. Thus, $G_2 \underset{K}{*} (H \times K)$ is not $\mathcal{C}$-residual and likewise G. The same result is obtained similarly when considering the subgroup H not $\mathcal{C}$-separable.

2. Conversely, let now subgroups H and K be $\mathcal{C}$-separable in G_1 and G_2 respectively. And let prove that the group $G = G_1 \underset{[H,K]}{*} G_2$ is $\mathcal{C}$-residual. Consider $(R_i)_{i\in I}$, the family of all normal subgroups of finite index in $G_1 \underset{H}{*} (H \times K)$ with $G_1 \underset{H}{*} (H \times K)/R_i \in \mathcal{C}$ for all $i \in I$ and let $(S_j)_{j\in J}$ be the family of all normal subgroups of finite index in $G_2 \underset{K}{*} (H \times K)$ with $G_2 \underset{K}{*} (H \times K)/R_j \in \mathcal{C}$ for all $j \in J$. Denote by Λ the subset of $I \times J$ that consists of the various pairs (i, j) such that the subgroups R_i and R_j are $(H \times K, H \times K)$-compatible and put $R_\lambda = R_i$ and $S_\lambda = S_j$ for an arbitrary element $\lambda = (i, j) \in \Lambda$.
   Let X and Y be arbitrary normal subgroups of finite index in G_1 and G_2 respectively with $G_1/X \in \mathcal{C}$ and $G_2/Y \in \mathcal{C}$. Since the groups G_1 and G_2 are $\mathcal{C}$-residual and their subgroups H and K are $\mathcal{C}$-separable, then, using ([8], Lemma 1.), it follows that for every nonidentity element g of $G_1 \underset{H}{*} (H \times K)$, there exists an element $\lambda_g \in \Lambda$ such that

   $$G_1 \cap R_{\lambda_g} = X \text{ and } (H \times K) \cap R_{\lambda_g} = (X \cap H).(Y \cap K) \text{ with } g \notin R_{\lambda_g}.$$

   Moreover, if g does not belong to $H \times K$, the subgroup R_{λ_g} can be chosen so that $g \notin (H \times K)R_{\lambda_g}$. Consequently, $\underset{\lambda\in\Lambda}{\cap} R_\lambda = 1$ and $\underset{\lambda\in\Lambda}{\cap} (H \times K)R_\lambda = H \times K$.

Therefore, the family $(R_\lambda)_{\lambda\in\Lambda}$ is a $H \times K$-filtration. Similarly, we obtain that the family $(S_\lambda)_{\lambda\in\Lambda}$ is a $H \times K$-filtration.
Thus, for every $\lambda \in \Lambda$, the map

$$(H \times K)R_\lambda/R_\lambda \longrightarrow (H \times K)S_\lambda/S_\lambda$$

from the subgroup $(H \times K)R_\lambda/R_\lambda$ of the quotient group $G_1 \underset{H}{*} (H \times K)/R_\lambda$ onto the subgroup $(H \times K)S_\lambda/S_\lambda$ of the quotient group $G_2 \underset{K}{*} (H \times K)/S_\lambda$, determined by the rule
$\varphi_{R_\lambda,S_\lambda}(xR_\lambda) = xS_\lambda$ $(x \in H \times K)$, is well defined and clearly an isomorphism since R_λ and S_λ are $(H \times K, H \times K)$-compatible. Therefore, we constuct the group
$G_{R_\lambda,S_\lambda} = G_1 \underset{H}{*} (H \times K)/R_\lambda \underset{(H\times K)R_\lambda/R_\lambda=(H\times K)S_\lambda/S_\lambda}{*} G_2 \underset{K}{*} (H \times K)/S_\lambda.$
The natural mappings from the group $G_1 \underset{H}{*} (H \times K)$ onto the quotient group $G_1 \underset{H}{*} (H \times K)/R_\lambda$ and from the group $G_2 \underset{K}{*} (H \times K)$ onto the quotient group $G_2 \underset{K}{*} (H \times K)/S_\lambda$ extend to a homomorphism $\rho_{R_\lambda,S_\lambda}$ from the group $G = (G_1 \underset{H}{*} (H \times K)) \underset{H\times K}{*} (G_2 \underset{K}{*} (H \times K))$ onto the group $G_\lambda = G_{R_\lambda,S_\lambda}$.
Note that the families $(R_\lambda)_{\lambda\in\Lambda}$ and $(S_\lambda)_{\lambda\in\Lambda}$ are closed under finite intersections, i.e., for any λ_1, $\lambda_2 \in \Lambda$, there is an index $\lambda \in \Lambda$ such that
$R_{\lambda_1} \cap R_{\lambda_2} = R_\lambda$ and $S_{\lambda_1} \cap S_{\lambda_2} = S_\lambda$.
Therefore, if g is a nonidentity element of G, then, considering a reduced form of g and the fact that the families $(R_\lambda)_{\lambda\in\Lambda}$ and $(S_\lambda)_{\lambda\in\Lambda}$ are $H \times K$-filtration, there exists $\lambda \in \Lambda$ such that the image of g under the homomorphism $\rho_\lambda = \rho_{R_\lambda,S_\lambda}$ differs from 1. Indeed, let $g \in G$.
• If $g \in G_1 \underset{H}{*} (H \times K)$, put $\lambda \in \Lambda$ such that $g \notin R_\lambda$. Note that, this choice is possible since $\underset{\lambda\in\Lambda}{\cap} R_\lambda = 1$. See that, $\rho_\lambda(g) = gR_\lambda \neq R_\lambda$. Similarly, we prove that there exists $\lambda \in \Lambda$ such that $\rho_\lambda(g) = gS_\lambda \neq S_\lambda$ if $g \in G_2 \underset{K}{*} (H \times K)$.
• If $g \notin G_1 \underset{H}{*} (H \times K) \cup G_2 \underset{K}{*} (H \times K)$, write $g = x_1y_1x_2y_2.....x_ny_n$ with $x_i \in G_1 \underset{H}{*} (H \times K)$, $y_i \in G_2 \underset{K}{*} (H \times K)$, $x_i, y_i \notin H \times K$, $1 \leq i \leq n$. We choice a convenient $\lambda \in \Lambda$ such that $x_i \notin (H \times K)R_\lambda$ and $y_i \notin (H \times K)S_\lambda$, $1 \leq i \leq n$ as follows.
Put $\lambda_1 \in \Lambda$ such that $x_i \notin (H \times K)R_{\lambda_1}$. Note that this choice is possible since $(R_\lambda)_{\lambda\in\Lambda}$ is $H \times K$-filtration and closed under finite intersection. Similarly, put $\lambda_2 \in \Lambda$ such that $y_i \notin H \times KS_{\lambda_2}$, $1 \leq i \leq n$.
Put then $\lambda \in \Lambda$ such that $R_\lambda = R_{\lambda_1} \cap R_{\lambda_2}$, $S_\lambda = S_{\lambda_1} \cap S_{\lambda_2}$.
See that, $\rho_\lambda(g) = x_1R_\lambda y_1S_\lambda x_2R_\lambda y_2S_\lambda...x_nR_\lambda y_nS_\lambda \neq R_\lambda = S_\lambda$ in G_λ.
Now, throught Lemma 3.1, $G_\lambda = G_{R_\lambda,S_\lambda}$ is $\mathcal{C}$-residual. Indeed, G_λ is the free

product of two groups of $\mathcal{C}$ with amalgamated subgroup
$(H \times K)R_\lambda/R_\lambda = (H \times K)S_\lambda/S_\lambda$ which is central in $G_1 \underset{H}{*} (H \times K)/R_\lambda$.
To see that $(H \times K)R_\lambda/R_\lambda = (H \times K)S_\lambda/S_\lambda$ is central in $G_1 \underset{H}{*} (H \times K)/R_\lambda$, let $u \in G_1 \underset{H}{*} (H \times K)/R_\lambda$. That is $u = xR_\lambda$ with $x \in G_1 \underset{H}{*} (H \times K)$. Assume that $x = g_1k_1g_2k_2 \ldots g_nk_n$, in its reduced form in $G_1 \underset{H}{*} (H \times K)/R_\lambda$ with $g_i \in G_1$ and $k_i \in K$.
Let $v = yR_\lambda \in (H \times K)R_\lambda/R_\lambda$ with $y = hk \in H \times K$ $(h \in H, k \in K)$.

$$\begin{aligned} uv &= xR_\lambda yR_\lambda \\ &= xyR_\lambda \\ &= g_1k_1g_2k_2 \ldots g_nk_nhkR_\lambda \\ &= hkg_1k_1g_2k_2 \ldots g_nk_nR_\lambda \text{ since } H \text{ is central in } G_1, K \text{ is abelian} \\ &\text{and commute with } G_1 \\ &= yxR_\lambda \\ &= yR_\lambda xR_\lambda \\ &= vu \end{aligned}$$

Now, since $\rho_\lambda(g) \neq 1$ and G_λ is $\mathcal{C}$-residual, it follows that there exists a homomorphism l from G_λ to a group of $\mathcal{C}$ such that for every nonidentity element g of G we have $l\rho_\lambda(g) \neq 1$, a nonidentity image. Consequently, G is $\mathcal{C}$-residual. And the Theorem is proven.

4. PROOF OF THEOREM 1.2

We first prove the following lemma.

Lemma 4.1
Let $\mathcal{C}$ be the class of all finite solvable groups. Let G_1 and G_2 be two pro-$\mathcal{C}$ groups with respective closed subgroups H and K. Then, the pro-$\mathcal{C}$ topology of $G = G_1 \underset{[H,K]}{*} G_2$ induces on $G_1 \underset{H}{*} (H \times K)$, $(H \times K) \underset{K}{*} G_2$ and $H \times K$ their respective pro-$\mathcal{C}$ topologies.

Proof. Set,
$\mathcal{N} = \{N \lhd_f G = G_1 \underset{[H,K]}{*} G_2 : N \cap G_i$ is open in $G_i,\ i = 1, 2$ and $G/N \in \mathcal{C}\}$,
$\mathcal{N}_1 = \{N \lhd_f G_1 \underset{H}{*} (H \times K) : N \cap G_1$ is open in $G_1,\ N \cap (H \times K)$ is open in $H \times K$ and $(G_1 \underset{H}{*} (H \times K))/N \in \mathcal{C}\}$,
$\mathcal{N}_2 = \{N \lhd_f G_2 \underset{K}{*} (H \times K) : N \cap G_2$ is open in $G_2,\ N \cap (H \times K)$ is open in $H \times K$ and $(G_2 \underset{K}{*} (H \times K))/N \in \mathcal{C}\}$, and

$\mathcal{N}_{induced} = \{N \cap (G_1 \underset{H}{*} (H \times K)) : \ N \in \mathcal{N}\}$.
Let prove that $\mathcal{N}_1 = \mathcal{N}_{induced}$.

1. $\mathcal{N}_{induced} \subset \mathcal{N}_1$ obviously.

2. Let now prove that $\mathcal{N}_1 \subset \mathcal{N}_{induced}$. Let $N \in \mathcal{N}_1$. Let determine $M \in \mathcal{N}$ such that
   $M \cap (G_1 \underset{H}{*} (H \times K)) = N$. To do it, it suffices to determine $M' \in \mathcal{N}$ such that $M' \cap (G_1 \underset{H}{*} (H \times K)) \leq N$. Indeed, if such $M' \in \mathcal{N}$ exists, then $M' \cap (G_1 \underset{H}{*} (H \times K)) \in \mathcal{N}_{induced}$. Consequently, $N \in \mathcal{N}_{induced}$ as a subgroup of $G_1 \underset{H}{*} (H \times K)$ containing the nonempty open set $M' \cap (G_1 \underset{H}{*} (H \times K))$. See ([4], Proposition 32). It follows then that there exists $M \in \mathcal{N}$ such that $M \cap (G_1 \underset{H}{*} (H \times K)) = N$ as needed.
   Let now construct $M' \in \mathcal{N}$ such that $M' \cap (G_1 \underset{H}{*} (H \times K)) \leq N$.
   Clearly, $N \cap G_1$ is open in G_1, $N \cap G_1 \lhd_f G_1$ and $G_1/N \cap G_1 \in \mathcal{C}$. Since the pro-$\mathcal{C}$ topology of G induces on G_1 its pro-$\mathcal{C}$ topology (see Proposition 2.1), it follows that there exists $M_1 \in \mathcal{N}$ such that $M_1 \cap G_1 = N \cap G_1$.
   Similarly, there exists $M_K \in \mathcal{N}$ such that $M_K \cap K = N \cap K$.
   Set $M' = M_1 \cap M_K$.

   (a) Let prove that $M' \in \mathcal{N}$.

      i. It is obvious that $M' \lhd_f G$

      ii. For any $i = 1, 2$, we have $M' \cap G_i = M_1 \cap M_K \cap G_i = (M_1 \cap G_i) \cap (M_K \cap G_i)$. By the definition of M_1 and M_K as elements of $\mathcal{N}$ we have clearly that $M_1 \cap G_i$ and $M_K \cap G_i$ are open in G_i and so is their intersection $M' \cap G_i$.

      iii. Let prove that $G/M' \in \mathcal{C}$.
      Since $G/M_1 \in \mathcal{C}$ and $G/M_K \in \mathcal{C}$, then $G/M_1 \cap M_K \in \mathcal{C}$ by [22] when considering $\mathcal{C}$ as a root-class.

      By i., ii. and iii., $M' \in \mathcal{N}$.

   (b) It remains to prove that $M' \cap (G_1 \underset{H}{*} (H \times K)) \leq N$, i.e, $M' \cap (G_1 \underset{H}{*} (H \times K))$ is a subgroup of N.
   Here, we use the presentation of groups by the generators and relations. Let,

$$G_1 = \langle S | D \rangle, \ H = \langle Q | V \rangle \text{ with } Q \subset S,$$

$$G_2 = \langle T | E \rangle, \ K = \langle P | R \rangle \text{ with } P \subset T,$$

$$G = \langle S \cup T | D \cup E,\ xy = yx \,\forall\, x \in Q,\ \forall\, y \in P \rangle,$$

$$G_1 \underset{H}{*} (H \times K) = \langle S \cup P | D \cup R,\ \sigma_1(h) = \tau_1(h) \,\forall\, h \in Q \rangle \text{ with}$$

$\sigma_1 : H \to G_1$ and $\tau_1 : H \to H \times K,$ the embedding maps

$$N = \langle A \cup B | C \rangle \text{ with } A \subset S \text{ and } B \subset P,$$

$$M_1 = \langle I \cup J | F \rangle \text{ with } I \subset S \text{ and } J \subset T,$$

$$M_K = \langle L \cup O | X \rangle \text{ with } L \subset S \text{ and } O \subset T,$$

$$M' = M_1 \cap M_K = \langle (I \cup J) \cap (L \cup O) | W \rangle.$$

Since

$$M_1 \cap G_1 = N \cap G_1 \Rightarrow (I \cup J) \cap S = (A \cup B) \cap S$$

and

$$M_K \cap K = N \cap K \Rightarrow (L \cup O) \cap P = (A \cup B) \cap P,$$

it follows that:

$$\begin{aligned}
M' \cap (G_1 \underset{H}{*} (H \times K)) &= M_1 \cap M_K \cap (G_1 \underset{H}{*} (H \times K)) \\
&= \langle [(I \cup J) \cap (L \cup O)] \cap (S \cup P) | Z \rangle \\
&= \langle [(I \cup J) \cap (L \cup O) \cap S] \cup [(I \cup J) \cap (L \cup O) \cap P] | Z \rangle \\
&= \langle [(A \cup B) \cap S \cap (L \cup O)] \cup [(I \cup J) \cap (A \cup B) \cap P] | Z \rangle \\
&= \langle (A \cup B) \cap [(S \cap (L \cup O)) \cup ((I \cup J) \cap P)] | Z \rangle \\
&= \langle Y | Z \rangle \text{ with } Y \subset A \cup B, \\
&\leq N.
\end{aligned}$$

By 1. and 2. we conclude that $\mathcal{N}_1 = \mathcal{N}_{induced}$, i.e., the pro-$\mathcal{C}$ topology of G induces on $G_1 \underset{H}{*} (H \times K)$ its own pro-$\mathcal{C}$ topology.
We argue similarly to prove that the pro-$\mathcal{C}$ topology of G induces on $(H \times K) \underset{K}{*} G_2$ and on $H \times K$ their own pro-$\mathcal{C}$ topologies. And the Lemma is demonstrated.

□

Proof of Theorem 1.2 Consider,
$\mathcal{N} = \{N \lhd_f G = G_1 \underset{[H,K]}{*} G_2 : N \cap G_i \text{ is open in } G_i,\ i = 1, 2,\ G/N \in \mathcal{C}\},$
$\mathcal{N}_1 = \{N \lhd_f G_1 \underset{H}{*} (H \times K) : N \cap G_1$ is open in $G_1, N \cap (H \times K)$ is open in $H \times K$ and $(G_1 \underset{H}{*} (H \times K))/N \in \mathcal{C}\},$
$\mathcal{N}_2 = \{N \lhd_f G_2 \underset{K}{*} (H \times K) : N \cap G_2$ is open in $G_2, N \cap (H \times K)$ is open in $H \times K$ and $(G_2 \underset{K}{*} (H \times K))/N \in \mathcal{C}\}$ and ,

$\mathcal{N}_3 = \{N \lhd_f (H \times K) : N \cap H$ is open in $H,$ and $N \cap K$ is open in $K,$ $(H \times K)/N \in \mathcal{C}\}.$
We have:

$$G_1 \underset{[H,K]}{\amalg} G_2 = \widehat{G_1 \underset{[H,K]}{*} G_2}^{\mathcal{N}}, \tag{1}$$

$$\widehat{G_1 \underset{[H,K]}{*} G_2}^{\mathcal{N}} = \overbrace{(G_1 \underset{H}{*} (H \times K)) \underset{H\times K}{*} ((H \times K) \underset{K}{*} G_2)}^{\mathcal{N}}, \tag{2}$$

$$\overbrace{(G_1 \underset{H}{*} (H \times K)) \underset{H\times K}{*} ((H \times K) \underset{K}{*} G_2)}^{\mathcal{N}} = \overbrace{G_1 \underset{H}{*} (H \times K)}^{\mathcal{N}_1} \underset{\widehat{HK}^{\mathcal{C}}}{\amalg} \overbrace{(H \times K) \underset{K}{*} G_2}^{\mathcal{N}_2}, \tag{3}$$

$$\overbrace{G_1 \underset{H}{*} (H \times K)}^{\mathcal{N}_1} \underset{\widehat{HK}^{\mathcal{C}}}{\amalg} \overbrace{(H \times K) \underset{K}{*} G_2}^{\mathcal{N}_2} = (G_1 \underset{H}{\amalg} (H \times K)) \underset{H\times K}{\amalg} ((H \times K) \underset{K}{\amalg} G_2). \tag{4}$$

Equation (1) is the construction of free pro-$\mathcal{C}$ products of pro-$\mathcal{C}$ groups with commuting subgroups which is the generalization of free profinite products of profinite groups with commuting subgroups presented by G. Mantika and D. Tieudjo in [10].
Equation (2) is obtained by writting the free abstract product of abstract groups with commuting subgroups as a double amalgamation presented by E. Loginova in [8].
Equation (3) is obtained by [20] using two reasons:

1. $G = G_1 \underset{[H,K]}{*} G_2$ induces on $G_1 \underset{H}{*} (H \times K)$, $(H \times K) \underset{K}{*} G_2$ and $H \times K$ their respective pro-$\mathcal{C}$ topologies, see Lemma 4.1, and
2. $G = G_1 \underset{[H,K]}{*} G_2$ is $\mathcal{C}$-residual since G_1 and G_2 are $\mathcal{C}$-residual and the subgroups H and K are $\mathcal{C}$-separated (see Theorem 1.1).

Equation (4) is obtained by the construction of free pro-$\mathcal{C}$ products of pro-$\mathcal{C}$ groups with amalgamation presented by L. Ribes and P. Zalesskii in [20], and using the equality $\widehat{H \times K}^{\mathcal{C}} = H \times K$ (by hypotesis). Thus, the Theorem is proven.

5. PROOF OF THEOREM 1.3

Lemma 5.1
Let $\mathcal{C}$ be the class of all finite solvable groups. Let G_1 and G_2 be two pro-$\mathcal{C}$ groups with respective closed subgroups H and K. Let $G = G_1 \underset{[H,K]}{\amalg} G_2$ be the free pro-$\mathcal{C}$ product of the pro-$\mathcal{C}$ groups G_1 and G_2 with commuting subgroups H and K. Then in G,

$$G_1 \underset{[H,K]}{\amalg} (H \times K) = G_1 \amalg K = G_1 \underset{H}{\amalg} (H \times K) = (G_1 \amalg K) \underset{H\times K}{\amalg} (H \times K).$$

Proof. It suffices to prove that in $G_1 \underset{[H,K]}{*} G_2$, we have

$$G_1 \underset{[H,K]}{*} (H \times K) = G_1 * K = G_1 \underset{H}{*} (H \times K) = (G_1 * K) \underset{H \times K}{*} (H \times K).$$

Indeed, assume that the above equalities hold. Then the following sets are the same:

$\mathcal{N}_a = \{N \lhd_f G_1 \underset{[H,K]}{*} (H \times K) : N \cap G_1$ is open in $G_1,\ N \cap K$ is open in $K,\ (G_1 \underset{[H,K]}{*} (H \times K))/N \in \mathcal{C}\}$,

$\mathcal{N}_b = \{N \lhd_f G_1 * K : N \cap G_1$ is open in $G_1,\ N \cap K$ is open in $K,\ (G_1 * K)/N \in \mathcal{C}\}$,

$\mathcal{N}_c = \{N \lhd_f G_1 \underset{H}{*} (H \times K) : N \cap G_1$ is open in $G_1,\ N \cap K$ is open in $K,\ (G_1 \underset{H}{*} (H \times K))/N \in \mathcal{C}\}$,

$\mathcal{N}_d = \{N \lhd_f (G_1 * K) \underset{H \times K}{*} (H \times K) : N \cap G_1$ is open in $G_1,\ N \cap K$ is open in K, $((G_1 * K) \underset{H \times K}{*} (H \times K))/N \in \mathcal{C}\}$. Consequently the following completions are equals:

$$\widehat{G_1 \underset{[H,K]}{*} (H \times K)}^{\mathcal{N}_a} = \widehat{G_1 * K}^{\mathcal{N}_b} = \widehat{G_1 \underset{H}{*} (H \times K)}^{\mathcal{N}_c} = \widehat{(G_1 * K) \underset{H \times K}{*} (H \times K)}^{\mathcal{N}_d},$$

i.e.,

$$G_1 \underset{[H,K]}{\amalg} (H \times K) = G_1 \amalg K = G_1 \underset{H}{\amalg} (H \times K) = (G_1 \amalg K) \underset{H \times K}{\amalg} (H \times K)$$

as needed.

So, let prove that in $G_1 \underset{[H,K]}{*} G_2$, we have:

$$G_1 \underset{[H,K]}{*} (H \times K) = G_1 * K = G_1 \underset{H}{*} (H \times K) = (G_1 * K) \underset{H \times K}{*} (H \times K).$$

Assume that $G_1 = \langle S|D \rangle$, $H = \langle Q|V \rangle$ with $Q \subset S$ and $K = \langle P|R \rangle$.

$$\begin{aligned} G_1 \underset{[H,K]}{*} (H \times K) &= \langle S \cup (Q \cup P)|D \cup R,\ xy = yx,\ \forall x \in Q,\ \forall y \in P \rangle \\ &= \langle S \cup P|D \cup R,\ xy = yx,\ \forall x \in Q,\ \forall y \in P \rangle. \end{aligned}$$

• $G_1 * K = \langle S \cup P|D \cup R >$. Now, in $G_1 \underset{[H,K]}{*} G_2$, $\forall x \in Q,\ \forall y \in P,\ xy = yx$. Then,

$$G_1 * K = \langle S \cup P|D \cup R,\ xy = yx\ \forall x \in Q,\ \forall y \in P \rangle = G_1 \underset{[H,K]}{*} (H \times K)$$

and the first egality is proven.

• Let $\sigma : H \to G_1 \underset{[H,K]}{*} G_2$ be the inclusion map. Let $i_1 : H \to G_1$ and $i_2 : H \to H \times K$ be the corestrictions of σ on G_1 and $H \times K$ respectively.

$$G_1 \underset{H}{*} (H \times K) = \langle S \cup P|D \cup R,\ xy = yx\ \forall x \in Q,\ \forall y \in P,\ i_1(h) = i_2(h)\ \forall h \in H \rangle.$$

Since in $G_1 \underset{[H,K]}{*} G_2,\ \forall h \in H,\ i_1(h) = i_2(h)$ then,

$$G_1 \underset{H}{*} (H \times K) = \langle S \cup P | D \cup R,\ xy = yx\ \forall x \in Q,\ \forall y \in P\rangle = G_1 \underset{[H,K]}{*} (H \times K)$$

and the second egality is proven.

• Let $\alpha : H \times K \to G_1 \underset{[H,K]}{*} G_2$ be the inclusion map. Let $i_3 : H \times K \to G_1 * K$ and $i_4 : H \times K \to H \times K$ be the corestrictions of α on $G_1 * K$ and $H \times K$ respectively. $(G_1 * K) \underset{H\times K}{*} (H \times K) = \langle S \cup P | D \cup R,\ xy = yx\ \forall x \in Q,\ \forall y \in P,\ i_3(h) = i_4(h)\ \forall h \in (H \times K)\rangle$. Since in $G_1 \underset{[H,K]}{*} G_2$, $\forall h \in (H \times K),\ i_3(h) = i_4(h)$ then,

$(G_1 * K) \underset{H\times K}{*} (H \times K) = \langle S \cup P | D \cup R,\ xy = yx\ \forall x \in Q,\ \forall y \in P\rangle = G_1 \underset{[H,K]}{*} (H \times K)$
and the third egality is proven.

□

We are now ready to prove Theorem 1.3.

Proof of Theorem 1.3

1)⇒ 2) Assume that $G = G_1 \underset{[H,K]}{\amalg} G_2$. Since the conditions in Theorem 1.2 are satisfied, we write G as double pro-$\mathcal{C}$ amalgamation. That is:

$$G = G_1 \underset{[H,K]}{\amalg} G_2 = (G_1 \underset{H}{\amalg} (H \times K)) \underset{H\times K}{\amalg} ((H \times K) \underset{K}{\amalg} G_2).$$

Following Ribes and Zalesskii through ([20], Theorem 9.3.1), it follows that the natural homomorphism

$$\phi_G : \mathrm{Der}_{H\times K}(G, A) \longrightarrow \mathrm{Der}_{H\times K}(G_1 \underset{H}{\amalg} (H\times K), A) \times \mathrm{Der}_{H\times K}(G_2 \underset{K}{\amalg} (H\times K), A)$$

$(f \longmapsto (f|_{G_1 \underset{H}{\amalg} (H\times K)}, f|_{G_2 \underset{K}{\amalg} (H\times K)}))$ is an isomorphism for all $[[\mathbb{Z}_{\hat{\mathcal{C}}} G]]-$modules A $\in \mathcal{C}$.
Now, through Lemma 5.1, we have:
$\mathrm{Der}_{H\times K}(G_1 \underset{H}{\amalg} (H \times K), A) \times \mathrm{Der}_{H\times K}(G_2 \underset{K}{\amalg} (H \times K), A) = \mathrm{Der}_{H\times K}(G_1 \underset{H\times K}{\amalg} (H \times K), A) \times \mathrm{Der}_{H\times K}(G_2 \underset{H\times K}{\amalg} (H \times K), A)$.
Also, by ([20], Theorem 9.3.1), the natural homomorphism
$\phi^1_{(G_1 \amalg K) \underset{H\times K}{\amalg} (H\times K)} : \mathrm{Der}_{H\times K}((G_1 \amalg K) \underset{H\times K}{\amalg} (H \times K), A) \longrightarrow \mathrm{Der}_{H\times K}(G_1 \amalg K, A) \times \mathrm{Der}_{H\times K}(H \times K, A)$ is an isomorphism for all $[[\mathbb{Z}_{\hat{\mathcal{C}}} G]]-$modules A $\in \mathcal{C}$.
Obviously, $\mathrm{Der}_{H\times K}(H \times K, A) = 0$. Now, let the isomorphism

$$\delta_1 :\ \mathrm{Der}_{H\times K}(G_1 \amalg K, A) \times 0 \longrightarrow\ \mathrm{Der}_{H\times K}(G_1 \amalg K, A).$$

We obtain the isomorphism

$$\delta_1 \phi^1_{(G_1 \amalg K) \underset{H\times K}{\amalg} (H\times K)} : \ \mathrm{Der}_{H\times K}(G_1 \underset{H}{\amalg} (H\times K), A) \longrightarrow \ \mathrm{Der}_{H\times K}(G_1 \amalg K, A).$$

Similary, we obtain the isomorphism

$$\delta_2 \phi^2_{(G_2 \amalg H) \underset{H\times K}{\amalg} (H\times K)} : \ \mathrm{Der}_{H\times K}(G_2 \underset{K}{\amalg} (H\times K), A) \longrightarrow \ \mathrm{Der}_{H\times K}(G_2 \amalg H, A).$$

The following diagram illustrate this situation.

$$\begin{tikzcd}
\mathrm{Der}_{H\times K}(G,A) \arrow[rr, "\phi_G", "\simeq"'] \arrow[dddddd, "\psi_G"', "\simeq"] & & \mathrm{Der}_{H\times K}(G_1 \underset{H}{\amalg} (H\times K), A)\times \mathrm{Der}_{H\times K}(G_2 \underset{K}{\amalg} (H\times K), A) \arrow[dl, "P_1"'] \arrow[ddd, "P_2"] \arrow[ddddddll, "\varphi", "\simeq"'] \\
& \mathrm{Der}_{H\times K}(G_1 \underset{H}{\amalg} (H\times K), A) = \mathrm{Der}_{H\times K}((G_1 \amalg K) \underset{H\times K}{\amalg} (H\times K), A) \arrow[d, "\phi^1_{(G_1\amalg K)\underset{H\times K}{\amalg}(H\times K)}"', "\simeq"] & \\
& \mathrm{Der}_{H\times K}(G_1 \amalg K, A) \times \underbrace{Der_{H\times K}(H\times K, A)}_{0} \arrow[dd, "\delta_1"', "\simeq"] & \\
& & \mathrm{Der}_{H\times K}(G_2 \underset{K}{\amalg} (H\times K), A) = \mathrm{Der}_{H\times K}((G_2 \amalg H) \underset{H\times K}{\amalg} (H\times K), A) \arrow[dd, "\phi^2_{(G_2\amalg H)\underset{H\times K}{\amalg}(H\times K)}", "\simeq"'] \\
& \mathrm{Der}_{H\times K}(G_1 \amalg K, A) & \\
& & \mathrm{Der}_{H\times K}(G_2 \amalg H, A) \times \underbrace{Der_{H\times K}(H\times K, A)}_{0} \arrow[d, "\delta_2"', "\simeq"] \\
\mathrm{Der}_{H\times K}(G_1 \amalg K, A)\times \mathrm{Der}_{H\times K}(G_2 \amalg H, A) \arrow[rr, "P_4"] \arrow[uur, "P_3"] & & \mathrm{Der}_{H\times K}(G_2 \amalg H, A)
\end{tikzcd}$$

where P_1, P_2, P_3 and P_4 are the canonical projections. We can see that

$$\mathrm{Der}_{H\times K}(G_1 \underset{H}{\amalg} (H\times K), A) \times \mathrm{Der}_{H\times K}(G_2 \underset{K}{\amalg} (H\times K), A)$$

and

$$\mathrm{Der}_{H\times K}(G_1 \amalg K, A) \times \mathrm{Der}_{H\times K}(G_2 \amalg H, A)$$

as two direct products of the groups $\mathrm{Der}_{H\times K}(G_1 \amalg K, A)$ and $\mathrm{Der}_{H\times K}(G_2 \amalg K, A)$. So, by the unicity of the direct product of two groups, $\psi_G = \varphi\phi_G$ is an isomorphism since so are φ and ϕ_G.

2)$\Rightarrow$ 1) Assume that the natural homomorphism

$$\psi_G : \mathrm{Der}_{H\times K}(G, A) \longrightarrow \mathrm{Der}_{H\times K}(G_1 \amalg K, A) \times \mathrm{Der}_{H\times K}(G_2 \amalg H, A)$$

$(f \longmapsto (f|_{G_1 \amalg K}, f|_{G_2 \amalg H}),\ f \in \mathrm{Der}_{H\times K}(G, A))$ is an isomorphism for all $[[\mathbb{Z}_{\hat{\mathcal{C}}}G]]-$modules $A \in \mathcal{C}$. Using Lemma 5.1,

$$\phi_G : \mathrm{Der}_{H\times K}(G, A) \longrightarrow \mathrm{Der}_{H\times K}(G_1 \coprod_H (H\times K), A) \times \mathrm{Der}_{H\times K}(G_2 \coprod_K (H\times K), A)$$

is an isomorphism for all $[[\mathbb{Z}_{\hat{\mathcal{C}}}G]]-$modules $A \in \mathcal{C}$. Applying Theorem 9.3.1 (2 $\Rightarrow$ 1) in [20] we have:

$$G = (G_1 \coprod_H (H \times K)) \coprod_{H\times K} ((H \times K) \coprod_K G_2).$$

Since the conditions in Theorem 1.2 are satisfied, the following equality holds:

$$(G_1 \coprod_H (H \times K)) \coprod_{H\times K} ((H \times K) \coprod_K G_2) = G_1 \coprod_{[H,K]} G_2.$$

Thus, $G = G_1 \coprod_{[H,K]} G_2$. And the Theorem is proven.

REFERENCES

[1] G. Baumslag, *On the residual finiteness of generalized free products of nilpotent groups*, Trans. Amer. Math. Soc., No. 2, (1963).

[2] N. Bourbaki, *General Topology*, Springer, Berlin (1989).

[3] T. Coulbois, *Propriétés de Ribes-Zalesskii, topologie profinie, produit libre et généralisations*, Thèse de Doctorat, Université Paris VII-Denis Diderot, (2000).

[4] B. Deschamps, *Groupes Profinis et Théorie de Galois*, Calrenton Press, Oxford (1998).

[5] E. Detomi, B. Klopsch and P. Shumyatsky, *Strong conciseness in profinite groups*, London Mathematical Society, (2020).

[6] M. Ferrara and M. Trombetti, *The pro-norm of a profinite group*, Israel Journal of Mathematics, (2023), Vol. 254

[7] D. Gildenhuys and E. Mackay, *Triple Cohomology and Galois Cohomology of Profinite Groups*, Communication in Algebra, Vol. 1, Issue 6 459-473 (1974).

[8] E. Loginova, *Residual finiteness of the free product of two groups with commuting sbgroups*, Siberian Mathematical journal, (1971).

[9] W. Magnus, A. Karass and D. Solitar, *Combinatorial Group Theory*, Pure and Applied Math. vol xiii, Wiley-International, New York-London-Sydney, (1966).

[10] G. Mantika and D. Tieudjo, *On the free profinite products of profinite groups with commuting sub-groups* , International Journal of Group Theory, Vol.5, No.2 (2016), p. 25-40.

[11] B.H. Neumann, *An essay on free products of groups with amalgamations*, Philos. Trans. Roy. Soc. London Ser. A 246, 503-554, (1954).

[12] B.H. Neumann, *Permutational products of groups*, J. Austral, Math. (1960), 299-310.

[13] J. Neukirch, *Freie Produkte pro-endlicher Gruppen und ihre Kohomologie*, Arch. Mathematik (Basel), (1971).

[14] J. Nielsen, *Die Isomorphismengruppe der freien Gruppen*, Mathematische Annalen, (1924).

[15] L. Ribes, *Amalgamed products of profinite groups*, Mathematische Zeitschrift, (1971).

[16] L. Ribes, *Amalgamed products of profinite groups: counterexamples*, American Mathematical Society. Vol. 37, (1973).

[17] L. Ribes, *Cohomological characterization of amalgamated products of groups*, Pure and Applied Algebra (1974).

[18] L. Ribes, *On a cohomology theory for pairs of groups*, Proc. Am. Maths. Soc. 21 (1969).

[19] L. Ribes, *Profinite graphs and groups*, Springer International Publishing, (2017).

[20] L. Ribes and P. A. Zalesskii, *Profinite Groups*, Springer, Berlin, (2010).

[21] J.P. Serre, *Galois cohomology*, Springer-Verlag, (1997).

[22] D. Tieudjo, *Root-class residuality of some free construction*, JP Journal of algebra, Number theory and applications. (2010).

[23] J.S. Wilson, *Profinite groups*, Clarendon Press, Oxford (1998).