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Approximation of Common Fixed Points of Two Commuting Generalized Nonexpansive Mappings

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Abstract

We discuss some generalizations of nonexpansive mappings and prove additional properties of these generalizations, particularly focusing on two commuting mappings satisfying condition $B_{\gamma,\mu}$. New algorithm to approximate a common fixed point of two commuting mappings satisfying condition $B_{\gamma,\mu}$ is introduced. Strong convergence of the introduced algorithm to a common fixed point of the two commuting mappings satisfying condition $B_{\gamma,\mu}$ is also proved. Our results extend and improve some recent results in the literature.

Keywords: nonexpansive mapping, Condition (C), Condition $B_{\gamma,\mu}$, Commuting mappings, Fixed point.

Mathematics subject Classification: 47H05, 47H10, 47J25, 49J40, 54H25, 91B99

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1. INTRODUCTION

The generalization of contraction mappings and the study of related fixed point theorems with different practical applications in nonlinear functional analysis have found great importance since 1920s. The celebrated Banach contraction principle[3] with its wide range of applications pioneered the field and attracted many mathematicians to research on it. In 1965 Browder[5, 6], Göhde [12] and Kirk [15], independently, studied existence and approximation of fixed points of nonexpansive mappings which was one of the break through in the generalization of Banach contraction principle. In 1972, Goebel and Kirk[11] introduced asymptotically nonexpansive mappings and proved existence and approximation of fixed points of such self-mappings in Banach spaces. Several authors (see Agarwal et al.[1], Aksoy et al.[**?**], Betiuk-Pilarska and Benavides [4], Browder[5, 6], Dhompongsa et al. [8], Garcia-Falset et al.[9], Goebel and Kirk [10], Khamsi and Khan [13], Lael and Heidarpoor [16], Mishra[17], Mishra et al. [19], Mishra et al. [18], Pant and Shukla [22], Patir et al [20, 21], Sangago[23], Suzuki [25, 26], Ullah et al. [28] and the references therein) have contributed immensely in this field, and different new classes of generalized nonexpansive mappings with interesting properties have been developed in this context. Researching of the practical significance of the metric fixed point approach in solving problems of applied sciences such us signal processing, inverse problems, equilibrium problems, game theory in market economy, optimization and so on come to the center stage in recent decades.

It is the purpose of this article to analyze and generalize some of recent results in the generalization of nonexpansive mappings with particular attention to the mappings introduced by Suzuki [25, 26], and further investigated by Patir et al.[21] and Thakur et al.[27]. Basic definitions and terminologies we use throughout the article follow hereunder.

Throughout this article, $\mathbb{N}$ and $\mathbb{R}$ stand for the set of natural numbers and the set of all real numbers, respectively. For a sequence $\{x_n\}$ of a normed space B and a point x in $B,$ the strong convergence of $\{x_n\}$ to x is denoted by $x_n \longrightarrow x$ and the weak convergence of $\{x_n\}$ to x is denoted by $x_n \rightharpoonup x.$

Let X be a nonempty set and $G : X \to X$ be a mapping. We say that a point $x \in X$ is said to be a fixed point of G when $Gx = x$. $Fix(G)$ denotes the set of all fixed points of G; that is,

$$Fix(G) = \{x \in X : Gx = x\}.$$

Definition 1.1 ([7, 11, 14])**.** *Let B be a real Banach space. Let K be a nonempty subset of B and $G : K \to K$. We say G is*

1. *a* ***contraction*** *mapping if there exists* $r \in [0,1)$ *such that*

$$\|Gx - Gy\| \leq r\,\|x-y\|\,, \ \text{ for all } \ x,y \in K. \tag{1.1}$$

2. *a* ***nonexpansive*** *mapping if*

$$\|Gx - Gy\| \leq \|x-y\|\,, \ \text{ for all } \ x,y \in K. \tag{1.2}$$

3. *an* ***asymptotically nonexpansive*** *mapping if there exists a sequence* $\{r_n\}$ *in* $[1,\infty)$ *such that* $\lim\limits_{n\to\infty} r_n = 1$ *and*

$$\|G^n x - G^n y\| \leq r_n\,\|x-y\|\,, \ \text{ for all } \ x,y \in K. \tag{1.3}$$

4. *a* ***quasi-nonexpansive*** *mapping if* $Fix(G) \neq \emptyset$, *and*

$$\|Gx - z\| \leq \|x-z\| \ \text{ for all } \ z \in Fix(G),\ x \in K. \tag{1.4}$$

From the above definitions, it follows that a nonexpansive mapping must be quasi-nonexpansive and asymptotically nonexpansive. Existence and approximation of fixed points of these mappings were also proved (see [2], [11], [15], and the references therein).

Recently new classes of generalized nonexpansive mappings were introduced by Suzuki [26] in 2008, Garcia-Falset et al. [9] in 2011 and Patir et al. [21] in 2018 as stated in the following definitions and proved fixed point theorems for their generalizations.

Definition 1.2 (Suzuki [26])**.** *Let* K *be a nonempty subset of the Banach space* B *and* $G : K \longrightarrow K$. *Then G is said to satisfy condition (C) if for all* $x,y \in K$

$$\frac{1}{2}\,\|x - Gx\| \leq \|x-y\| \quad \Rightarrow \quad \|Gx - Gy\| \leq \|x-y\|\,. \tag{1.5}$$

Suzuki (Propostion 1 and Proposition 2 of [26]) proved that the condition (C) is weaker than nonexpansiveness and stronger than quasi-nonexpansiveness.

Definition 1.3 (Garcia-Falset et al. [9])**.** *Let* K *be a nonempty subset of the Banach space* B *and* $G : K \longrightarrow K$. *Then G is said to satisfy condition* (C_λ), *where* $\lambda \in (0,1)$, *if for all* $x,y \in K$

$$\lambda\,\|x - Gx\| \leq \|x-y\| \quad \Rightarrow \quad \|Gx - Gy\| \leq \|x-y\|\,. \tag{1.6}$$

It was shown by Garcia-Falset et al. [9] that Condition (C) is a particular case of Condition (C_λ) with $\lambda = \frac{1}{2}$. Hence a nonexpansive self-mapping satisfies the condition (C_λ) for each $\lambda \in (0,1)$.

Definition 1.4 (Patir et al. [21]). *Let K be a nonempty subset of the Banach space B and $G : K \longrightarrow K$. Then G is said to satisfy condition $B_{\gamma,\mu}$ if there exists $\gamma \in [0,1],\ \mu \in [0, \frac{1}{2}]$ with $2\mu \leq \gamma$ such that for all $x, y \in K$*

$$\begin{aligned} &\gamma \|x - Gx\| \leq \|x - y\| + \mu \|y - Gy\| \\ &\quad \Rightarrow \quad \|Gx - Gy\| \leq (1-\gamma)\ \|x - y\| + \mu(\|x - Gy\| + \|y - Gx\|). \end{aligned} \tag{1.7}$$

Patir et al. [21] constructed examples to justify that their generalization was more general than that of Condition (C) and Condition (C_λ). Both authors justified that the inclusions were strict. In case $Fix(T) \neq \emptyset$, each of the conditions (C), (C_λ) and $B_{\gamma,\mu}$ implies quazi-nonexpansiveness of self-mapping G.

Suzuki [25] stated and proved the following characterization for two commuting nonexpansive mappings.

Proposition 1.5. *[25] Let K be a closed convex subset of the Banach space B. Let $G_1, G_2 : K \longrightarrow K$ be commuting nonexpansive mappings (i.e., $G_1 \circ G_2 = G_2 \circ G_1$). Let $\{x_n\}$ be a sequence in K that converges strongly to some $z \in K$. If $\{\alpha_n\}$ is a sequence in $(0, \frac{1}{2})$ converging to 0 such that*

$$\lim_{n\to\infty} \frac{\|(1-\alpha_n)G_1x_n + \alpha_n G_2 x_n - x_n\|}{\alpha_n} = 0, \tag{1.8}$$

then z is a common fixed point of G_1 and G_2.

In the same article Suzuki [25] extended Proposition 1.5 systematically from two to three and then for a finite family of commuting nonexpansive mappings; and then proved the following fixed point theorem.

Theorem 1.6. *[25] Let K be a compact convex subset of a Banach space B. Let $\{T_n : n \in \mathbb{N}\}$ be an infinite family of commuting nonexpansive mappings on K. Fix $\lambda \in (0,1)$. Let $\{\alpha_n\}$ be a sequence in $[0, \frac{1}{2}]$ satisfying*

$$\liminf_{n\to\infty} \alpha_n = 0, \quad \limsup_{n\to\infty} \alpha_n > 0, \quad \lim_{n\to\infty} [\alpha_{n+1} - \alpha_n] = 0.$$

Define a sequence $\{x_n\}$ in K $x_1 \in K$ and

$$x_{n+1} = \lambda \left(1 - \sum_{k=1}^{n-1} \alpha_n^k\right) T_1 x_n + \lambda \sum_{k=2}^{n} \alpha_n^{k-1} T_k x_n + (1-\lambda) x_n$$

for $n \in \mathbb{N}$. Then $\{x_n\}$ converges strongly to a common fixed point of $\{T_n : n \in \mathbb{N}\}$.

Theorem 1.7. *[21] Let K be a nonempty subset of the Banach space B. Let G be a self-mapping and satisfies the condition $B_{\gamma,\mu}$ on K. For $x_0 \in K$, let a sequence $\{x_n\}$ in K be defined by;*

$$x_{n+1} = \lambda G x_n + (1-\lambda)x_n, \tag{1.9}$$

where $\lambda \in [\gamma, 1) - \{0\}$and $n \in \mathbb{N} \cup \{0\}$. Then $||Gx_n - x_n|| \to 0$ as $n \to \infty$.

Motivated and inspired by the above results we further investigate and generalize these extensions of nonexpansive mappings. We also prove the convergence of some iterative algorithms to fixed points of such mappings with mild assumptions on the parameters.

Methodology: Well developed analytic as well as fixed point theoretical methods to prove our results are implemented. Mainly the key existing methods in the literature to prove our results are taken from [21, 24, 25] and references therein.

2. PRELIMINARIES

We collect here basic concepts and technical lemmas that can be used in the sequel.

Let K be a nonempty closed convex subset of a Banach space B and let $G : K \to K$. A sequence $\{x_n\}$ in K said to be almost fixed point sequence for G if $\lim_{n\to\infty} \|Gx_n - x_n\| = 0$..

Definition 2.1. *[14]A Banach space B is said to be uniformly convex, if for every $\epsilon, 0 < \epsilon \leq 2$ there exists a $\delta = \delta(\epsilon) > 0$ such that*

$$x, y \in B, \quad \|x\| \leq 1, \quad \|y\| \leq 1, \quad \& \quad \|x-y\| \geq \epsilon \quad \text{implies} \quad \left\|\frac{x+y}{2}\right\| \leq 1-\delta.$$

We use the following characterization of uniformly convex spaces in proving our main results.

Lemma 2.2. *[24] Let B be a uniformly convex Banach space. Assume that $0 < b \leq t_n \leq c < 1, n = 1, 2, 3, \cdots$. Let the sequences $\{x_n\}$ and $\{y_n\}$ in B be such that*

$$\limsup_{n\to\infty} \|x_n\| \leq \nu, \quad \limsup_{n\to\infty} \|y_n\| \leq \nu, \ \text{ and } \lim_{n\to\infty} \|t_n x_n + (1-t_n)y_n\| = \nu, \text{where } \nu \geq 0.$$

Then

$$\lim_{n\to\infty} \|x_n - y_n\| = 0.$$

Lemma 2.3. *[1] Let B be a uniformly convex Banach space and K be a nonempty closed convex subset of B. Let $G : K \to B$ a mapping satisfying the condition $B_{\gamma,\mu}$ on*

K with $2\mu \leq \gamma,\ \gamma \in [0,1]$ and $\mu \in [0, \frac{1}{2}]$. Then, for any $\epsilon > 0$, there exists positive number $M(\epsilon) > 0$ such that $||x - Gx|| < \epsilon$ for all $x \in co(\{x_0, x_1\})$, where $x_0, x_1 \in K$ with $||x_0 - Gx_0|| \leq M(\epsilon)$ and $||x_1 - Gx_1|| \leq M(\epsilon)$.

Lemma 2.4. *[1] Let B be a uniformly convex Banach space and K be a nonempty closed convex bounded subset of B. Let $G : K \to B$ a mapping satisfying the condition $B_{\gamma,\mu}$ on K with $2\mu \leq \gamma,\ \gamma \in [0,1]$ and $\mu \in [0, \frac{1}{2}]$. Then, $I - G$ is demiclosed on K.*

The following properties of a mapping that satisfies condition $B_{\gamma,\mu}$ were proved in Patir et al. [21].

Lemma 2.5. *[21] Let K be a nonempty subset of the Banach space B. Let $G : K \to K$ satisfy the condition $B_{\gamma,\mu}$ on K. Then, for all $x, y \in K$ and for $\theta \in [0,1]$,*

(i) $\left\|Gx - G^2x\right\| \leq \|x - Gx\|$,

(ii) at least one of the following ((a) and (b)) holds:

(a) $\frac{\theta}{2}\|x - Gx\| \leq \|x - y\|$,

(b) $\frac{\theta}{2}\left\|Gx - G^2x\right\| \leq \|Gx - y\|$.

The condition (a) implies

$$\|Gx - Gy\| \leq (1 - \frac{\theta}{2})\|x - y\| + \mu(\|x - Gy\| + \|y - Gx\|)$$

and the condition (b) implies

$$\left\|G^2x - Gy\right\| \leq (1 - \frac{\theta}{2})\|Gx - y\| + \mu(\|Gx - Gy\| + \left\|y - G^2x\right\|).$$

$$(iii)\quad \|x - Gy\| \leq (3 - \theta)\|x - Gx\| + (1 - \frac{\theta}{2})\|x - y\| + \mu(2\|x - Gx\| + \|x - Gy\| + \|y - Gx\| + 2||Gx - G^2x||).$$

Lemma 2.6. *[21] For a nonempty subset K of a Banach space B, let $G : K \to B$ be a mapping satisfying $B_{\gamma,\mu}$ condition. If p is a fixed point of G on K, then for all $x \in K$,*

$$\|Gx - p\| \leq \|x - p\|.$$

3. MAIN RESULTS

We prove the following technical lemma that plays an important role in proving the main fixed point theorem of this paper. This lemma was proved for nonexpansive mappings by Suzuki [25].

Lemma 3.1. *Let K be a closed convex subset of the Banach space B. Let $G_1, G_2 : K \longrightarrow K$ be mappings satisfying the condition $B_{\gamma,\mu}$, where $2\mu < \gamma$, with $G_1 \circ G_2 = G_2 \circ G_1$ on K. Let $\{x_n\}$ be a sequence in K that converges strongly to some $z \in K$. If $\{\alpha_n\}$ is a sequence in $(0, \frac{1}{2})$ converging to 0 such that*

$$\lim_{n\to\infty} \|(1-\alpha_n)G_1x_n + \alpha_n G_2 x_n - x_n\| = 0, \tag{3.1}$$

then z is a common fixed point of G_1 and G_2.

Proof. For $n \in \mathbb{N}$ it follows from (i) and (iii) of Lemma 2.5 that

$$\begin{aligned}
\|z - G_1x_n\| &\le (3-\gamma)\|z - G_1z\| + \frac{2-\gamma}{2}\|z - x_n\| \\
&\quad + \mu\left(2\|z - G_1z\| + \|z - G_1x_n\| + \|x_n - G_1z\| + 2\left\|G_1z - G_1^2z\right\|\right) \\
&\le (3-\gamma+4\mu)\|z - G_1z\| + (1-\gamma)\|z - x_n\| + \mu\|z - G_1x_n\| + \mu\|x_n - G_1z\|.
\end{aligned} \tag{3.2}$$

It follows from (3.2) that

$$\|z - G_1x_n\| \le \frac{3-\gamma+4\mu}{1-\mu}\|z - G_1z\| + \frac{1-\gamma}{1-\mu}\|z - x_n\| + \frac{\mu}{1-\mu}\|x_n - G_1z\| \tag{3.3}$$

Because $\{x_n\}$ is a bounded sequence, it follows from (3.3) that $\{G_1x_n\}$ is a bounded sequence. By similar argument we conclude that $\{G_2x_n\}$ is also a bounded sequence.

For each $n \in \mathbb{N}$ it follows from triangle inequality that

$$\|(1-\alpha_n)G_1x_n + \alpha_n G_2x_n - x_n\| \ge (1-\alpha_n)\|G_1x_n - x_n\| - \alpha_n\|x_n - G_2x_n\|. \tag{3.4}$$

Thus for each $n \in \mathbb{N}$, we obtain from (3.4) that

$$\|G_1x_n - x_n\| \le \frac{1}{1-\alpha_n}\|(1-\alpha_n)G_1x_n + \alpha_n G_2x_n - x_n\| + \frac{\alpha_n}{1-\alpha_n}\|x_n - G_2x_n\|. \tag{3.5}$$

Using (3.1), the assumption $a_n \to 0$ as $n \to \infty$, and boundedness of the sequence $\{\|x_n - G_2x_n\|\}$, it follows from (3.5) that

$$\lim_{n\to\infty}\|G_1x_n - x_n\| = 0. \tag{3.6}$$

To show that z is a fixed point of G_1 we utilize (ii) of Lemma 2.5.

Case 1. There exists a strictly increasing sequence $\{n_k\}_{k=1}^{\infty}$ of natural numbers such that

$$\frac{\gamma}{2}\left\|x_{n_k}-G_1x_{n_k}\right\| \leq \left\|x_{n_k}-z\right\| \;\; \forall k\in\mathbb{N}. \tag{3.7}$$

It follows from Lemma 2.5 (ii)(a) and (3.7) that

$$\begin{aligned}\left\|G_1z-G_1x_{n_k}\right\| &\leq (1-\frac{\gamma}{2})\left\|z-x_{n_k}\right\|+\mu\left(\left\|z-G_1x_{n_k}\right\|+\left\|x_{n_k}-G_1z\right\|\right)\\ &\leq (1-\frac{\gamma}{2}+\mu)\left\|z-x_{n_k}\right\|+2\mu\left\|x_{n_k}-G_1x_{n_k}\right\|+\mu\left\|G_1x_{n_k}-G_1z\right\|.\end{aligned} \tag{3.8}$$

We get from (3.8) that

$$\left\|G_1z-G_1x_{n_k}\right\| \leq \frac{(1-\frac{\gamma}{2}+\mu)}{1-\mu}\left\|z-x_{n_k}\right\|+\frac{2\mu}{1-\mu}\left\|x_{n_k}-G_1x_{n_k}\right\|. \tag{3.9}$$

It follows from (3.6), (3.9) and convergence of $\{x_n\}$ to z that

$$\lim_{k\to\infty}\left\|G_1x_{n_k}-G_1z\right\|=0. \tag{3.10}$$

For each $k\in\mathbb{N}$ we have

$$\left\|z-G_1z\right\| \leq \left\|z-x_{n_k}\right\|+\left\|x_{n_k}-G_1x_{n_k}\right\|+\left\|G_1x_{n_k}-G_1z\right\|. \tag{3.11}$$

Using (3.6) and convergence of $\{x_n\}$ to z, and letting $k\to\infty$ in (3.11), we get

$$G_1z=z. \tag{3.12}$$

Therefore, z is a fixed point of G_1.

Case 2. There exists a strictly increasing sequence $\{n_k\}_{k=1}^{\infty}$ of natural numbers such that

$$\frac{\gamma}{2}\left\|G_1x_{n_k}-G_1^2x_{n_k}\right\| \leq \left\|G_1x_{n_k}-z\right\| \;\; \forall k\in\mathbb{N}. \tag{3.13}$$

It follows from Lemma 2.5[(i) & (ii)(b)] and (3.13) that

$$\left\|G_1z-G_1^2x_{n_k}\right\| \leq (1-\frac{\gamma}{2})\left\|z-G_1x_{n_k}\right\|+\mu\left(\left\|G_1z-G_1x_{n_k}\right\|+\left\|z-G_1^2x_{n_k}\right\|\right) \tag{3.14}$$

$$\leq (1-\frac{\gamma}{2}+3\mu)\left\|x_{n_k}-G_1x_{n_k}\right\|+(1-\frac{\gamma}{2}+\mu)\left\|x_{n_k}-z\right\|+\mu\left\|G_1^2x_{n_k}-G_1z\right\|.$$

We get from (3.14) that

$$\left\|G_1z-G_1^2x_{n_k}\right\| \leq \frac{(1-\frac{\gamma}{2}+\mu)}{1-\mu}\left\|x_{n_k}-G_1x_{n_k}\right\|+\frac{(1-\frac{\gamma}{2}+\mu)}{1-\mu}\left\|x_{n_k}-z\right\|. \tag{3.15}$$

It follows from (3.6), (3.15) and convergence of $\{x_n\}$ to z that

$$\lim_{k\to\infty} \left\| G_1^2 x_{n_k} - G_1 z \right\| = 0. \tag{3.16}$$

For each $k \in \mathbb{N}$, it follows from repeated application of triangle inequality and Lemma 2.5(i) that

$$\begin{aligned} \|z - G_1 z\| &\leq \|z - x_{n_k}\| + \|x_{n_k} - G_1 x_{n_k}\| + \left\|G_1 x_{n_k} - G_1^2 x_{n_k}\right\| + \left\|G_1^2 x_{n_k} - G_1 z\right\| \\ &\leq \|z - x_{n_k}\| + 2\|x_{n_k} - G_1 x_{n_k}\| + \left\|G_1^2 x_{n_k} - G_1 z\right\|. \end{aligned} \tag{3.17}$$

Using (3.6), (3.16), and letting $k \to \infty$ in (3.17), we get

$$G_1 z = z.$$

Therefore, z is a fixed point of G_1.

We note that

$$(G_1 \circ G_2)z = (G_2 \circ G_1)z = G_2 z. \tag{3.18}$$

For each $n \in \mathbb{N}$ we have

$$\begin{aligned} &\|G_2 z - x_n\| \leq \|G_2 z - (1-\alpha_n)G_1 x_n - \alpha_n G_2) x_n\| + \|(1-\alpha_n)G_1 x_n + \alpha_n G_2) x_n - x_n\| \\ &\leq (1-\alpha_n)\|G_2 z - G_1 x_n\| + \alpha_n \|G_2 z - G_2 x_n\| + \|(1-\alpha_n)G_1 x_n + \alpha_n G_2) x_n - x_n\|. \end{aligned} \tag{3.19}$$

Because $\gamma \|G_2 z - G_1(G_2 z)\| = 0 \leq \|G_2 z - x_n\| + \mu \|x_n - G_1 x_n\|$, it follows from $B_{\gamma,\mu}$ condition that

$$\begin{aligned} \|G_2 z - G_1 x_n\| &= \|G_1(G_2 z) - G_1 x_n\| \\ &\leq (1-\gamma)\|G_2 z - x_n\| + \mu\left[\|G_2 z - G_1 x_n\| + \|x_n - G_2 z\|\right] \\ &\leq (1-\gamma+\mu)\|G_2 z - x_n\| + \mu\|G_2 z - G_1 x_n\|. \end{aligned} \tag{3.20}$$

It follows from (3.20) that

$$\|G_2 z - G_1 x_n\| \leq \left(\frac{1-\gamma+\mu}{1-\mu}\right)\|G_2 z - x_n\|. \tag{3.21}$$

We get from (3.19) and (3.21) that

$$\begin{aligned} &\left[1 - (1-\alpha_n)\left(\frac{1-\gamma+\mu}{1-\mu}\right)\right]\|G_2 z - x_n\| \\ &\quad \leq \alpha_n \|G_2 z - G_2 x_n\| + \|(1-\alpha_n)G_1 x_n + \alpha_n G_2) x_n - x_n\|. \end{aligned} \tag{3.22}$$

Since $\|G_2z - G_2x_n\|$ is a bounded sequence, letting $n \to \infty$ in (3.22) we get

$$\left(\frac{\gamma - 2\mu}{1-\mu}\right) \|G_2z - z\| \leq 0. \tag{3.23}$$

Because $\dfrac{\gamma - 2\mu}{1-\mu} > 0$, we have $G_2z = z$. Hence z is a common fixed point of G_1 and G_2.

Let K be a nonempty closed convex subset of a given Banach space B. Let $\mu \in [0, \frac{1}{2}]$ and $\gamma \in [0,1]$ such that $2\mu \leq \gamma$. Let $G_1, G_2 : K \to K$ be commutating mappings (that is; $G_1 \circ G_2 = G_2 \circ G_1$,) satisfying the condition $B_{\gamma,\mu}$. Let $\{\alpha_n\}_{n=0}^{\infty}$ be a sequence in $\left(0, \frac{1}{2}\right)$ and $\lambda \in (\gamma, 1)$. Let us define a sequence $\{x_n\}$ in K by the iteration

$$\begin{cases} x_0 \in K \\ y_n = (1-\alpha_n)G_1x_n + \alpha_n G_2 x_n \\ x_{n+1} = \lambda y_n + (1-\lambda)x_n, \;\; n = 0, 1, 2, \cdots \end{cases} \tag{3.24}$$

Lemma 3.2. *If $F = Fix(G_1) \cap Fix(G_2) \neq \emptyset$, then the sequence $\{x_n\}$ defined in (3.24) is bounded.*

Proof. Let $p \in F$. Then it follows from Lemma 2.6 that for each $x \in K$

$$\|G_1x - p\| \leq \|x - p\| \quad \& \quad \|G_2x - p\| \, \|x - p\|.$$

Thus for each $n = 0, 1, 2, \cdots$, we have

$$\begin{aligned} \|x_{n+1} - p\| &\leq \lambda \|y_n - p\| + (1-\lambda)\|x_n - p\| \\ &\leq \lambda[(1-\alpha_n)\|G_1x_n - p\| + \alpha_n \|G_2x_n - p\|] + (1-\lambda)\|x_n - p\| \\ &\leq \lambda[(1-\alpha_n)\|x_n - p\| + \alpha_n \|x_n - p\|] + (1-\lambda)\|x_n - p\| \\ &\leq \lambda \|x_n - p\| + (1-\lambda)\|x_n - p\| \\ &\leq \|x_n - p\|. \end{aligned}$$

Therefore, $\{\|x_n - p\|\}$ is a decreasing sequence, and so that $\{x_n\}$ is a bounded sequence.

Lemma 3.3. *Let B be a uniformly convex Banach space and K be a nonempty closed convex subset of B. Let G_1 and G_2 be mappings satisfying the condition $B_{\gamma,\mu}$ with $G_1 \circ G_2 = G_2 \circ G_1$ on K. Suppose that $F = Fix(G_1) \cap Fix(G_2) \neq \emptyset$. Let $\{\alpha_n\}$ be a sequence in $(0, \frac{1}{2})$ converging to 0, and $\lambda \in (\gamma, 1)$. Then for $x_0 \in K$ the sequence defined in (3.24) satisfies*

$$\lim_{n\to\infty} \|(1-\alpha_n)G_1x_n + \alpha_n G_2x_n - x_n\| = 0.$$

Proof. Let $p \in F$. Put $w_n = y_n - p$ and $z_n = x_n - p$. It follows from the proof of Lemma 3.2 that for some $\nu \geq 0$,

$$\lim_{n\to\infty} \|z_n\| = \lim_{n\to\infty} \|x_n - p\| = \nu. \tag{3.25}$$

Moreover, we note that

$$\begin{aligned} &\limsup_{n\to\infty} \|w_n\| = \limsup_{n\to\infty} \|y_n - p\| \leq \nu \qquad (3.26)\\ &\limsup_{n\to\infty} \|(1-\lambda)w_n + \lambda z_n\| = \lim_{n\to\infty} \|(1-\lambda)(y_n - p) + \lambda(x_n - p)\| \\ &= \lim_{n\to\infty} \|x_{n+1} - p\| = \nu. \end{aligned}$$

It follows from (3.25), (3.26), (**??**), and Lemma 2.2 that

$$\lim_{n\to\infty} \|w_n - z_n)\| = 0,$$

and so that

$$\begin{aligned} \lim_{n\to\infty} \|(1-\alpha_n)G_1x_n + \alpha_nG_2x_n - x_n\| &= \lim_{n\to\infty} \|(y_n - p) - (x_n - p)\| \\ &= \lim_{n\to\infty} \|w_n - z_n)\| = 0. \end{aligned} \tag{3.27}$$

For a nonempty compact convex subset K of a uniformly convex Banach space B, we have the following fixed point result.

Theorem 3.4. *Let K be a nonempty compact convex subset a uniformly convex Banach space B. Let G_1 and G_2 be commuting self-mappings satisfying the condition $B_{\gamma,\mu}$ with $F = Fix(G_1) \cap Fix(G_2) \neq \emptyset$. Let $\{\alpha_n\}$ be a sequence in $(0, \frac{1}{2})$ converging to 0, and $\lambda \in (\gamma, 1)$. For $x_0 \in K$, let $\{x_n\}$ be a sequence in K defined in (3.24). Then $\{x_n\}$ converges strongly to a common fixed point of G_1 and G_2.*

Proof. It follows from Lemma 3.2 that the sequence $\{x_n\}$ is bounded. By the compactness of K, there is a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ and there is some $p \in K$ such that

$$\lim_{j\to\infty} x_{n_j} = p.$$

By Lemma 3.3 we get

$$\lim_{n\to\infty} \left\|(1-\alpha_{n_j})G_1x_{n_j} + \alpha_nG_2x_{n_j} - x_{n_j}\right\| = 0.$$

It follows from Lemma 3.1 that p is a common fixed point of G_1 and G_2; that is, $p \in F$. We get from the proof of Lemma 3.2 that

$$\lim n \to \infty \|x_n - p\| = \lim_{j\to\infty} \left\|x_{n_j} - p\right\| = 0.$$

Therefore, $\{x_n\}$ converges strongly to a common fixed point p of G_1 and G_2.

Theorem 3.4 generalizes the following fixed point theorems of Suzuki [26] and Patir et al. [21].

Theorem 3.5. *(Suzuki [26]) Let G be a mapping on a compact convex subset K of a Banach space B. Assume that G satisfies condition (C). Define a sequence $\{x_n\}$ in K by $x_0 \in K$ and*

$$x_{n+1} = \lambda G x_n + (1-\lambda) x_n$$

for $n = 0, 1, 2, \cdots$, where λ is a real number belonging to $[\frac{1}{2}, 1)$. Then $\{x_n\}$ converges strongly to a fixed point of G.

Theorem 3.6. *(Patir et al. [21]) Let K be a compact and convex subset of a Banach space B. Let G be a self-mapping on K satisfying the condition $B_{\gamma,\mu}$. For $x_0 \in K$, let $\{x_n\}$ be a sequence in K defined as*

$$x_{n+1} = \lambda G x_n + (1-\lambda) x_n$$

for $n = 0, 1, 2, \cdots$, where λ is sufficiently small. Then $\{x_n\}$ converges strongly to a fixed point of G.

4. NUMERICAL EXAMPLES

Example 4.1. *On the subset $K = [0, 4]$ of the Banach space $\mathbb{R}$, define $G_1, G_2 : K \to K$ as*

$$G_1 x = \begin{cases} 0, & 0 \le x < 4, \\ 1, & x = 4 \end{cases} \tag{4.1}$$

$$G_2 x = \begin{cases} 0, & 0 \le x < 4, \\ 2, & x = 4 \end{cases} \tag{4.2}$$

Then it can be easily shown that, for $\gamma = \frac{3}{4}$ and $\mu = \frac{1}{4}$, the mappings G_1 and G_2 satisfy the condition $B_{\gamma,\mu}$ (see Patir et al. [21]). Moreover, $G_1 \circ G_2 = G_2 \circ G_1$; that is, G_1 and G_2 are commuting mappings. We note that $F = Fix(G_1) \cap Fix(G_2) = \{0\} \neq \emptyset$.

Now to implement our algorithm take $\lambda = \frac{4}{5}$ and $\alpha_n = \frac{1}{2^{n+1}}$ for $n = 0, 1, 2, 3, \cdots$ and

$x_0 = 3$. *Then we have*

$$
\begin{aligned}
x_1 &= \lambda\left[(1-\alpha_0)G_1(x_0) + \alpha_0 G_2(x_0)\right] + (1-\lambda)x_0 = \frac{3}{5^1} \\
x_2 &= \lambda\left[(1-\alpha_1)G_1(x_1) + \alpha_1 G_2(x_1)\right] + (1-\lambda)x_1 = \frac{3}{5^2} \\
x_3 &= \lambda\left[(1-\alpha_2)G_1(x_2) + \alpha_2 G_2(x_2)\right] + (1-\lambda)x_2 = \frac{3}{5^3} \\
&\vdots \\
x_{n+1} &= \lambda\left[(1-\alpha_n)G_1(x_n) + \alpha_n G_2(x_n)\right] + (1-\lambda)x_n = \frac{3}{5^{n+1}} \\
&\vdots
\end{aligned}
$$

Hence $\{x_n\}$ *converges to the common fixed point* $p = 0$ *in* K.

In case we start at $x_0 = 4$ *we have*

$$
\begin{aligned}
x_1 &= \lambda\left[(1-\alpha_0)G_1(x_0) + \alpha_0 G_2(x_0)\right] + (1-\lambda)x_0 = 2 = \frac{2}{5^{1-1}} \\
x_2 &= \lambda\left[(1-\alpha_1)G_1(x_1) + \alpha_1 G_2(x_1)\right] + (1-\lambda)x_1 = \frac{2}{5} = \frac{2}{5^{2-1}} \\
x_3 &= \lambda\left[(1-\alpha_2)G_1(x_2) + \alpha_2 G_2(x_2)\right] + (1-\lambda)x_2 = \frac{2}{5^{3-1}} \\
&\vdots \\
x_{n+1} &= \lambda\left[(1-\alpha_n)G_1(x_n) + \alpha_n G_2(x_n)\right] + (1-\lambda)x_n = \frac{2}{5^{n}} \\
&\vdots
\end{aligned}
$$

Hence $\{x_n\}$ *converges to the common fixed point* $p = 0$ *in* K.

Example 4.2. *(See [21]) Consider* $B = \ell^2_{\mathbb{R}}$ *and* $K \subseteq \ell^2$ *defined as*

$$K = \left\{\{\xi_i\} \in \ell^2_{\mathbb{R}} : |\xi_1| \leq \frac{1}{2},\ \ \xi_i = 0, \forall i \geq 2\right\} = \left\{(\xi_1, 0, 0, \cdots) : \xi_1 \in \mathbb{R},\ \ |\xi_1| \leq \frac{1}{2}\right\}$$

Define $G : K \longrightarrow K$ *by*

$$Gx = \left\{\xi_1^2, 0, 0, \cdots\right\},$$

where $x = (\xi_1, 0, 0, \cdots) = \{\xi_i\} \in K$.

Let $x = (\xi_1, 0, 0, \cdots)$ $y = (\eta_1, 0, 0, \cdots)$ *be in* K. *Then*

$$\|Gx - Gy\|_2 = \left|\xi_1^2 - \eta_1^2\right| = |\xi_1 - \eta_1|\,|\xi_1 + \eta_1| \leq |\xi_1 - \eta_1| = \|x - y\| \tag{4.3}$$

Thus we observe that G *is a nonexpansive self-mapping on* K; *and hence satisfies the condition* $B_{\gamma,\mu}$.

Let $\lambda = \frac{1}{2}$, *and let the initial point be* $x_0 = \{\frac{1}{3}, 0, 0, \cdots\}$. *Then it follows from the algorithm (*$G_1 = G_2 = G$*) we get*

$$\begin{aligned}
x_1 &= \lambda G x_0 + (1-\lambda)x_0 = \left(\frac{2}{3^2}, 0, 0, 0, \cdots\right)\\
x_2 &= \lambda G x_1 + (1-\lambda)x_1 = \left(\frac{11}{3^4}, 0, 0, 0, \cdots\right)\\
x_3 &= \lambda G x_2 + (1-\lambda)x_2 = \left(\frac{506}{3^8}, 0, 0, 0, \cdots\right)\\
&\vdots
\end{aligned}$$

Hence $\{x_n\}$ *converges to the fixed point* $p = (0, 0, 0, \cdots$ *in* K.

5. CONCLUSION

In this paper, we have generalized the approach of fixed point searching from [25], by taking the condition $B_{\gamma,\mu}$ and the process by properly using the two mappings into its structure. In other words, we brought together both the Suziki's strong nonexpansive mappings and the Patir et al. [21] condition $B_{\gamma,\mu}$ under the same iteration process. Under the resulted iteration process, we proved approximation of a common fixed point of two mappings satisfying the condition $B_{\gamma,\mu}$. Generalization of Suzuki [25] results for finite and infinite family of mappings satisfying the condition $B_{\gamma,\mu}$ can be investigated by interested mathematicians for the future.

Availability of data and materials
Not applicable.

Conflicts of interest
The authors declare that they have no conflict of competing interests.

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