

Vertex Covering Transversal Geodetic Number of a Graph

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Abstract

Let $G = (V, E)$ be a simple connected graph. A subset $S \subseteq V$ is called a geodetic set if every vertex of G lies on a shortest path between some pair of vertices $u, v \in S$. The geodetic number of G , denoted by $g(G)$, is the minimum cardinality among all geodetic sets of G . A subset $C \subseteq V(G)$ is called a vertex cover if every edge of G is incident to at least one vertex in C . A vertex cover is said to be minimum if $|C| < |C'|$ for any other vertex cover $C' \subseteq V(G)$ with $C' \neq C$.

In this paper, we introduce and study a new parameter, the vertex covering transversal geodetic number of a graph. We explore this parameter across various graph classes and analyze its structural and combinatorial properties in detail.

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1. INTRODUCTION

We consider a graph $G = (V, E)$, as a finite undirected connected graph without any loops or multiple edges throughout this research article. The number of vertices in G is called the order of G , denoted by n . The distance $d(u, v)$ between two vertices u and v in a connected graph G is the length of a shortest u - v path in G [1, 3].

For a vertex v of G , the *eccentricity* $e(v)$ is the distance between v and a vertex farthest from v . The minimum eccentricity among the vertices of G is called the *radius* and the maximum eccentricity is called the *diameter of G* and are denoted by $\text{rad } G$ and $\text{diam } G$ respectively [2, 3]. A vertex v is an extreme vertex of a graph G if the subgraph induced by its neighbours is complete.

A subset $C \subseteq V(G)$ is called a vertex covering set if every edge of G is incident with at least one vertex in C . Also C is said to be a minimum vertex covering set if $|C| < |C'|$ for any other vertex covering set $C' \subseteq V(G)$ with $C' \neq C$. The minimum cardinality of a vertex covering set is denoted by $\alpha_0(G)$. A vertex covering set of minimum cardinality is called as α_0 -set.

A set $S \subseteq V(G)$ is called a *geodetic set* if every vertex of G lies on a shortest $u-v$ path for some $u, v \in S$. The minimum cardinality among all geodetic sets is called *geodetic number* and is denoted by $g(G)$. A geodetic set of minimum cardinality is called as g -set.

Geodetic sets and Geodetic number are introduced and analyzed in [4, 5]. Vertex covering transversal domination number is introduced in [7] and Vertex covering transversal domination number of regular graphs are analyzed in [8]. The maximum independent sets (β_0 -sets) are the complements of the minimum vertex covering sets (α_0 -sets) in a graph. In this paper, we introduce the parameter *vertex covering transversal geodetic number* of a graph. We analyze it in some standard graphs and also its characteristics in detail.

The following existing theorems are referred whenever needed.

Theorem 1.1. [4] *Each extreme vertex of a graph G belongs to every geodetic set of G . In particular, each end vertex of G belongs to every geodetic set of G .*

Theorem 1.2. [4] *For any cycle C_n ($n \geq 3$),*

$$g(C_n) = \begin{cases} 3 & \text{if } n \text{ is odd} \\ 2 & \text{if } n \text{ is even} \end{cases}$$

Theorem 1.3. [4] *For a connected graph G , $g(G) = p$ if and only if $G = K_p$.*

Theorem 1.4. [4] *If $W_{1,n}$ is a wheel graph with $n \geq 3$, then*

$$g(W_{1,n}) = \begin{cases} 4 & \text{if } n = 3 \\ \lceil \frac{n}{2} \rceil & \text{if } n \geq 4 \end{cases}$$

2. VERTEX COVERING TRANSVERSAL GEODETIC NUMBER

Definition 2.1. Let $G=(V, E)$ be a simple connected graph with at least three vertices. A geodetic set $S \subseteq V(G)$ which intersects every minimum vertex covering set (α_0 -set) is called a vertex covering transversal geodetic set. The minimum cardinality of a vertex covering transversal geodetic set of G is called the vertex covering transversal geodetic number of G and is denoted by $g_{vct}(G)$.

Example 2.2. Consider the graph G shown in Figure 1.

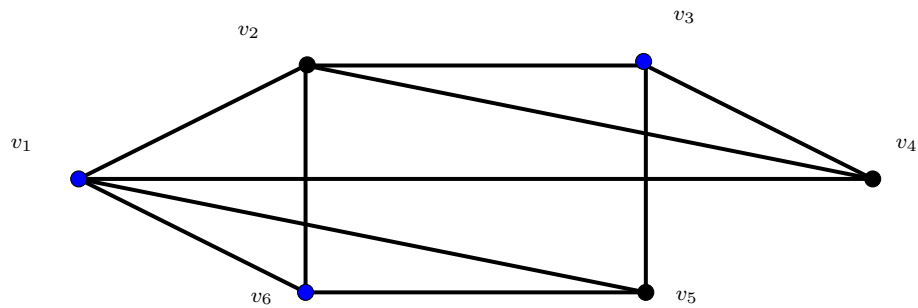


Figure 1: Graph G

For the graph G , $S_1 = \{v_1, v_2, v_3, v_5\}$, $S_2 = \{v_1, v_2, v_3, v_6\}$, $S_3 = \{v_1, v_2, v_4, v_5\}$, $S_4 = \{v_1, v_3, v_4, v_6\}$ and $S_5 = \{v_2, v_4, v_5, v_6\}$ are the minimum vertex covering sets (α_0 -sets).

Let $S = \{v_1, v_3, v_6\}$. Then S is a geodetic set that intersects S_1, S_2, S_3, S_4 and S_5 in G . Hence S is a vertex covering transversal geodetic set of minimum cardinality in G .

Therefore $g_{vct}(G) = 3$.

Remark 2.3. In example 2.2, the set $S = \{v_1, v_3, v_6\}$ is identified as the minimum geodetic set for the graph G shown in Figure 1. Therefore the geodetic number of G is $g(G) = 3$. Hence the geodetic number and the vertex covering transversal geodetic number are equal in this graph.

Remark 2.4. It is also observed that in example 2.2, the set $\{v_2, v_4, v_5\}$ is another vertex covering transversal geodetic set of minimum cardinality in G . It is noted that $\{v_2, v_4, v_5\}$ is the complement set of $S = \{v_1, v_3, v_6\}$ in this graph.

Example 2.5. Let G be the graph shown in Figure 2.

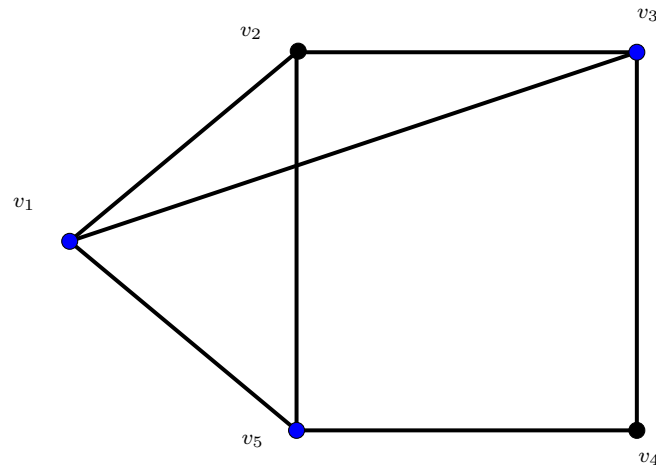


Figure 2: Graph G

Here $S_1 = \{v_1, v_2, v_4\}$, $S_2 = \{v_1, v_3, v_5\}$ and $S_3 = \{v_2, v_3, v_5\}$ are the minimum vertex covering sets (α_0 -sets) of G .

Then $S = \{v_1, v_3, v_5\}$ is a geodetic set which intersects every α_0 -set of G .

So S is a vertex covering transversal geodetic set of minimum cardinality in G .

Therefore $g_{vct}(G) = 3$.

Remark 2.6. It is noted that in example 2.5, the only minimum geodetic set for the graph G shown in Figure 2 is the set $S = \{v_3, v_5\}$ and thus $g(G) = 2$. Hence, the geodetic number and the vertex covering transversal geodetic number are not equal for this graph.

Example 2.7. Consider the graph G shown in Figure 3.

It is examined that $S_1 = \{v_1, v_2, v_4, v_5, v_7\}$, $S_2 = \{v_1, v_2, v_4, v_6, v_7\}$, $S_3 = \{v_1, v_3, v_4, v_6, v_8\}$, $S_4 = \{v_1, v_3, v_5, v_6, v_8\}$, $S_5 = \{v_2, v_3, v_5, v_6, v_8\}$ and $S_6 = \{v_2, v_4, v_5, v_7, v_8\}$ are the α_0 -sets of G .

Let $S = \{v_2, v_6\}$. Then S is a geodetic set that intersects every α_0 -set in G .

So S is a vertex covering transversal geodetic set of minimum cardinality in G .

Therefore $g_{vct}(G) = 2$.

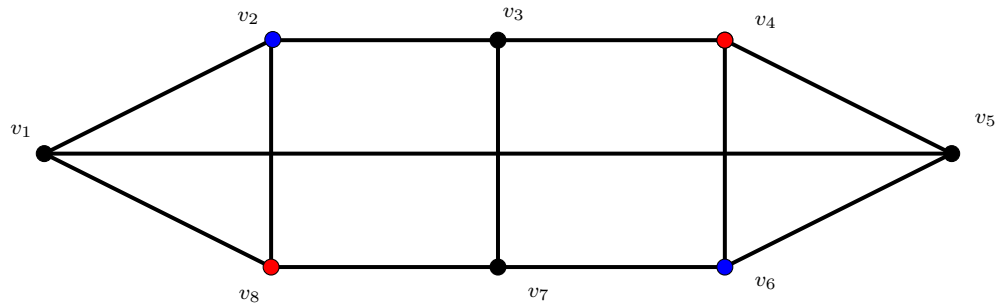
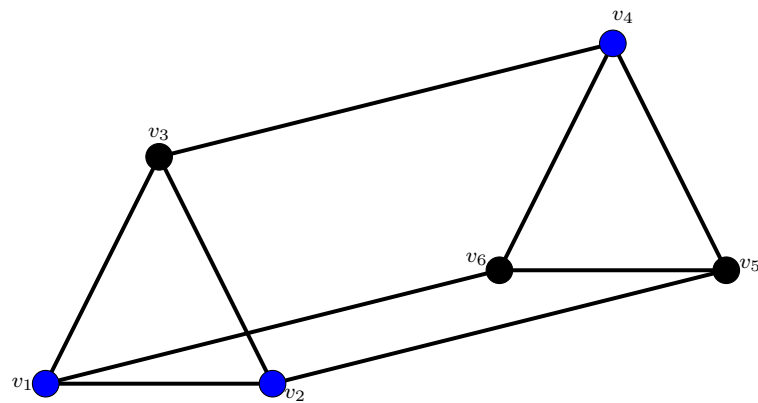


Figure 3: Graph G

Remark 2.8. It is observed that in example 2.7, $\{v_4, v_8\}$ is also a vertex covering transversal geodetic set of minimum cardinality in G .

3. VERTEX COVERING TRANSVERSAL GEODETIC NUMBER OF STANDARD GRAPHS

Example 3.1. Consider the Triangular prism graph G shown in Figure 4 which is 3-regular.

Figure 4: Triangular prism graph G

Here $C_1 = \{v_1, v_2, v_4, v_5\}$, $C_2 = \{v_1, v_2, v_4, v_6\}$, $C_3 = \{v_1, v_3, v_4, v_5\}$, $C_4 = \{v_1, v_3, v_5, v_6\}$, $C_5 = \{v_2, v_3, v_4, v_6\}$ and $C_6 = \{v_2, v_3, v_5, v_6\}$ are the α_0 -sets of G .

It is clear that $S = \{v_1, v_2, v_4\}$ is a geodetic set of minimum cardinality in G that also intersects every α_0 -set of G mentioned above.

So S is a vertex covering transversal geodetic set of minimum cardinality in G .

Therefore $g_{vct}(G) = 3$.

Remark 3.2. It is observed that in Example 3.1, $S^c = \{v_3, v_5, v_6\}$ is also a vertex covering transversal geodetic set of minimum cardinality in G .

Similarly, the set $\{v_1, v_3, v_5\}$ and its complement are also vertex covering transversal geodetic sets of minimum cardinality in G .

Likewise, the set $\{v_2, v_3, v_6\}$ and its complement are also vertex covering transversal geodetic sets of minimum cardinality in G .

Example 3.3. Consider Peterson graph G shown in Figure 5 which is 3-regular.

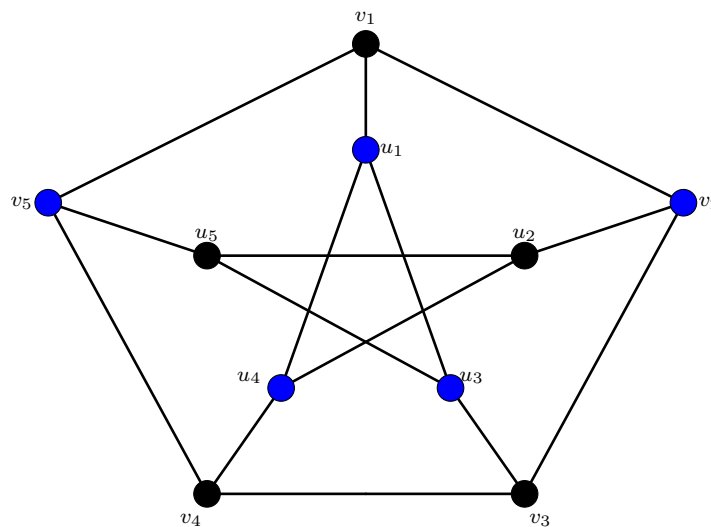


Figure 5: Peterson graph G

$C_1 = \{u_1, u_2, u_3, v_2, v_4, v_5\}$, $C_2 = \{u_1, u_2, u_5, v_1, v_3, v_4\}$, $C_3 = \{u_1, u_4, u_5, v_2, v_3, v_5\}$, $C_4 = \{u_2, u_3, u_4, v_1, v_3, v_5\}$ and $C_5 = \{u_3, u_4, u_5, v_1, v_2, v_4\}$ are the α_0 -sets of G .

It is clear that $S = \{u_1, u_3, u_4, v_2, v_5\}$ is a vertex covering transversal geodetic set of minimum cardinality in G .

Therefore $g_{vct}(G) = 5$.

Remark 3.4. It can be observed that the geodetic number of Peterson graph G is 4 as $\{u_3, u_4, v_2, v_5\}$ is a geodetic set of minimum cardinality in G .

Theorem 3.5. For the complete graph K_n on n vertices, $g_{vct}(K_n) = n$.

Proof: Since K_n is a complete graph on n vertices, $rad(K_n) = diam(K_n) = 1$. If a graph G is such that $rad(G) = diam(G) = 1$, then each vertex of G is joined to every other vertex by an edge. So any shortest path between two vertices in G is of length 1. Therefore the only geodetic set in G is the entire vertex set V .

Hence if $V(K_n) = \{v_1, v_2, \dots, v_n\}$, then this set itself is the unique geodetic set.

Also $V - \{v_i\}$, $i = 1, 2, \dots, n$ are the α_0 -sets of K_n . Obviously, each of these α_0 -sets are contained in V .

So V is the unique vertex covering transversal geodetic set in K_n .

Therefore $g_{vct}(K_n) = n$. ■

Result 3.6. If G is a star containing n vertices as shown in Figure 6, then $g_{vct}(G) = n$ since the entire vertex set $\{u, v_1, v_2, \dots, v_{n-1}\}$ is a geodetic set which intersects the unique α_0 -set $\{u\}$ in G .

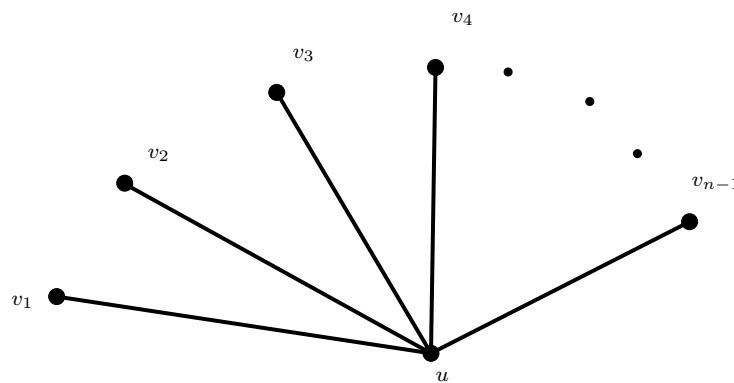


Figure 6: Graph G

Remark 3.7. It can be observed that the geodetic number of the star graph G mentioned in Result 3.6 is $n - 1$ as $\{v_1, v_2, \dots, v_{n-1}\}$ is the geodetic set of minimum cardinality in G .

Remark 3.8. It can be noted that the graph G shown in Figure 6 is the complete bipartite graph $K_{1,n-1}$ which is also known as a star.

Theorem 3.9. If $K_{m,n}$ is a complete bipartite graph with $m, n \geq 2$, then

$$g_{vct}(K_{m,n}) = \begin{cases} 2 & \text{if } m = 2 \text{ and } n > 2 \\ 3 & \text{if } m = 2, n = 2 \text{ and if } m = 3, n > 3 \\ 4 & \text{if } m, n \geq 3 \end{cases}$$

Proof: $K_{m,n}$ is a complete bipartite graph with the vertex set V partitioned into two disjoint subsets $V_1 = \{u_1, u_2, \dots, u_m\}$ and $V_2 = \{v_1, v_2, \dots, v_n\}$. Each vertex in V_1 is connected to every vertex in V_2 by an edge.

Case 1: $m = 2$ and $n > 2$

Then $V_1 = \{u_1, u_2\}$ and $V_2 = \{v_1, v_2, \dots, v_n\}$

Since $n > 2$, V_1 (of size 2) is the unique minimum vertex covering set of $K_{m,n}$.

Also V_1 itself is a geodetic set of minimum cardinality.

Therefore, V_1 is the unique vertex covering transversal geodetic set of size 2 which is also of minimum cardinality in $K_{m,n}$.

Hence $g_{vct}(K_{m,n}) = 2$ in this case.

Case 2: $m = 2, n = 2$ and $m = 3, n > 3$

For $m = 2, n = 2$, both V_1 and V_2 (each of size 2) are α_0 -sets of $K_{m,n}$.

Let $S = \{u_1, u_2, v_1\}$. Then S is a geodetic set that intersects both V_1 and V_2 .

So S is a vertex covering transversal geodetic set of minimum cardinality in this case.

Therefore $g_{vct}(K_{m,n}) = 3$.

For $m = 3, n > 3$, let $V_1 = \{u_1, u_2, u_3\}$ and $V_2 = \{v_1, v_2, \dots, v_n\}$

Since $n > 3$, V_1 (of size 3) is the unique α_0 -set of $K_{m,n}$.

Also V_1 itself is a geodetic set of minimum cardinality.

Therefore, V_1 is the unique vertex covering transversal geodetic set of minimum cardinality in $K_{m,n}$.

Hence $g_{vct}(K_{m,n}) = 3$ in this case.

Case 3: $m, n \geq 3$

If $m = n$, then both V_1 and V_2 are minimum vertex covering sets of size n in $K_{m,n}$.

If $m \neq n$, then the smaller of the two (either V_1 or V_2) is the unique minimum vertex covering set in $K_{m,n}$.

Now define $S = \{u_i, u_j, v_k, v_l\}$ for any $i, j = 1, 2, \dots, m$ and $k, l = 1, 2, \dots, n$ with $i \neq j$ and $k \neq l$.

Since S contains exactly 2 vertices from both V_1 and V_2 , it intersects both.

Obviously S is a geodetic set of minimum cardinality.

Thus S is a vertex covering transversal geodetic set of minimum cardinality 4 and so $g_{vct}(K_{m,n}) = 4$ in this case. ■

Remark 3.10. The geodetic number of a complete bipartite graph $K_{r,s}$ is $g(K_{r,s}) = \min \{4, r\}$ where $2 \leq r \leq s$ [6].

Theorem 3.11. If P_n ($n \geq 2$) is a path on n vertices, then $g_{vct}(P_n) = 3$.

Proof: Let $V(P_n) = \{v_1, v_2, \dots, v_n\}$.

If n is odd, the set $C = \{v_2, v_4, \dots, v_{n-1}\}$ is the unique α_0 -set in the path graph P_n .

If n is even, the sets $S_1 = \{v_1, v_3, \dots, v_{n-1}\}$, $S_2 = \{v_2, v_4, \dots, v_n\}$, $S_3 = \{v_2, v_3, v_5, v_7, \dots, v_{n-1}\}$ and $S_4 = \{v_2, v_4, \dots, v_{n-2}, v_{n-1}\}$ are the α_0 -sets in P_n .

Define $S = \{v_1, v_2, v_n\}$. Then S is a geodetic set in P_n which intersects every α_0 -set for odd as well as even n .

So S is a vertex covering transversal geodetic set of minimum cardinality in P_n .

$\therefore g_{vct}(P_n) = 3$.

■

Theorem 3.12. *If C_{2n+1} is an odd cycle with $n \geq 1$, then $g_{vct}(C_{2n+1}) = 3$.*

Proof: Let $V = \{1, 2, 3, \dots, 2n+1\}$ be the vertex set of the odd cycle C_{2n+1} .

For an odd cycle with $2n+1$ vertices, there are $2n+1$ α_0 -sets, each containing $n+1$ vertices. Denote these α_0 -sets by S_j ; $j = 1, 2, \dots, 2n+1$.

Now, define $T_i = \{i, i+1, i+n+1\}$, $i = 1, 2, \dots, n$. Then each T_i is a geodetic set of minimum cardinality in C_{2n+1} .

We claim that T_i intersects every α_0 -set S_j of C_{2n+1} where $j = 1, 2, \dots, 2n+1$.

Suppose that $T_i \cap S_j = \phi$ for some j .

Then the vertices $i, i+1, i+n+1$ do not belong to S_j .

So S_j contains $n+1$ vertices from the remaining $2n-2$ vertices in C_{2n+1} . Since S_j is a minimum vertex covering set, every edge of C_{2n+1} is incident with at least one vertex in S_j .

But the edge $(i, i+1)$ in C_{2n+1} is not incident with any vertex in S_j as both the vertices i and $i+1$ do not belong to S_j . This leads to a contradiction as S_j is a minimum vertex covering set of C_{2n+1} .

Hence, for $i = 1, 2, \dots, n$, $T_i \cap S_j \neq \phi$ for all $j = 1, 2, \dots, 2n+1$. Therefore T_i intersects every α_0 -set S_j of C_{2n+1} .

Thus each T_i is a vertex covering transversal geodetic set of minimum cardinality in C_{2n+1} .

$\therefore g_{vct}(C_{2n+1}) = 3$.

■

Remark 3.13. *Additionally, it is noted that $T_j = \{j, j+n, (j+n+1)(\text{mod}(2n+1))\}$, $j = 1, 2, \dots, n+1$ are also vertex covering transversal geodetic sets of minimum cardinality in C_{2n+1} .*

Theorem 3.14. *If C_{2n} is an even cycle with $n \geq 2$, then*

$$g_{vct}(C_{2n}) = \begin{cases} 2 & \text{if } n \text{ is odd} \\ 3 & \text{if } n \text{ is even} \end{cases}$$

Proof: Let $V(C_{2n}) = \{1, 2, 3, \dots, 2n\}$.

Obviously, the sets $S_1 = \{1, 3, 5, \dots, 2n-1\}$ and $S_2 = \{2, 4, 6, \dots, 2n\}$ are the only two α_0 -sets of C_{2n} .

Case 1: n is odd.

Define $T_i = \{i, i+n\}$, $i = 1, 2, \dots, n$.

Then each T_i is a geodetic set of minimum cardinality in C_{2n} .

Also if $i \in S_1$, then $i+n \in S_2$ and vice versa.

This implies that each T_i intersects both S_1 and S_2 .

Hence each T_i is a vertex covering transversal geodetic set of minimum cardinality in C_{2n} .

Therefore $g_{vct}(C_{2n}) = 2$ if n is odd.

Case 2: n is even.

Define $W_i = \{i, i+1, i+n\}$, $i = 1, 2, \dots, n$.

Then each W_i is a geodetic set in C_{2n} .

If $i \in S_1$, then $i+1 \in S_2$ and vice versa.

Thus each W_i intersects both S_1 and S_2 .

Hence each W_i is a vertex covering transversal geodetic set of minimum cardinality in C_{2n} .

Therefore $g_{vct}(C_{2n}) = 3$ if n is even. ■

Remark 3.15. It is observed that the sets $W_j = \{j, j+n, (j+n+1)(\text{mod}(2n))\}$, $j = 1, 2, \dots, n$ can also be identified as vertex covering transversal geodetic sets of minimum cardinality in C_{2n} when n is even.

Theorem 3.16. If $W_{1,n}$ is a wheel graph with $n \geq 3$, then

$$g_{vct}(W_{1,n}) = \begin{cases} 4 & \text{if } n = 3 \\ \lceil \frac{n}{2} \rceil & \text{if } n \text{ is odd} \\ \frac{n}{2} + 1 & \text{if } n \text{ is even} \end{cases}$$

Proof: Let $V(W_{1,n}) = \{u, v_1, v_2, v_3, \dots, v_n\}$.

Case 1: $n = 3$.

Here $S_1 = \{v_1, v_2, v_3\}$, $S_2 = \{v_1, v_2, u\}$, $S_3 = \{v_1, v_3, u\}$ and $S_4 = \{v_2, v_3, u\}$ are the α_0 -sets in $W_{1,3}$.

Then it is obvious that entire vertex $\{u, v_1, v_2, v_3\}$ set of $W_{1,3}$ is the unique geodetic set which intersects every α_0 -set. And so it is a vertex covering transversal geodetic set of minimum cardinality in $W_{1,3}$.

Therefore $g_{vct}(W_{1,n}) = 4$ if $n = 3$.

The wheel graph $W_{1,3}$ is as shown in the following Figure 7.

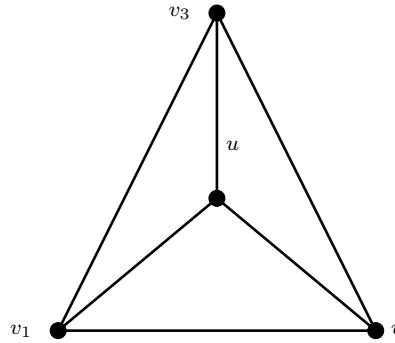


Figure 7: Wheel Graph $W_{1,3}$

Case 2: n is odd.

For odd n , the wheel $W_{1,n}$ with $n + 1$ vertices contains n α_0 -sets, each containing $\frac{n+1}{2} + 1$ vertices including the centre vertex u in each set. Denote these α_0 -sets by S_j ; $j = 1, 2, \dots, n$.

Now define $S = \{v_1, v_3, v_5, \dots, v_n\}$. Then S is a g -set in $W_{1,n}$.

We claim that S intersects every α_0 -set S_j of $W_{1,n}$ where $j = 1, 2, \dots, n$.

Suppose, for contradiction assume that $S \cap S_j = \phi$ for some j . Then all the $\frac{n+1}{2}$ vertices $v_1, v_3, v_5, \dots, v_n$ in S do not belong to S_j . That is, $S_j \subseteq S^c = \{u, v_2, v_4, v_6, \dots, v_{n-1}\}$. Of course, S^c contains $\frac{n+1}{2}$ vertices only. Therefore $|S_j| \leq \frac{n+1}{2}$ which is a contradiction since any α_0 -set of $W_{1,n}$ contains exactly $\frac{n+1}{2} + 1$ vertices.

$\therefore S \cap S_j \neq \phi$ for all j .

If $n = 7$, the wheel graph $W_{1,7}$ is as shown in the following Figure 8.

Hence S itself is a vertex covering transversal geodetic set of minimum cardinality in $W_{1,n}$ when n is odd. So $g_{vct}(W_{1,n}) = \lceil \frac{n}{2} \rceil$ in this case.

Case 2: n is even.

For even n , the wheel $W_{1,n}$ with $n + 1$ vertices contains only 2 α_0 -sets namely, $S_1 = \{u, v_1, v_3, v_5, \dots, v_{n-1}\}$ and $S_2 = \{u, v_2, v_4, v_6, \dots, v_n\}$.

When $n = 8$, the wheel graph $W_{1,8}$ is as shown in the following Figure 9.

It is obvious that the sets $\{v_2, v_4, v_6, \dots, v_n\}$ and $\{v_1, v_3, v_5, \dots, v_{n-1}\}$ are the only two g -sets in $W_{1,n}$ when n is even.

But these sets do not intersect S_1 and S_2 and so they are not vertex covering transversal

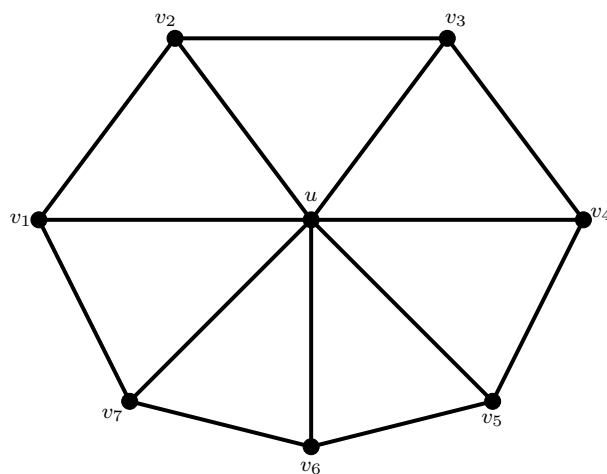


Figure 8: Wheel Graph: $W_{1,7}$

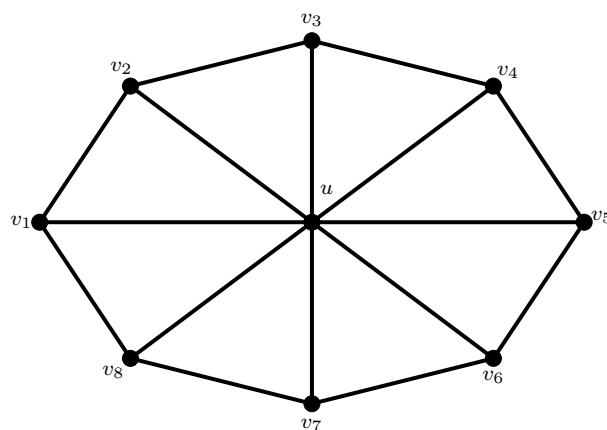


Figure 9: Wheel Graph: $W_{1,8}$

geodetic sets.

Hence define $S = \{v_1, v_2, v_4, v_6, \dots, v_n\}$. Then S intersects both S_1 and S_2 . So it is a vertex covering transversal geodetic set of minimum cardinality.

Thus $g_{vct}(W_{1,n}) = \frac{n}{2} + 1$ in this case.

■

Definition 3.17. *Hypercube*

For $n \geq 2$, the hypercube or n -dimensional cube Q_n is defined as the graph containing 2^n vertices whose vertex set is the set of ordered n -tuples of 0's and 1's in which two vertices are adjacent if their ordered n -tuples differ in exactly one position.[7]

Theorem 3.18. *If Q_n is a hypercube on n vertices with $n \geq 3$, then*

$$g_{vct}(Q_n) = \begin{cases} 2 & \text{if } n \text{ is odd} \\ 3 & \text{if } n \text{ is even} \end{cases}$$

Proof: The hypercube Q_n contains 2^n vertices and is n -regular. Each vertex in Q_n is represented by an n -tuple consisting of 0's and 1's. Two vertices in Q_n are adjacent if and only if their corresponding n -tuples differ in exactly one position.

For any $v \in Q_n$, let v^c denote its complement, obtained by replacing every 0 in v with a 1, and every 1 with a 0.

The weight of a vertex is defined as the number of 1's in its binary representation. There are exactly 2^{n-1} vertices of odd weight and 2^{n-1} vertices of even weight. Furthermore, each edge in Q_n connects one vertex of even weight with one vertex of odd weight.

Therefore, the set of all vertices of even weight forms a vertex covering set, and so does the set of all vertices of odd weight. As a result, Q_n is bipartite with bipartition sets $S_1 = \{\text{vertices of even weight}\}$ and $S_2 = \{\text{vertices of odd weight}\}$.

Moreover, S_1 and S_2 are the only two α_0 -sets of Q_n .

Case 1: n is odd.

Let $S = \{u, u^c\}$ where u is any vertex in Q_n .

For odd n , if $u \in I_1$, then $u^c \in I_2$ and vice-versa.

So S intersects both I_1 and I_2 .

Also, all vertices of Q_n lie on some geodesic between u and u^c , and so $S = \{u, u^c\}$ is a geodetic set.

Thus S is a vertex covering transversal geodetic set of minimum cardinality.

Hence $g_{vct}(Q_n) = 2$ if n is odd.

Case 2: n is even.

For any $u \in S_1$, the complement of u viz. $u^c \in S_1$ itself as n is even. Similarly, for any $u \in S_2$, it's obvious that $u^c \in S_2$.

So define $S = \{u, v, u^c\}$ where u and v are any two adjacent vertices in Q_n so that if $u \in S_1$, then $v \in S_2$ and vice versa.

It is clear that every vertex in Q_n lies on a shortest path between some pair of vertices in S . So S is a geodetic set which intersects both S_1 and S_2 . Thus S is a vertex covering transversal geodetic set of minimum cardinality in this case.

Hence $g_{vct}(Q_n) = 3$ if n is even. ■

4. BOUNDS OF g_{vct} AND EXISTENCE THEOREM ON g_{vct}

Theorem 4.1. *If $G = (V, E)$ is a simple connected graph on n vertices, then vertex covering transversal geodetic number of G satisfies the inequality $2 \leq g_{vct}(G) \leq n$.*

The following Theorem follows immediately from Theorem 1.1.

Theorem 4.2. *Let G be a simple connected graph. Then every extreme vertex of G belongs to every vertex covering transversal geodetic set of G . In particular, every end vertex of G is contained in every vertex covering transversal geodetic set of G .*

Theorem 4.3. *For positive integers r , d and $k \geq 2$ with $r \leq d \leq 2r$, there exists a connected graph G with $\text{rad } G = r$, $\text{diam } G = d$ and $g_{vct}(G) = k$.*

Proof: Suppose that $r = 1$. Then there are two possibilities: $d = 1$ or $d = 2$.

If $d = 1$, let us take $G = K_k$. Then by Theorem 3.5, we have $g_{vct}(G) = k$.

If $d = 2$, let us assume $G = K_{1,k-1}$. By Result 3.6, we have $g_{vct}(G) = k$.

Now suppose that $r \geq 2$. Then there are two cases $r = d$ and $r < d$.

Case 1: $r = d$.

For $k = 2, 3$, $G = C_{2r}$ or C_{2r+1} exhibit the desired properties for any value of $r \geq 2$.

Now let $r = 2$.

For $k = 4$, let $G = K_{m,n}$ with $m, n \geq 3$. Then $r = d = 2$ and $g_{vct}(G) = 4 = k$.

For $k \geq 5$, construct the graph G by starting with the 4-cycle $C_4 : x, u, y, v, x$ and adding $k - 2$ new vertices $x_1, x_2, \dots, x_{k-2}$, each adjacent to both x & u .

Then the α_0 -sets in G are $S_1 = \{x, u, v\}$ and $S_2 = \{x, u, y\}$.

Now $S = \{x_1, x_2, \dots, x_{k-2}, u, v\}$ is a geodetic set intersecting both S_1 and S_2 . Hence S forms a vertex covering transversal geodetic set of minimum cardinality in G .

So $g_{vct}(G) = k$.

The graph G is shown in the following Figure 10 for $k = 5$.

It is noted that $\{x_1, x_2, \dots, x_{k-2}, x, v\}$, $\{x_1, x_2, \dots, x_{k-2}, x, y\}$ and $\{u_1, u_2, \dots, u_{k-2}, y, v\}$ can also be identified as vertex covering transversal geodetic sets of minimum cardinality in G .

Now let $r = 3$. Construct the graph G as follows:

Begin with the 6-cycle $C_6 : x, u, y, v, z, w, x$. Then perform the following steps:

(1) Add $k - 3$ new vertices $u_1, u_2, \dots, u_{k-3}$, each joined to both x & y

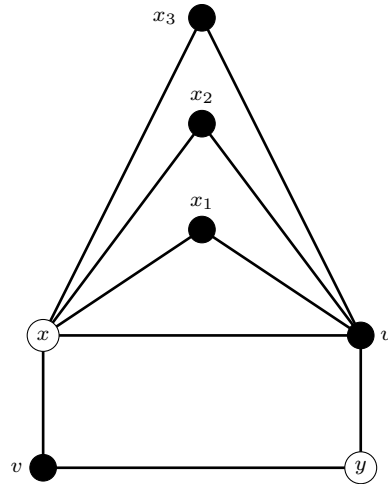


Figure 10: Graph G with $r = d = 2$ and $g_{vct}(G) = 5$

(2) Add $k - 3$ new vertices $w_1, w_2, \dots, w_{k-3}$ each joined to both to x & z

(3) Add a final set of $k - 3$ new vertices $x_1, x_2, \dots, x_{k-3}$ where x_i is joined to u_i and w_i for $i = 1, 2, \dots, k - 3$.

The graph G thus obtained is shown in Figure 11 for $k = 5$.

Let $S_1 = \{u, u_1, u_2, \dots, u_{k-3}\}$, $S_2 = \{v\}$ and $S_3 = \{w, w_1, w_2, \dots, w_{k-3}\}$

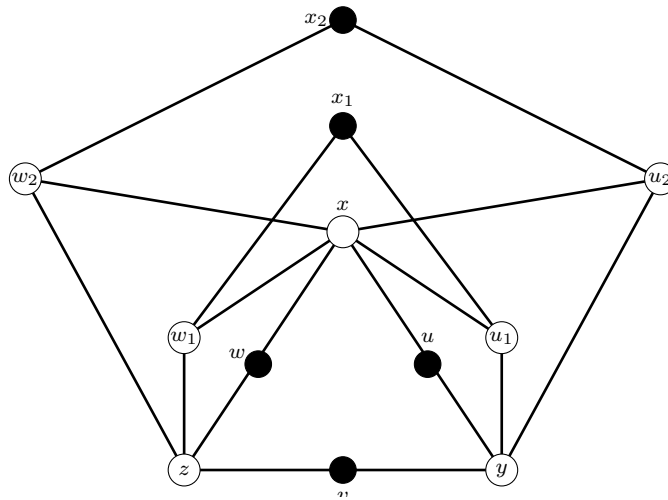


Figure 11: Graph G with $r = d = 3$ and $g_{vct}(G) = 5$

Then $I = S_1 \cup S_2 \cup S_3$ is the unique β_0 -set in G.

Hence its complement $S = \{x, y, z, x_1, x_2, \dots, x_{k-3}\}$ is the unique α_0 -set in G.

It is obvious that $T = \{x_1, x_2, \dots, x_{k-3}, u, v, w\}$ is a minimum geodetic set of G. Also T intersects the unique α_0 -set S . So T is a vertex covering transversal geodetic set of minimum cardinality.

Hence $g_{vct}(G) = k$.

Next, consider the case $r = 4$.

For each i with $1 \leq i \leq k - 4$, let $F_i = \{u_{i1}, u_{i2}\}$ and $H_i = \{w_{i1}, w_{i2}\}$ be two copies of the path P_2 .

Construct the graph G as follows:

Start with the 8-cycle $C_8: v_1, v_2, \dots, v_8, v_1$. Then perform the following additions:

- (1) For each $i = 1, 2, \dots, k - 4$, join u_{i1} to v_2 and u_{i2} to v_4
- (2) For each $i = 1, 2, \dots, k - 4$, join w_{i1} to v_6 and w_{i2} to v_8
- (3) Then add $k - 4$ new vertices $x_1, x_2, \dots, x_{k-4}$ and join x_i with u_{i1} & w_{i1} for each $i = 1, 2, \dots, k - 4$.

The graph G thus obtained is shown in Figure 12 for $r = 4$ & $k = 6$.

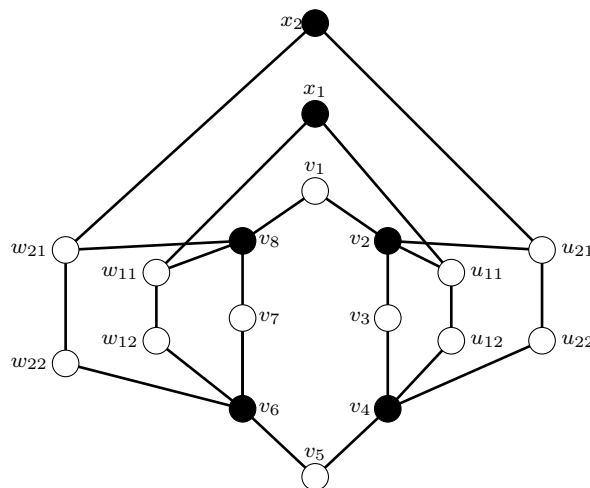


Figure 12: Graph G with $r = d = 4$ and $g_{vct}(G) = 6$

It is obvious that $I = \{u_{12}, u_{22}, \dots, u_{(k-4)2}\} \cup \{w_{12}, w_{22}, \dots, w_{(k-4)2}\} \cup \{x_1, x_2, \dots, x_{k-4}\} \cup \{v_1, v_3, v_5, v_7\}$ is the unique β_0 -set of G .

Then its complement $S = \{v_2, v_4, v_6, v_8, u_{11}, u_{21}, \dots, u_{(k-4)1}, w_{11}, w_{21}, \dots, w_{(k-4)1}\}$ is the unique α_0 -set in G .

Define $T = \{v_2, v_4, v_6, v_8\} \cup \{x_1, x_2, \dots, x_{k-4}\}$. Then it is a geodetic set intersecting the unique α_0 -set S .

Therefore T is a vertex covering transversal geodetic set of minimum cardinality k and so $g_{vct}(G) = k$.

When $r \geq 5$, there are two sub cases.

Subcase 1.1: $k = 2p + 1 \geq 3$ is odd.

For each i with $1 \leq i \leq 2p - 2$,

let $z_{i1}, z_{i2}, \dots, z_{i(2r-5)}$ form a path isomorphic to P_{2r-5} .

We construct a graph G starting with the even cycle C_{2r} : $v_1, v_2, \dots, v_{2r}, v_1$ and joining z_{i1} to v_{2r} & $z_{i(2r-5)}$ to v_2 for each $i = 1, 2, \dots, 2p - 2$.

Also $I = \{z_{i1}, z_{i3}, \dots, z_{i(2r-5)}; 1 \leq i \leq 2p - 2\} \cup \{v_1, v_3, \dots, v_{2r-1}\}$ is the unique β_0 -set of G . Then its complement $S = \{z_{i2}, z_{i4}, \dots, z_{i(2r-6)}; 1 \leq i \leq 2p - 2\} \cup \{v_2, v_4, \dots, v_{2r}\}$ is the unique α_0 -set of G .

Define $T = \{z_{i(\lfloor \frac{r}{2} \rfloor + 1)}; (1 \leq i \leq 2p - 2)\} \cup \{v_2, v_{r+1}, v_{2r}\}$. Then T is a geodetic set of minimum cardinality that intersects unique α_0 -set S and thus is a vertex covering transversal geodetic set. Therefore, $g_{vct}(G) = 2p - 2 + 3 = 2p + 1 = k$.

The graph G is shown in Figure 13 for $r = d = 6$ & $k = 7$.

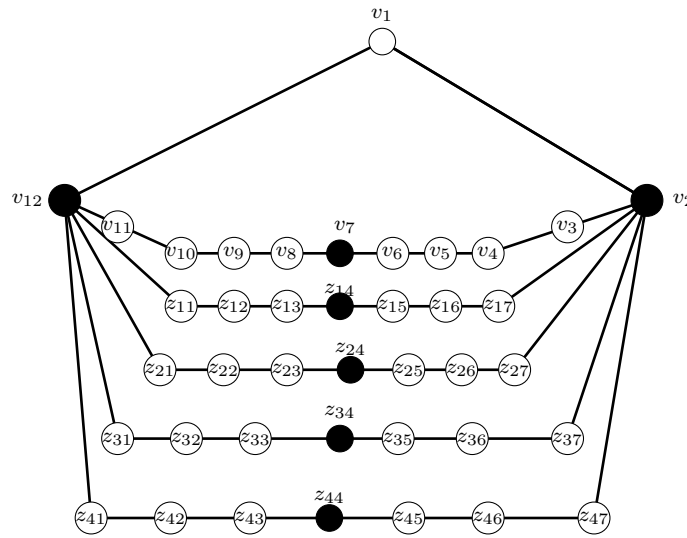


Figure 13: Graph G with $r = d = 6$ and $g_{vct}(G) = 7$

Subcase 1.2: $k = 2p + 2 \geq 4$ is even.

Now, construct the graph G by extending the graph from Subcase 1.1 (for $k=2p+1$) by adding a new vertex u and joining it to the vertices v_2 and v_{2r} .

Now, $I = \{z_{i1}, z_{i3}, \dots, z_{i(2r-5)}; 1 \leq i \leq 2p - 1\} \cup \{u, v_1, v_3, \dots, v_{2r-1}\}$ is the unique β_0 -set of G . Then its complement $S = \{z_{i2}, z_{i4}, \dots, z_{i(2r-5)}; 1 \leq i \leq 2p - 1\} \cup \{v_2, v_4, \dots, v_{2r}\}$ is the unique α_0 -set of G .

Define $T = \{z_{i(\lfloor \frac{r}{2} \rfloor + 1)}; (1 \leq i \leq 2p - 1)\} \cup \{v_2, v_{r+1}, v_{2r}\}$. Then T is a geodetic set of minimum cardinality that intersects S and thus is a vertex covering transversal geodetic set. Therefore, $g_{vct}(G) = 2p - 1 + 3 = 2p + 2 = k$.

The graph G is shown in the Figure 14 for $r = d = 5$ and $k = 6$.

Case 2: $r < d$.

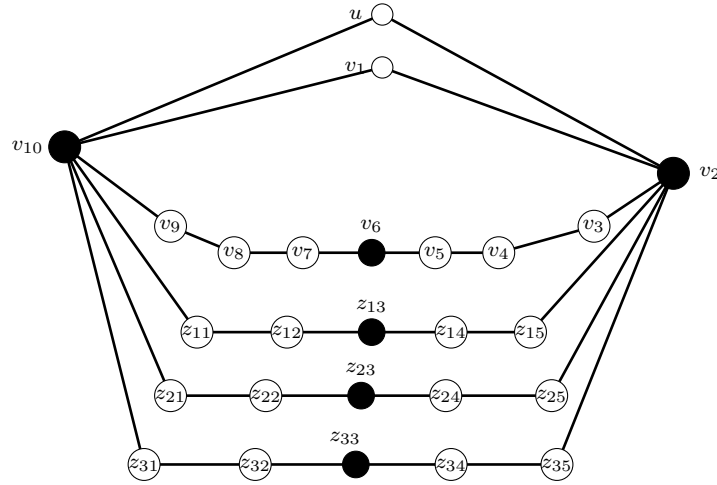


Figure 14: Graph G with $r = d = 5$ and $g_{vct}(G) = 6$

Let C_{2r} be a cycle with vertices $v_1, v_2, \dots, v_{2r}$, where the edges form a closed loop: $C_{2r} = (v_1, v_2, \dots, v_{2r}, v_1)$.

Let P_{d-r+1} be a path with vertices: $u_0, u_1, u_2, \dots, u_{d-r}$ where the edges connect consecutive vertices: $P_{d-r+1} = u_0, u_1, u_2, \dots, u_{d-r}$.

Then identify vertex v_1 from the cycle with vertex u_0 from the path, effectively merging the two graphs at this common vertex.

Finally, create the graph G by adding $k-3$ new vertices $w_1, w_2, \dots, w_{k-3}$ and connecting each of them to vertex u_{d-r-1} in the path.

Now $\text{rad. } G = r$ and $\text{diam. } G = d$.

The graph G thus obtained is as shown in Figure 15.

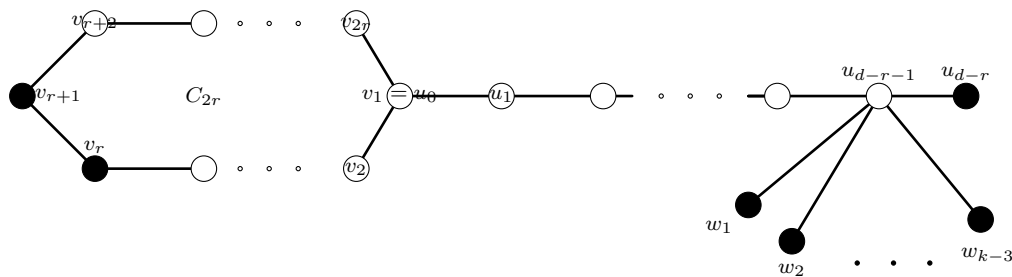


Figure 15: Graph G with $\text{rad. } G = r$, $\text{diam. } G = d$ and $g_{vct}(G) = k$

Let $S = \{u_{d-r}, w_1, w_2, \dots, w_{k-3}\}$ be a set consisting of $k-2$ end vertices of the graph

G. By Theorem 1.1, it follows that S is necessarily a subset of every geodetic set in G . Moreover, S is a subset of every maximum independent set in G .

So $S \subseteq I$ for every β_0 -set I .

This shows that S is not contained in any α_0 -set of G .

However, in C_{2r} , the sets $S_1 = \{v_1, v_3, v_5, \dots, v_{2r-1}\}$ and $S_2 = \{v_2, v_4, v_6, \dots, v_{2r}\}$ are the only two α_0 -sets. So these sets are necessarily the subsets of any α_0 -set in G .

Consider the set $S' = S \cup \{v_r, v_{r+1}\}$. This set forms a geodetic set.

Also $v_r \in S_1$ and $v_{r+1} \in S_2$ if r is odd and $v_r \in S_2$ and $v_{r+1} \in S_1$ if r is even. Thus S' intersects every α_0 -set of G , making it a vertex covering transversal geodetic set of minimum cardinality.

Hence $g_{vct}(G) = k$.

■

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