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Revisit: Gravitational Lensing in Brane World Black Hole

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Abstract

The bending of light rays in both weak and strong field approximations on brane world black hole has been investigated. We shall implement the Rindler-Ishak and Bozza's method in the more general enviroment of the solution given by Dadhich et. al. The solution resembles the Reissner Nördstrom solution but there is no electric field, instead we encounter a tidal charge parameter q arising via gravitational effects from the fifth dimension. Error has been corrected upto $O(M)$. Signatures of the extra dimension, which appears in the bending formulas are discussed. The lensing observables and the lensing coefficients are calculated and it is compared with other spherically symmetric solutions in General Relativity (GR) having same mass. It is found that brane world black hole have significantly different signatures than GR black holes.
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1. INTRODUCTION

Since few decades, physical consequences of extra space-like dimensions has attracted an intensive theoretical investigations. The existence of extra dimensions in our universe is an old idea formulated by Kaluza and Klien [1]. After their contribution, a large number of researchers developed this old concept [2] over the years. The important factor behind studying the scenario of extra dimension was to solve the problem so called "Hierarchy problem".

ADD [3] proposed a model where the extra dimensions $(4+d)$ were the usual compact ones. But later it was found to have some disadvantages. Subsequently, Randall and Sundrum (RS) [4] came up with a new idea where the extra dimension ”d” may be large and non-compact. In this RS type, gravity can be localized on a single 3-brane for $d = 1,$ even when the fifth (extra) dimension is infinite. The non compact localization arises via the exponential ”warp factor” in the non-factorizable geometry. Apart from the non compactness, the four dimensional slice may have a conformal dependence on the fifth (extra) dimension. In this sense, the brane world cosmological model says that the world we are living is a subspace (called brane world) of the $4 + d$ dimensional space-time (called bulk), where only gravity can propagate. Although this model has not yet been ruled out by experiments.

Discoveries of black holes at the centre of the galaxies, including our own galaxy, has grown an interest in studying the black holes as gravitational lens. In General Relativity, Gravitational Lensing the first application ever studied, is the only possible way to study black holes. The full theory of weak field approximation of Gravitational Lensing, where the deflection angle was supposed to be very small, has achieved a great success in explaining the physical observations. However the Scientific community was not fully satisfied by the scheme of weak field limit as it fails to distinguish between various different solutions that are asymptotically flat. So they started looking for a strong field perspective where the light rays passes close to the photon sphere giving a large deviation. This strong deflection limit consisting in a logarithmic approximation was introduced for Schwarzschild space-time by Bozza et. al. [5]. It was rediscovered several times [6] and finally Bozza et. al. [7] came out with an analytical formula for deflection angle and was generalized to any static, spherically symmetric spacetimes. After his general formulation in strong limit, lots of studies can be seen in strong lensing history. E.F. Eiroa et. al. studied the strong field limit taking Riessner Nördstrom black hole as lens [8]. A. Bhadra [9] applied this analytic technique to GMGHS charged black hole of string theory. Strong deflection limit of Kerr Black Hole has also been studied [10]. Not only the black holes, Nandi et. al. [11] extended the strong deflection limit to traversable Lorentzian wormholes.

Lensing of brane world black holes have also been studied in both weak and strong field limit. Kar and Sinha [12] investigated the weak field limit of bending of light rays on brane geometries and discussed about the signatures found in the bending formula. Rotating and non-rotating brane world black holes were analyzed as gravitational lenses [13-15]. All these investigations indeed provide us a huge quantity of informations about the signatures of those abstract astrophysical objects.

The main purpose of this paper is to study the brane black hole as gravitational lens in both weak field and strong field regime and examine how the four dimensional bending of light formula for on-brane carry the signature of extra dimension. The paper is organized as follows. In sec. 2 we introduce the metric of brane black hole. In sec. 3 we find the deflection angle in weak field limit using Rindler Ishak method. In sec. 4 we outline Bozza's method for strong deflection limit and apply it to Brane world black hole. We conclude in sec. 5. Throughout the paper we take units such that $G = c = 1$ unless specifically restored.

2. BRANE WORLD BLACK HOLE AS GRAVITATIONAL LENS

To check how does the extra dimensions and the warp factor effect our four dimensional spacetime, we investigate the bending angle of light trajectories on weak and strong field approximation in the four dimensional brane black hole. To do so, we consider the black hole solution of the Einstein equation given by Dadich et. al. [16]. But before introducing the metric let us briefly discuss about the line element in the brane world scenario.

The effective Einstein equation on the brane is given by,

$$G_{\mu\nu} = -\Lambda g_{\mu\nu} + \kappa^2 T_{\mu\nu} + \widetilde{\kappa}^4 S_{\mu\nu} - \varepsilon_{\mu\nu} \tag{2.1}$$

where tildes denotes bulk quantities and $\kappa^2 = \frac{8\pi}{M_p^2}$. The first two terms in the above expression for $G_{\mu\nu}$ gives the standard 4D Einstein equation. The third term $S_{\mu\nu}$ involve the square of the brane energy momentum tensor $T_{\mu\nu}$ and the last term $\varepsilon_{\mu\nu}$ is the limit on the brane of the projected bulk Weyl tensor $\varepsilon_{AB} = \widetilde{C}_{ABCD} n^C n^D$ [17].

The relation between energy scales to each other and with cosmological constant are as,

$$M_p = \sqrt{\frac{3}{4\pi}} \left(\frac{\widetilde{M_p^2}}{\sqrt{\lambda}} \right) \widetilde{M_p}; \quad \Lambda = \frac{4\pi}{\widetilde{M_p^3}} \left[\widetilde{\Lambda} + \left(\frac{4\pi}{3\widetilde{M_p^3}} \right) \lambda^2 \right]. \tag{2.2}$$

The fundamental 5-dimensional Planck mass $\widetilde{M_p}$ enters via $\widetilde{\kappa} = \frac{8\pi}{\widetilde{M_p^3}}$. λ is the brane tension, $\widetilde{\Lambda}$ is the bulk cosmological constant. Typically, the fundamental Plank scale is much smaller than the effective scale in the brane world $\left(\widetilde{M_p} << M_p \right)$.

Here we introduce one of the on brane black hole solution of the Einstein equation (2.1) obtained by Dadhich et. al. [16]. It is given as,

$$ds^2 = -A(r)dt^2 + \frac{dr^2}{A(r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{2.3}$$

where

$$A(r) = 1 - \frac{2M}{M_p^2}\frac{1}{r} + \frac{q}{\widetilde{M_p^2}}\frac{1}{r^2} \tag{2.4}$$

and $q = Q\widetilde{M_p^2}$ is a dimensionless tidal charge parameter. This solution has the form of the Reissner-Nördstrom solution, but there is *no electric field* on the brane. Instead, the non local coulomb effects imprinted by the bulk Weyl tensor have induced a "*tidal*" *charge* parameter q. It depends on the mass M on the brane, i.e., $q = q(M)$.

This 4-dimensional on brane black hole depends on the sign of q. For $q \geqslant 0$ there is a similarity to the general relativity Reissner-Nördstrom solution with two horizons:

$$r_{\pm} = \frac{M}{M_p^2}\left[1 \pm \sqrt{1 - q\frac{M_p^4}{M^2\widetilde{M_p^2}}}\right] \leq r_s = \frac{2M}{M_p^2} \tag{2.5}$$

both lying inside the Schwarzschild horizon (r_s) and there is an upper limit on q.

$$0 \leq q \leq q_{\max} = \left(\frac{\widetilde{M_p}}{M_p}\right)\left(\frac{M}{M_p}\right)^2$$

In our case, i.e., $q < 0$, which is impossible for general relativity Reissner-Nördstrom case, ensures only one horizon lying outside the Schwarzschild hoizon (r_s)

$$r_{+} = \frac{M}{M_p^2}\left[1 + \sqrt{1 - q\frac{M_p^4}{M^2\widetilde{M_p^2}}}\right] > r_s \tag{2.6}$$

The electric field in the Reissner-Nördstrom solution acts to weaken the gravitational field and the same is true for the Brane black hole with $q > 0$. In contrary, the $q < 0$ gives the opposite effect, i.e., bulk effects tends to strengthen the gravitational field. Furthermore, $q < 0$ ensures that the singularity is spacelike, as in the Schwarzschild solution, whereas $q > 0$ leads to a timelike singularity.

3. WEAK DEFLECTION LIMIT

Let us consider the most general form of a static spherically symmetric spacetime as,

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + C(r)d\Omega^2 \tag{3.1}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ is the metric on a unit sphere. The deflection angle is given by [18],

$$\begin{aligned} \alpha(r_0) &= -\pi + I(r_0) \\ I(r_0) &= \int_{r_0}^{\infty} \frac{2\sqrt{B(r)}dr}{\sqrt{C(r)}\sqrt{\left(\frac{C(r)}{C(r_0)}\right)\left(\frac{A(r_0)}{A(r)}\right) - 1}} \end{aligned} \tag{3.2}$$

where r_0 is the distance of closest approach of the light light ray to the source.

Comparing the above general metric with our considered metric (2.3) we see,

$$A(r) = 1 - \frac{2M}{M_p^2}\frac{1}{r} + \frac{q}{\widetilde{M_p^2}}\frac{1}{r^2} \tag{3.3}$$

$$B(r) = \frac{1}{A(r)} \tag{3.4}$$

$$C(r) = r^2 \tag{3.5}$$

Kar and Sinha [12] has already shown us the deflection angle using the above integration method and the expression obtained by them is,

$$\alpha(r_0) = \frac{4M}{M_p^2}\frac{1}{r_0} + \frac{3\pi\,|q|}{4\widetilde{M_p^2} r_0^2} \tag{3.6}$$

which is only upto $O(M)$. Here r_0 is the distance of closest approach . Here we will try to find the deflection formula upto $O(M^2)$ using a different method, so that the bending formula will be more exact. Also it will be an extension of the work done by them.

To find the deflection angle in weak deflection limit we employ the method formulated by Rindler and Ishak (For details Reader can refer to reference no [19]). Ample of works [20] has been done using it with great success. So using this method in this article we will not only obtained the exact result upto $O(M^2)$ but doing so it will indeed support the earlier results to some extent.

The path equation of the light rays on the equatorial plane for the metric (2.3) is,

$$\frac{d^2u}{d\phi^2} + u = \frac{3M}{M_p^2}u^2 - \frac{2q}{\widetilde{M_p^2}}u^3 \tag{3.7}$$

where $u = \frac{1}{r}$.Solving the above equation perturbatively upto $O(M^2)$ we get,

$$\begin{aligned} u &= \frac{\cos\phi}{R} + \frac{q\left(-9\cos\phi + \cos 3\phi - 12\phi\sin\phi\right)M_p^2 - 8MR\left(-3+\cos 2\phi\right)\widetilde{M_p^2}}{16R^3 M_p^2\widetilde{M_p^2}} \\ &+ \frac{1}{256R^5 M_p^4\widetilde{M_p^4}}\Bigg[q^2\Big\{24(8-3\phi^2)\cos\phi - 48\cos 3\phi + \cos 5\phi \\ &\quad +384\phi\sin\phi - 36\phi\sin 3\phi\Big\}M_p^4 \\ &\quad -16MqR\Big\{87 - 40\cos 2\phi + \cos 4\phi - 12\phi\sin 2\phi\Big\}M_p^2\widetilde{M_p^2} \\ &\quad +48M^2R^2\Big\{10\cos\phi + \cos 3\phi + 20\phi\sin\phi\Big\}\widetilde{M_p^4}\Bigg] \end{aligned} \tag{3.8}$$

Here R is the perpendicular distance from the Lens. The method of Rindler and Ishak is based on the invariant formula for the cosine of the angle ψ between two directions d and δ,

$$\cos\psi = \frac{g_{ij}d^i\delta^j}{(g_{ij}d_id_j)^{\frac{1}{2}}(g_{ij}\delta_i\delta_j)^{\frac{1}{2}}} \tag{3.9}$$

where g_{ij} are the metric components given by the metric (2.3), $d = (dr, d\phi) = (B_0, 1)d\phi$, $\delta = (1, 0)\delta r$ and $B_0(r, \phi) = \frac{dr}{d\phi} = (-r^2)\frac{du}{d\phi}$. Now using the metric (2.3) and the eqn (3.9) and after few manipulations we arrive to,

$$\tan\psi = \frac{rA^{\frac{1}{2}}(r)}{|B_0(r, \phi)|} \tag{3.10}$$

The bending is defined as $\alpha = 2(\psi - \phi)$ for different possible values of ϕ. Here we take only the case $\phi = 0$. Thus putting $\phi = 0$ in $B_0(r, \phi) = (-r^2)\frac{du}{d\phi}$ and $r = \frac{1}{u}$ in eqn (3.8) and inserting this values in eqn (3.10) we finally get the bending angle as,

$$\alpha(R) = \left(\frac{4}{RM_p^2} - \frac{16q}{R^3M_p^2\widetilde{M_p^2}}\right)M + \left(\frac{15\pi}{4R^2M_p^4} - \frac{117\pi q}{64R^4M_p^4\widetilde{M_p^2}}\right)M^2 - \frac{3\pi q}{4R^2\widetilde{M_p^2}} \tag{3.11}$$

$$= \frac{4M}{RM_p^2} + \frac{15\pi M^2}{4R^2M_p^4} - q\left(\frac{16M}{R^3M_p^2\widetilde{M_p^2}} + \frac{117\pi M^2}{64R^4M_p^4\widetilde{M_p^2}} + \frac{3\pi}{4R^2\widetilde{M_p^2}}\right) \tag{3.12}$$

Comparing the result (3.11) with the one obtained by Kar and Sinha [12], we can see that, upto $O(M)$ we have obtained an extra term $\left(-\frac{16qM}{R^3M_p^2\widetilde{M_p^2}}\right)$ in addition to the one obtained by Kar and SInha [12], which also shows the exactness of the bending expression (eqn 3.11) . Also as an extension of their work, the bending expression has been calculated upto $O(M^2)$ which is also an exact one. The presence of q-terms $\left\{-q\left(\frac{16M}{R^3M_p^2\widetilde{M_p^2}} + \frac{117\pi M^2}{64R^4M_p^4\widetilde{M_p^2}} + \frac{3\pi}{4R^2\widetilde{M_p^2}}\right)\right\}$ in the expression signifies attractive effect of the "tidal charge" to the bending as in our case q is negative..

4. STRONG DEFLECTION LIMIT

4.1. BOZZA'S Method

Here we first outline Bozza's method to ensure the clarity to readers that what quantities have been calculated to reach to the final lensing observables. To do so, we start with the generic spherically symmetric static spacetime

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + C(r)d\Omega^2 \tag{4.1}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ is the metric on a unit sphere.

The photon sphere $r = r_{ps}$ is the largest positive root of the equation

$$\frac{C'(r)}{C(r)} = \frac{A'(r)}{A(r)} \tag{4.2}$$

where prime denotes derivative with respect to r. This photon sphere radius is taken as the starting point of the strong field expansion. A photon coming from infinity with some impact parameter u gets deviated while approaching some gravitating source. It will reach the closest approach distance r_0 from the gravitating source before emerging to another direction. By the conservation of angular momentum, the closest approach distance r_0 is related to the impact parameter u by the relation

$$u = \sqrt{\frac{C(r_0)}{A(r_0)}} \tag{4.3}$$

where the subscript 0 indicates that the function is evaluated at r_0.

From the geodesic equation, the deflection angle in terms of the closest approach distance is given by [18],

$$\begin{aligned} \widehat{\alpha}(r_0) &= -\pi + I(r_0) \\ I(r_0) &= \int_{r_0}^{\infty} \frac{2\sqrt{B(r)}dr}{\sqrt{C(r)}\sqrt{\left(\frac{C(r)}{C(r_0)}\right)\left(\frac{A(r_0)}{A(r)}\right) - 1}} \end{aligned} \tag{4.4}$$

Here the closest approach distance (r_0) and the deflection angle ($\widehat{\alpha}$) are inversely proportional as decrease in r_0 increases the deflection angle. In this context, for certain value of r_0, $\widehat{\alpha}$ will become 2π which means the light ray will make a complete loop around the lensing object and if r_0 decreases further the light ray will wind up several times around the lens before reaching the observer. Finally when $r_0 = r_{ps}$, corresponding to an impact parameter $u = u_{ps}$ the deflection angle diverges and the light ray will get captured by the gravitating source.

Bozza's has shown us that this divergence is logarithmic for all spherically symmetric metrics which gives an analytical expansion of the deflection close to the divergence in the form

$$\widehat{\alpha}(r_0) = -a \log\left(\frac{r_0}{r_{ps}} - 1\right) + b + O(r_0 - r_{ps}) \tag{4.5}$$

here all the coefficients depend on the metric functions evaluated at $r = r_{ps}$. Equation (4.5) is the strong field limit of the light deflection angle.

To solve the integration (4.4) we start by defining a new variable,

$$z = \frac{A(r) - A(r_0)}{1 - A(r_0)} \tag{4.6}$$

and rewrite the integral as,

$$I(r_0) = \int_0^1 R(z, r_0) f(z, r_0) dz \tag{4.7}$$

$$R(z, r_0) = \frac{2\sqrt{BA}}{CA'}[1 - A_0]\sqrt{C_0} \tag{4.8}$$

$$f(z, r_0) = \frac{1}{\sqrt{A_0 - [(1 - A_0)z + A_0]\frac{C_0}{C}}} \tag{4.9}$$

The function $R(z, r_0)$ is regular for all values of z and r_0 but $f(z, r_0)$ diverges for $z \to 0$. So the integral (4.7) is splitted into two parts,

$$I(r_0) = I_D(r_0) + I_R(r_0) \tag{4.10}$$

where

$$I_D(r_0) = \int_0^1 R(0, r_{ps}) f_0(z, r_0) dz \tag{4.11}$$

contains the divergence and

$$I_R(r_0) = \int_0^1 \left[R(z, r_0) f(z, r_0) - R(0, r_{ps}) f_0(z, r_0)\right] dz \tag{4.12}$$

is regular in z and r_0 as the divergence part gets substracted. We note that all the functions without the subscript 0 are evaluated at $r = A^{-1}[(1-A_0)z+A_0]$. The function $f_0(z, r_0)$ is the expansion of the divergent function $f(z, r_0)$ within the square root upto second order in z

$$f_0(z, r_0) = \frac{1}{\sqrt{xz + yz^2}} \tag{4.13}$$

where

$$x = \frac{1 - A_0}{C_0 A_0'}\left[C_0' A_0 - C_0 A_0'\right] \tag{4.14}$$

$$y = \frac{(1 - A_0)^2}{2C_0^2 A_0'^3}\left[2C_0 C_0' A_0'^2 + (C_0 C_0'' - 2C_0'^2) A_0 A_0' - C_0 C_0' A_0 A_0''\right] \tag{4.15}$$

here prime denotes differentiation with respect to r.

As the angular separation of the image from the lens is $\tan\theta = \frac{u}{D_{OL}}$, where D_{OL} is the distance between lens and the observer. The deflection angle (4.5) is rewritten in terms of this variable as,

$$\widehat{\alpha} = -\overline{a} \ln \left(\frac{\theta D_{OL}}{u_{ps}} - 1 \right) + \overline{b} \tag{4.16}$$

$$u \simeq \theta D_{OL} \; (assuming \; small \; \theta) \tag{4.17}$$

where the coefficients are

$$\overline{a} = \frac{R(0, r_{ps})}{2\sqrt{y_{ps}}} \tag{4.18}$$

$$\overline{b} = -\pi + b_R + \overline{a} \ln \left(\frac{2y_{ps}}{A_{ps}} \right) \tag{4.19}$$

$$\begin{aligned} b_R &= \int_0^1 g(z, r_{ps}) dz \\ g(z, r_{ps}) &= R(z, r_{ps}) f(z, r_{ps}) - R(0, r_{ps}) f_0(z, r_{ps}) \end{aligned} \tag{4.20}$$

$$y_{ps} = y \mid_{r_0 = r_{ps}} \tag{4.21}$$

$$A_{ps} = A(r = r_{ps})$$

Using the relation (4.17) between angular separation θ of the images and the impact parameter u, Bozza has proposed three strong lensing observables as,

$$\theta_\infty = \frac{u_{ps}}{D_{OL}} \tag{4.22}$$

$$s = \theta_\infty \exp \left(\frac{\overline{b}}{\overline{a}} - \frac{2\pi}{\overline{a}} \right) \tag{4.23}$$

$$r = \exp \left(\frac{2\pi}{\overline{a}} \right) \tag{4.24}$$

where θ_∞ is the asymptotic position approached by a set of images in the limit of a large number of loops the rays make around the photon sphere, s is the angular separation between the outermost image resolved as a single image and the set of other asymptotic images, all packed together, while r is the ratio between the flux of the first image and the flux coming from all the other images.

In our next section, we shall calculate the set of strong field lensing coefficients $\{\overline{a}$, $\overline{b}$, $u_{ps}\}$ which is the "identity card" of the concerned lens as said by Nandi et. al. in their paper [21] and the resultant observables θ_∞, s, r, applying all the above formulas to the Braneworld black hole.

4.2. APPLICATION TO BRANEWORLD BLACK HOLE:

For our purpose, we take the generic spherically symmetric spacetime metric given by (2.3) and (2.4). Thus using (3.3), (3.5) and (4.2) we see in our case the radius of the photon sphere as,

$$r_{ps} = \frac{3M\widetilde{M_p} + \sqrt{-8qM_p^4 + 9M^2\widetilde{M_p^2}}}{2M_p^2\widetilde{M_p}} \tag{4.25}$$

The relation between the impact parameter u and the distance of closest approach $r = r_0$ is given by,

$$u = \frac{r_0}{\sqrt{1 - \frac{2M}{M_p^2}\frac{1}{r_0} + \frac{q}{\widetilde{M_p^2}}\frac{1}{r_0^2}}} \tag{4.26}$$

If $r_0 = r_{ps}$, then the minimum impact parameter u_{ps} is given by (4.25) and (4.26)

$$u_{ps} = \frac{\sqrt{2q}\left\{3M\widetilde{M_p} + \sqrt{-8qM_p^4 + 9M^2\widetilde{M_p^2}}\right\}}{\widetilde{M_p}\sqrt{4qM_p^4 + M\widetilde{M_p}\left(-3M\widetilde{M_p} + \sqrt{-8qM_p^4 + 9M^2\widetilde{M_p^2}}\right)}} \tag{4.27}$$

Applying the formulas (4.8), (4.13), (4.14) and (4.15) in Dadich et al. solution (2.3), (2.4) we obtain,

$$R(z, r_0) = \frac{\left(2M\widetilde{M_p^2}r_0 - qM_p^2\right)\left(2Mr_0\widetilde{M_p^2} + qzM_p^2\right)}{M\widetilde{M_p^2}r_0\left\{(3z-2)\,qM_p^2 + 2M\widetilde{M_p^2}r_0\right\}} \tag{4.28}$$

$$f\left(z, r_0\right) \sim f_0(z, r_0) = \frac{1}{\sqrt{xz + yz^2}}$$

$$x = \frac{\left(2M\widetilde{M_p^2}r_0 - M_p^2q\right)\left\{M_p^2\left(2q + r_0^2\widetilde{M_p^2}\right) - 3M\widetilde{M_p^2}r_0\right\}}{M_p^2\widetilde{M_p^2}r_0^2\left(M\widetilde{M_p^2}r_0 - M_p^2q\right)} \tag{4.29}$$

$$y = \frac{\left(M_p^2q - 2M\widetilde{M_p^2}r_0\right)^2\left\{4M_p^4q^2 - 9MM_p^2\widetilde{M_p^2}qr_0 + M\widetilde{M_p^4}r_0^2\left(6M - r_0M_p^2\right)\right\}}{4M_p^2\widetilde{M_p^2}r_0^2\left(M\widetilde{M_p^2}r_0 - M_p^2q\right)^3} \tag{4.30}$$

and finally from (4.18) and (4.19) we obtain,

$$\overline{a} = \frac{M_p\widetilde{M_p}r_{ps}\sqrt{\left(M_p^2 q - M\widetilde{M_p^2}r_{ps}\right)}}{\sqrt{-4M_p^4q^2 - 6M^2\widetilde{M_p^4}r_{ps}^2 + MM_p^2\widetilde{M_p^2}r_{ps}\left(9q + r_{ps}^2\widetilde{M_p^2}\right)}} \tag{4.31}$$

$$\begin{aligned} \overline{b} = & -\pi + b_R + \overline{a}\ln\left[\frac{(M_p^2 q - 2M\widetilde{M_p^2}r_{ps})^2}{2\left(M\widetilde{M_p^2}r_{ps} - M_p^2 q\right)^3}\right. \\ & \left.\times \frac{\left\{4M_p^4q^2 - 9MM_p^2\widetilde{M_p^2}qr_{ps} + M\widetilde{M_p^4}r_{ps}^2\left(6M - r_{ps}M_p^2\right)\right\}}{\left\{M_p^2 q + \widetilde{M_p^2}r_{ps}\left(r_{ps}M_p^2 - 2M\right)\right\}}\right] \end{aligned} \tag{4.32}$$

where r_{ps} is given by eqn (4.25).

To calculate b_R we expand the integrand (4.20) in powers of q as it cannot be calculated analytically. Thus we get

$$b_R = b_{R,0} + b_{R,1}q + O(q^2) \tag{4.33}$$

where $b_{R,0}$ is the value of the coefficient in case of Schwarzschild metric and its value is

$$b_{R,0} = 0.949603 \tag{4.34}$$

and the value of the coefficient $b_{R,1}$ is

$$b_{R,1} = -\frac{0.398477M_p^4}{M^2\widetilde{M_p^2}} \tag{4.35}$$

Thus

$$b_R = 0.949603 - \frac{0.398477qM_p^4}{M^2\widetilde{M_p^2}} \tag{4.36}$$

Hence from eqn (4.16) the final expression for the deflection angle in terms of image position θ in strong field limit is,

$$\widehat{\alpha} = -\overline{a}\ln\left(\frac{\theta D_{OL}}{u_{ps}} - 1\right) + \overline{b} \tag{4.37}$$

where u_{ps}, $\overline{a}$ and $\overline{b}$ are given by (4.27), (4.31) and (4.32).

Also using the values of u_{ps}, $\overline{a}$ and $\overline{b}$ we can calculate the resultant observables θ_∞, s, r from (4.22), (4.23) and (4.24) respectively.

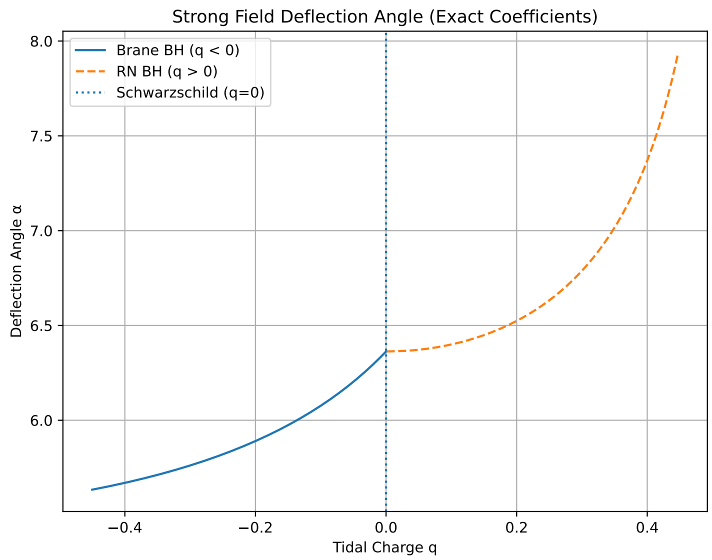

Figure 1: Variation of the deflection is shown with respect to the tidal charge parameter q evaluated at $u = u_{ps} + 0.003$. The solid blue line represents the Brane Black hole solution, dashed red line represents the Reissner Nördstrom solution and the point where the graphs intersects represents Schwarzschild bending.

5. NUMERICAL ANALYSIS

It will be an interesting work to compare the results obtained in the present work with those corresponding to the spherically symmetric black hole solutions in General Relativity, i.e., Schwarzschild and the Reissner Nördstrom geometries. For that purpose, we consider a massive black hole residing at the centre of our galaxy with mass $M = 2.8 \times 10^6 M_\odot$ [22] and $D_{OL} = 8.5\,\text{kpc}$. The table below shows the values of the lensing coefficients and the observables when the central black hole of our galaxy is portrayed as a Brane black hole, Schwarzschild black hole and Reissner Nördstrom black hole. To reduce complexity we set $2M = 1$ and $M_p = 1$.

	Brane Black Hole		Schwarzschild	R-N Solution	
	$q = -0.2\widetilde{M}_p^2$	$q = -0.1\widetilde{M}_p^2$	$q = 0$	$q = Q^2\ (Q = 0.1)$	$q = Q^2\ (Q = 0.2)$
$\overline{a}$	0.939274	0.964079	1	1.00456	1.01974
$\overline{b}$	−0.318384	−0.379345	−0.40027	−0.39935	−0.39651
u_{ps}/r_s	2.90299	2.75961	2.59821	2.58062	2.52649
$\theta_\infty\ (\mu as)$	18.8498	17.9187	16.8708	16.7565	16.405
$s\ (\mu as)$	0.016708	0.017865	0.021112	0.021635	0.0234359
r_m (magnitude)	7.26293	7.07606	6.82188	6.79094	6.68985

Table 1: Strong field lensing coefficients and observables for the black hole at the centre of our galaxy under different hypotheses for the space-time geometry. Here $2M = 1$, $r_s = \frac{2GM}{c^2}$ is the Schwarzschild radius, $M_p = 1$ and $r_m = 2.5 \times \log_{10}(r)$.

6. RESULTS AND CONCLUSION

In this paper, we have studied the gravitational lensing by the four dimensional brane black hole solution of Einstien Equation given by Dadhich et al. This black hole solution is characterized by the parameter q, the tidal charge parameter whose value should be less than zero in order to preserve the spacelike nature of the singularity in the geometry. Both weak and strong deflection limits have been analyzed in detail.

Firstly, in weak deflection limit, deflection angle has been calculated upto $O(M^2)$ using the well known Rindler-Ishak method. Actually this result is an extension cum modified work of the paper given by Kar and Sinha [12]. If we check the result (3.11) upto $O(M)$, we have obtained an extra term $\left(-\frac{16qM}{R^3 M_p^2 \widetilde{M_p^2}}\right)$ in addition to the result obtained by Kar and Sinha [12]. It is evident that the Rindler-Ishak method is a well established method [19, 20]. So we can confirm about the exactness of the deflection expression given by (3.11). The presence of $q-$terms $\left\{-q\left(\frac{16M}{R^3 M_p^2 \widetilde{M_p^2}} + \frac{117\pi M^2}{64R^4 M_p^4 \widetilde{M_p^2}} + \frac{3\pi}{4R^2 \widetilde{M_p^2}}\right)\right\}$ in our result (3.12) signifies the attractive effect of tidal charge q to the bending as q is negative in our case. Observationally, if we verify the presence of $q-$terms, it will surely test the effective Einstein equations on the brane.

Secondly, in strong deflection limit, deflection angle has been calculated using the Bozza's method. The value of the lensing observables θ_∞, s, r have also been calculated using the expressions of the strong field lensing coefficients $\{\overline{a}, \overline{b}, u_{ps}\}$ also termed as identity card of the concerned lens. Finally, values of the lensing coefficients and the observables have been tabulated and compared with two other black hole solutions. From the table, it is evident that, as the tidal charge increases, the lensing coefficient $\overline{a}$ increases but the minimum impact parameter u_{ps} decreases which appears to behave similar to the Reissner Nördstrom solution. But the lensing coefficient $\overline{b}$ decreases as the tidal charge increases and it is contrary to the Reissner Nördstrom case. Also, the angular position of the relativistic images θ_∞ and the relative magnitude r_m decreases and the angular separation s increases as the tidal charge increases, which is alike the Reissner Nördstrom case. The above table also tells us that, the resolution of two extremely faint images separated is $\sim\ 0.017\ \mu as$, which is very small to detect. However, one would need a resolution of $\sim\ 0.01\ \mu as$ to detect the effect of extra dimensions, if such refinement in technology be possible in near future.

To identify the lensing behavior of the brane black hole, we plot the deflection angle of the black hole and compare it with the Reissner Nördstrom solution (Figure 1). It is observed that the behavior is completely different from that of the Reissner Nördstrom case. These results may help to distinguish brane black hole solution from general relativity black hole solution using astronomical observations developed in near future. It is also easy to see that our result reduce to Schwarzschild bending as $q = 0$ which can be seen in figure 1 as the intersecting point.

Using well-established methods, we have demonstrated that the four-dimensional light-bending formula for brane geometry contains signatures of extra dimensions in both the weak-field and strong-field regimes. In the future, with the availability of more advanced observational data and improved technologies, these results may play an important role in identifying and verifying the effects of extra dimensions through gravitational lensing observations.

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