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Elastic Buckling of Columns By Differential Quadrature Method

J. Girish
Civil Engineering Department, KIET School of Engineering and Technology,
Ghaziabad – Meerut Highway, Ghaziabad – 201 206, India
girish.iitkgp@gmail.com
http://www.kiet.edu

Abstract

In this paper, the first five buckling loads of elastic columns are presented by using differential quadrature method. The weighing coefficients for derivative approximations are obtained by using explicit formulae. Boundary conditions are implemented through the discrete grid points by constraining the displacements, bending moments and rotations. The buckling loads for different boundary conditions are reported along with the convergence of results with grid points. Typical grid points with uniform spacing and non uniform spacing are examined for solution accuracy. The theoretical formulations and solution procedures of the method are illustrated through solving numerical examples. The accuracy and validity of the results are examined by direct comparison with the known values.

Keywords: Buckling, Differential Quadrature, Weighting Coefficients, Boundary Conditions

Introduction

Differential Quadrature (DQ) method is an efficient discretization technique for solving linear and nonlinear ordinary and partial differential equations with initial, boundary or mixed initial boundary conditions. Accurate numerical solutions can be obtained using a considerably small number of grid points and hence relatively less computational effort. DQ method was introduced by Bellman and Casti [1971] and later on by Bellman *et al*. [1972]. This method has been applied successfully to the various problems.

The method of differential quadrature approximates the partial derivative of a function by a polynomial which is constructed as a weighted linear sum of the function values at a given discrete points. Thus, for time-independent problems, a partial differential equation is reduced to a set of algebraic equations. This method, similar to any polynomial approach, is dependent on the order of the approximating polynomial, i.e. with an increase in the order of the polynomial the accuracy of the solution increases. Possible oscillations arising from higher-order polynomials can be avoided by using numerical interpolation techniques.

The quadrature solutions of higher-order differential equation with multiple boundary conditions were introduced through the application of the DQM to structural mechanics problems by Bert *et al.* [1988] and Jang *et al.* [1989]. To solve the fourth-order differential equation of a beam or column which has two boundary conditions at each end, many investigators employed δ-technique wherein adjacent to the boundary points of the differential quadrature grid, points are chosen at small distances. The boundary point and its adjacent δ-points were used for implementing the multiple boundary conditions. Jang *et al.* [1989] proposed the so called δ-technique for the static analysis of beams and plates. The points adjacent to the boundary points of the differential quadrature grid points are chosen at small distance δ = 0.000001. However, an arbitrariness in the choice of the δ - value becomes apparent while implementing the boundary conditions. The δ - value cannot be large for acceptable solution accuracy, with small very small δ - value, the solution begins to oscillate. An alternative concept was suggested by Wang and Bert [1993] wherein it was proposed that the boundary conditions can be implemented in the weighting coefficient matrices themselves.

In the present investigation, the differential quadrature is used to compute the first five buckling loads of columns with various boundary conditions. Typical grid points with uniform spacing and non uniform spacing are examined for solution accuracy. The boundary conditions are implemented with and without using the δ - type grid technique. Results are compared with exact analytical solutions available in the literature. The applicability and accuracy of the differential quadrature method are demonstrated through the numerical examples.

Differential Quadrature Method

In Differential quadrature method, a partial derivative of a function with respect to a coordinate direction is expressed as a linear weighted sum of all the functional values at all grid points along that direction. The main principle is to determine the weighting coefficients for the discretization of a derivative of any order. To illustrate the method, consider a first order derivative of the function $f(x)$ with respect to x at a grid point x_i, is approximated by a linear sum of all the functional values in the whole domain, that is

$$f_{,x}(x_i) = \left.\frac{df}{dx}\right|_{x_i} = \sum_{j=1}^{N} a_{ij}\, f(x_j), \quad for\ i = 1, 2, \ldots\ldots, N \tag{1}$$

Where x_i: i = 1, 2, ….., N, are the sample points obtained by breaking the x – variable into N discrete values, $f(x_i)$ are the function values at these points, and a_{ij} represent the weighting coefficients attached to these function values and N is the number of grid points in the whole domain. The weighting coefficients a_{ij} are different at different locations of x_i. The determination of weighting coefficients in eqn. (1) is a key procedure in the DQ approximation. The test functions are chosen as

$$f_k(x) = x^{k-1}, \quad k = 1, \ldots\ldots\ldots, N \tag{2}$$

Obviously, eqn. (2) gives N test functions. For the weighting coefficients a_{ij} in eqn. (1), i and j are taken from 1 to N. Thus the total number of weighting coefficients is N x N. To obtain these weighting coefficients, the N test functions should be applied at N grid points $x_1, x_2, \ldots.., x_N$.

$$f_k(x) = \left\{ x^0 \ x^1 \ x^2 \ x^3 \ \ldots\ldots x^{N-1} \right\}, \quad k = 1, \ldots\ldots\ldots\ldots, N \tag{3}$$

The set of linear algebraic equations can be written as

$$(k-1).x_j^{k-2} = \sum_{j=1}^{N} a_{ij} x_j^{k-1} \tag{4}$$

Eqn. (4) represents N sets of N linear algebraic equations. The equation system has a unique solution because its matrix is of Vandermonde form. Eqn. (4) can be solved for weighting coefficients a_{ij} by Gauss elimination.

The first order derivative approximation formula given by eqn. (1) can be expressed in closed form by the following linear transformation for the derivative with respect to x:

$$\frac{df}{dx} \cong AF \tag{5}$$

Where $F = [f(x_i)]$ is the function matrix consisting of the function values at the grid points. $A = [a_{ij}]$ is the weighting coefficient matrix whose elements are attached to those function values in the x direction. The approximation formulae for higher order partial derivatives are obtained by iterating the linear transformations given by eqn. (5). Thus, for example, the higher order derivatives are:

$$\frac{d^2 f}{dx^2} \cong (A)^2 F \cong BF \tag{6}$$

$$\frac{d^3 f}{dx^3} \cong (A)^3 F \ or \ (AB)F \cong CF \tag{7}$$

$$\frac{d^4 f}{dx^4} \cong (A)^4 F \ or \ (B)^2 F \cong DF \tag{8}$$

Typical weighting coefficient matrices for three, four, five and seven sampling points are given in annexure I.

The main disadvantage of the above approach is that each time the degree of the polynomial (grid points - N) is changed, all of the work must be redone. The weighting coefficients obtained from other sampling points is of no help in computing the weighting coefficients for any other sampling points. It requires a considerable amount of effort to solve the system of equations for the coefficients. For a large grid spacing (high-degree polynomial), the system of equations can be ill conditioned, which causes large errors in the coefficients. A simpler, more direct procedure is desired.

Explicit Formulae for Differential Quadrature Coefficients

The explicit formulae for DQ coefficients can be obtained by considering the sample Lagrange interpolation polynomials as the test functions (Quan and Chang, [1989]):

$$f_k(x) = \frac{\phi_n(x)}{(x - x_k)\left[\frac{d}{dx}\phi_n(x_k)\right]} = \sum_{j=1}^{N} a_{ij} x_j^{k-1}, \quad k = 1, 2, ..., N \tag{9}$$

Where $\phi_n(x) = \prod_{m=1}^{N}(x - x_m)$ (10)

$$\frac{d}{dx}\phi_n(x_i) = \prod_{m=1, m\neq i}^{N}(x_i - x_m) \tag{11}$$

By applying eqn. (9) at N grid points, Quan and Chang proposed the following algebraic formulations to compute the weighting coefficients,

$$a_{ij} = \frac{1}{x_j - x_i}\prod_{\substack{m=1\\ m\neq i,j}}^{n}\frac{x_i - x_m}{x_j - x_m}, \tag{12}$$

$$b_{ij} = \frac{2}{x_j - x_i}\left(\prod_{\substack{m=1\\ m\neq i,j}}^{N}\frac{x_i - x_m}{x_j - x_m}\right)\left(\sum_{\substack{k=1\\ k\neq i,j}}^{n}\frac{1}{x_i - x_k}\right), \tag{13}$$

for $i \neq j$, and

$$a_{ii} = \sum_{\substack{k=1\\ k\neq i}}^{n}\frac{1}{x_i - x_k}, \tag{14}$$

$$b_{ii} = 2\sum_{\substack{k=1\\ k\neq i}}^{n-1}\left[\frac{1}{x_i - x_k}\left(\sum_{\substack{l=k+1\\ l\neq i}}^{N}\frac{1}{x_i - x_l}\right)\right], \tag{15}$$

for $i = j$

These general formulae are valid for arbitrary location of grid points x_i, i = 1,2, …,N. More significantly, the differential quadrature coefficients can now be calculated directly. The weighting coefficients presented in Appendix I can be verified directly by using these explicit formulae.

Buckling of Linear Elastic Columns

The governing differential equation for the buckling analysis of a column is [Timoshenko and Gere]

$$EI\frac{d^4w}{dx^4}+P\frac{d^2w}{dx^2}=0 \tag{16}$$

To determine the critical buckling load of the column, the in-plane compressive load P should be considered, while the transverse load is set to zero. The non-dimensional governing differential equation for buckling can be written as

$$\frac{d^4W}{dX^4}+\lambda\frac{d^2W}{dX^2}=0 \tag{17}$$

Where $X = x/L$, $W = w/L$ and $\lambda = PL^2/EI$.

The numerical model can be expressed by replacing the derivatives with the corresponding Differential Quadrature approximation formulae, using eqs. (6) and (8). DQ form of eqn. (17) can be expressed as

$$\sum_{j=1}^{N}D_{ij}W_j+\lambda\sum_{j=1}^{N}B_{ij}W_j=0;\quad i=1,2,3,......,N \tag{18}$$

Where D and B are the weighting coefficients of the fourth order derivative and second order derivative with respect to the variable *x*. On applying the DQ method with the boundary conditions at the above equation, we get a set of algebraic equations. The only way to have a nontrivial solution of these equations is to have the determinant of the coefficients equal to zero. The determinant is

$$\left|[D]-\lambda[B]\right|=0 \tag{19}$$

Where $\lambda = -Pl^2/EI$ and *P* is the buckling load.

Typical Grid Points

GP1: Equally spaced grid points

$x(i)=\left[\frac{i-1}{n-1}\right]$; i = 1,2,3,……..,N

GP2: Chebyshev Gauss Lobatto points

$x(i)=\frac{1}{2}\left[1-\cos\left(\frac{i-1}{n-1}\right)\pi\right]$; i = 1,2,3,……..,N

GP3: Equally spaced grid points with δ (δ = 0.00001) points

$x(1) = 0$, $x(2) = \delta$, $x(N-1) = 1-\delta$, $x(N) = 1$

$x(i)=\left[\frac{i-1}{n-1}\right]$; i = 3,4,5,……..,(N-2)

GP4: Chebyshev Gauss Lobatto points with δ (δ = 0.00001) points

$x(1) = 0$, $x(2) = \delta$, $x(N-1) = 1-\delta$, $x(N) = 1$

$$x(i)=\frac{1}{2}\left[1-\cos\left(\frac{i-1}{n-1}\right)\pi\right];\ i=3,4,5,\ldots\ldots..,(N\text{-}2)$$

Table 1: Coordinates of grid points for N = 9

Grids	$x(1)$	$x(2)$	$x(3)$	$x(4)$	$x(5)$	$x(6)$	$x(7)$	$x(8)$	$x(9)$
GP1	0	0.125	0.25	0.375	0.5	0.625	0.75	0.875	1
GP2	0	0.038	0.146	0.308	0.5	0.691	0.853	0.961	1
GP3	0	0.00001	0.25	0.375	0.5	0.625	0.75	0.99999	1
GP4	0	0.00001	0.146	0.308	0.5	0.691	0.853	0.99999	1

Results and Discussions

A computer program is written in MATLAB to generate weighting coefficients and to implement boundary conditions. Finally buckling loads (Eigen values) are obtained by solving eqn. (19). Numerical results are presented for a linearly elastic column with four sets of boundary conditions, i.e., (i) column with pinned ends , (ii) column with one end fixed and the other pinned, (iii) Fixed end column and (iv) column with one end fixed and the other free. The boundary conditions are:

(i) Pinned-Pinned

$$W=0 \text{ and } \frac{d^2W}{dX^2}=0 \text{ at } X=0;\ W=0 \text{ and } \frac{d^2W}{dX^2}=0 \text{ at } X=1 \tag{20 a}$$

(ii) Fixed-pinned

$$W=0 \text{ and } \frac{dW}{dX}=0 \text{ at } X=0;\ W=0 \text{ and } \frac{d^2W}{dX^2}=0 \text{ at } X=1 \tag{20 b}$$

(iii) Fixed – Fixed

$$W=0 \text{ and } \frac{dW}{dX}=0 \text{ at } X=0;\ W=0 \text{ and } \frac{dW}{dX}=0 \text{ at } X=1 \tag{20 c}$$

(iv)Fixed-Free

$$W=0 \text{ and } \frac{dW}{dX}=0 \text{ at } X=0;$$

$$\frac{d^2W}{dX^2}=0 \text{ and } \frac{d^3W}{dX^3}=0 \text{ at } X=1 \tag{20 d}$$

The results after applying boundary conditions for column are presented in Table 2.

Table 2: Comparison of column buckling loads

Boundary conditions	λ_{cr} exact	N	λ_{cr} GP1	λ_{cr} GP2	λ_{cr} GP3	λ_{cr} GP4
Pinned-pinned	9.8696	5	9.9677	9.8239	9.6000	9.6000
		6	9.9214	9.8672	9.6744	9.7869
		7	9.8674	9.8696	9.8752	9.8715
		8	9.8683	9.8696	9.8735	9.8703
		9	9.8696	9.8696	9.8695	9.8696
Fixed-Fixed	39.4784	7	34.5599	43.6363	36.0000	38.4002
		9	40.1651	39.4096	39.8403	39.5178
		11	39.4464	39.3887	39.4623	39.4766
		13	39.4795	39.4784	39.4789	39.4784
		15	39.4784	39.4784	39.4784	39.4784
Fixed-Free	2.4674	5			2.4001	2.4001
		7	4.3876	3.0498	2.4687	2.4681
		9	3.7376	2.7752	2.4674	2.4674
		11	3.4151	2.6590	2.4674	2.4674
		13	3.2227	2.5985		
		15	3.0951	2.5629		
		17	3.0043	2.5401		
		19	2.9364	2.5246		
		21	2.8837	2.5136		
Fixed-Pinned	20.19	5	21.1522	20.3357	18.9125	18.9125
		7	20.1514	20.1826	20.1884	20.1990
		9	20.1918	20.1908	20.1918	20.1906
		11	20.1907	20.1907	20.1907	20.1907
		13	20.1907	20.1907	20.1907	20.1907

Table 3: Buckling parameters $\lambda_{cr} = Pl^2 / EI$ for columns

Boundary conditions	First five buckling loads				
	1	2	3	4	5
Fixed-Free	2.4674 (N =9)	22.2067 (11)	61.6646 (11)	120.9117 (15)	199.8993 (15)
Pinned-Pinned	9.8696 (N=7)	39.4783 (11)	88.8241 (11)	157.9117 (13)	246.7408 (15)
Fixed-Fixed	39.4784 (N=13)	80.7629 (13)	157.9137 (15)	238.7158 (21)	355.3053 (25)
Fixed-Pinned	20.1907 (N=9)	59.6750 (11)	118.9013 (13)	197.6735 (13)	296.6093 (15)

In Table 2, the buckling loads for the columns are compared to the respective exact solutions. For Pinned – Pinned boundary condition, Chebyshev Gauss Lobatto points (GP2) produce more accurate results than the other points and with lesser grid points as well. In the case of Fixed – Fixed boundary conditions, the buckling loads are converged at N = 13 for GP2 and GP4 grid points. The convergence of results, in the case of cantilever column (Fixed – Free) is at N = 9 with GP3 and GP4 grid points. It was found that, GP2 and GP4 grid spacing produce convergence of results at N = 9. Table 3 illustrates the converged first five buckling loads of columns. It can be seen that the differential quadrature results compare very well with the exact results.

Conclusions

The differential quadrature method is presented here to solve fourth-order differential equations of a column for buckling loads. Four different grid spacing points are employed to determine the weighting coefficients required by the differential quadrature approximation without restricting the choice of mesh grids. Numerical calculations showed that accurate results can be achieved using considerably fewer grid points. The solution procedures and programming are very simple and easy. Further, boundary conditions are easily incorporated into the solution procedure. It can be concluded that due to its accuracy and simplicity differential quadrature method has great potential for wide use in structural analysis.

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