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Myhill Nerode Theorem For Fuzzy Automata (Min-Min Composition)

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Abstract

In this paper, Myhill Nerode theorem of finite automaton has been extended to fuzzy automaton where the composition considered is min-min composition. In the case of max-min composition, it has already been proved that if L is a fuzzy regular language, then for any $\alpha \in [0, 1]$, $L_\alpha = L(D_\alpha(M))$. In the case of max-product composition L_α is only a subset of $L(D_\alpha(M))$. But still Myhill Nerode theorem has been extended to max-product composition [4]. In the case of max-average composition, L_α is not even contained in $L(D_\alpha(M))$. This lead to lots of challenges and we had to resort to splitting to prove the analogue of Myhill Nerode Theorem for max-average composition. In a similar line, an attempt has been made in this paper to study the behavior of fuzzy automata under min-min composition and to prove the analogue of Myhill Nerode theorem for min - min composition.

Keywords: Monoid, min-min composition, finite automaton, equivalence class, fuzzy regular language, fuzzy automaton

Introduction

The *Myhill-Nerode theorem* [1] is a central result in formal language theory that provides necessary and sufficient conditions for a language to be regular, which are in terms of right congruences and congruences of finite index on a free monoid. The theorem shows that right congruences on a free monoid are very useful in the proof of existence and construction of the minimal deterministic automaton recognizing a given language, as well as in minimization of deterministic automata. *Jiri Mockr* [4] demonstrated that every fuzzy automaton can be expressed as a cascaded set of nondeterministic automaton for each distinct membership degrees of the words accepted by the fuzzy automaton. Basic definitions of fuzzy automaton, language accepted by the fuzzy automata and relationship between fuzzy automata and fuzzy languages are discussed in the papers by *Mordeson* [5], *Qui* [6] and *Pedrycz* [8]. Early work on applying Myhill-Nerode theorem for a fuzzy automaton using max-min composition is found in a paper [9] by *Ramaswamy*. The obtained results establish nice relationships between fuzzy languages, fuzzy automata and nondeterministic automata. The Myhill-Nerode theorem for fuzzy automata with min-max composition is discussed in [9]. In min-max case it is found that the α - cuts of the language accepted by fuzzy automaton and the language accepted by NFA for a given α-value are not equivalent. However, the theorem is proved using some additional constraints.

In this paper we develop a Myhill-Nerode theorem for fuzzy automaton using min-min composition. The Myhill-Nerode theorem is extended and proved for min-min composition. The procedure for construction of automata using right congruences is discussed and illustrated with an example. The rest of the paper is organized as follows. In Section 2 basic concepts of fuzzy automaton, fuzzy transition function, min-min composition and language accepted by fuzzy automaton are discussed. In Section 3, the Myhill Nerode theorem for fuzzy automata is stated and proved using min-min composition. The working of the theorem is illustrated using the example in Section 4. Finally, the Section 5 gives the results and discussions.

Basic Concepts

Let A be a finite non empty set. A fuzzy automaton over A is a 4-tuple M = (Q, f, I, F) where Q is a finite nonempty set, f is a fuzzy subset of Q × A × Q, I and F are fuzzy subsets of Q. In other words, f: Q × A × Q → [0, 1] and I, F: → [0, 1].

Let S be a free monoid with identity element e generated by A. If $s \in S$, then s can be written as $s = a_1a_2\ldots a_n$ where $a_i \in A$. Here n is called the length of s and we write |s| = n. We now extend f to a function f^*: Q × S × Q → [0, 1] defined as

$f^*(q, e, p) = 1$ if $q = p$,

$= 0$ if $q \neq p$

$f^*(q, sa, p) = \bigwedge_{r \in Q} [\, f^*(q, s, r) \wedge f(r, a, p)\,]$ $(s \in S, a \in A)$ where $f^*(q, s, r) > 0$, $f(r, a, p) > 0$

It can be shown that $f^*(q, a, p) = f(q, a, p)$ for all $p, q \in Q$ and for all $a \in A$.

Definition 1

Let M = (Q, f^*, I, F) be a fuzzy automaton over S. We define the language accepted by M denoted by L (M) to be a fuzzy subset of S defined as L (M) (s) = I o fs * o F for all $s \in S$. Here o denotes min-min composition.

In min-min composition we define

$I \circ f_s^* \circ F = \bigwedge_{p \in Q} [\, I(p) \wedge (f_s^* \circ F)(p)]$ where $I(p) > 0$, $(f_s^* \circ F)(p) > 0$

Here

$(f_s^* \circ F)(p) = \bigwedge_{r \in Q} [\, f_s^*(p, r) \wedge F(r)]$ where $F(r) > 0$, $f_s^*(p, r) > 0$.

Definition 2
A fuzzy subset L of S is said to be a fuzzy regular language if L= L(M) where M is a fuzzy automaton over S.

Myhill Nerode Theorem For Fuzzy Automata (Min-Min Composition)

Theorem: Let S be a monoid with identity element e and L be a fuzzy subset of S. Then the following statements are equivalent.

(i) L is a fuzzy regular language.

(ii) L can be expressed as a fuzzy union

$L = (\delta_1)_L \cup (\delta_2)_L \cup \ldots \cup (\delta_t)_L$

where $\delta_1, \delta_2 \ldots \delta_t \in [0, 1]$. For each i = 1, 2…t, $(\delta_i)_L = \delta_i \,.\, L_{\delta i}$ where $L_{\delta i} = \cup [s]_{\delta i}$. This union is a set theoretic union and $[s]_{\delta i}$ denotes the equivalence class of s of a right invariant equivalence relation of finite index in $L_{\delta i.}$

(iii) Define a relation R_L as follows.

If s, t $\in$ S, then s R_L t if and only if for all u $\in$ S and for all $\alpha \in [0, 1]$, $L(su) \geq \alpha$ only when $(tu) \geq \alpha$. Then R_L is a right invariant equivalence relation of finite index.

Proof of (i) → (ii)
Since L is a fuzzy regular language, we have L = L (M) where M = (Q, f^*, I, F) is a fuzzy automaton. Consider any $\alpha \in [0, 1]$. With M and α, we associate a non-deterministic automaton D_α (M) = (Q, d_α, I_α, F_α) where

d_α: Q x S $\rightarrow 2^Q$ is defined as $d_\alpha (q, s) = \{p \in Q \mid f^* (q, s, p) \geq \alpha\}$,

$I_\alpha = \{p \in Q \mid I (p) \geq \alpha\}$ and

$F_\alpha = \{p \in Q \mid F (p) \geq \alpha\}$.

In case of min-min composition it is found that $L_\alpha \subseteq L (D_\alpha (M))$ and we prove this as follows:

Let $s \in L_\alpha$. Then $L (s) = L (M) (s) \geq \alpha$. ie $(I \circ f_s^* \circ F) \geq \alpha$ which means

$$\underset{p \in Q}{\wedge} [(f_s^* \circ F) (p) \wedge I (p)] \geq \alpha$$

This means for any state $p \in Q$, $I (p) \geq \alpha$ and $(f_s^* \circ F) (p) \geq \alpha$. Now $(f_s^* \circ F) (p) \geq \alpha$ means

$$\underset{r \in Q}{\wedge} [(f_s^*(p, r) \wedge F(r)] \geq \alpha$$

Therefore we have $f_s^*(p, r) \geq \alpha$ and $F(r) \geq \alpha$ for any $r \in Q$. Now $f_s^*(p, r) \geq \alpha$ means

$$\underset{r \in Q}{\wedge} [f_s^*(p , r) \wedge F (r)] \geq \alpha \text{ for any } r \in Q.$$

This implies that $f_s^* (p , r) \geq \alpha$ and $F (r) \geq \alpha$ where $I (p) \geq \alpha$.

$F (r) \geq \alpha$ means $r \in F_\alpha$ and

$f_s^*(p, r) \geq \alpha$ means $r \in d_\alpha(p, s)$

Thus we get $d_\alpha(p, s) \cap F_\alpha \neq \phi$ where $p \in I_\alpha$. This proves that $L_\alpha \sqsubseteq L(D_\alpha(M))$.

Let $Q = \{q_0, q_1, q_2 \ldots q_n\}$. If $s \in S$, then L(s) can take any of the values from $I(q_1), \ldots I(q_n)$, $F(q_0)$, $F(q_1), \ldots F(q_n)$, $f(q_i, a, q_k)$ $(q_i, q_k \in Q, a \in A)$. Denote these values (after arranging them in non decreasing order) by $\delta_1, \delta_2 \ldots, \delta_t$. Then $\delta_1, \delta_2 \ldots, \delta_t \in [0, 1]$ and for each i, $L_{\delta i} \subseteq L(D_{\delta i}(M))$. Since $L(D_{\delta i}(M))$ is the regular language accepted by a nondeterministic finite automaton $D_{\delta i}(M)$ is a regular language we can apply Myhill-Nerode theorem of finite automaton as follows:

Let R_i be the right invariant equivalence relation of finite index in $L(D_{\delta i}(M))$. Let R_i' denote the restriction of R_i to $L_{\delta i}$. Clearly, R_i' is a right invariant equivalence relation of finite index in $L_{\delta i}$. Let $[s]_{\delta i}'$ denote the equivalence class of s in $L_{\delta i}$. Since these equivalence classes partition $L_{\delta i}$, it follows that $L_{\delta i} = \cup [s]_{\delta i}'$. This is true for i = 1, 2… t.

Now we will prove that $L = (\delta_1)_L \cup (\delta_2)_L \cup \ldots \cup (\delta_t)_L$ as follows:

Define $(\delta_i)_L = \delta_i . L_{\delta i}$. If $s \in S$ such that $L(s) \geq \delta_i$ $(s \in L_{\delta i})$, then $(\delta_i)_L(s) = \delta_i$, Otherwise $(\delta_i)_L(s) = 0$. We note that each $(\delta_i)_L$ is a fuzzy set. Let $s \in S$ and assume that $L(s) = \delta_i$. Now $L(s) = \delta_i \leq \delta_{i+1} \leq \ldots \leq \delta_t$. Again, $L(s) = \delta_i \geq \delta_{i-1} \geq \ldots \geq \delta_1$.

$$
\begin{aligned}
\text{Hence } ((\delta_1)_L \cup (\delta_2)_L \cup \ldots \cup (\delta_t)_L)(s) &= (\delta_i)_L(s) \vee (\delta_j)_L(s) \vee \ldots (\delta_k)_L(s) \\
&= \delta_i \vee \delta_j \vee \ldots \vee \delta_k \\
&= L(s).
\end{aligned}
$$

for $((\delta_i)_L(s), (\delta_j)_L(s), (\delta_k)_L(s)) > 0$

This proves that $L = (\delta_1)_L \cup (\delta_2)_L \cup \ldots \cup (\delta_t)_L$.

Proof of (ii) → (iii)

If $s \in S$, then $s R_L s$ because for all $u \in S$ and for all $\alpha \in [0,1]$, $L(su) \geq \alpha$ only when $L(su) \geq \alpha$ is obviously true. This proves that R_L is reflexive. Clearly, R_L is symmetric. If $s R_L t$ and $t R_L v$, then for all $u \in S$ and for all $\alpha \in [0, 1]$, $L(su) \geq \alpha$ only when $L(tu) \geq \alpha$ only when $L(vu) \geq \alpha$ proving that $s R_L v$. Hence R_L is transitive. R_L is thus an equivalence relation.

To prove R_L is right invariant, assume that $s R_L t$ and $u \in S$. We have to prove that $su R_L tu$. For this, we have to prove that for all $v \in S$ and $\alpha \in [0, 1]$, $L(suv) \geq \alpha$ only when $L(tuv) \geq \alpha$ which is the same as saying that $L(sz) \geq \alpha$ only when $L(tz) \geq \alpha$ where $z = uv$. But this is true since $s R_L t$.

We will now prove that R_L is of finite index. For $i = 1,2,\ldots,t$, let R_i denote the right invariant equivalence relation of finite index in $L_{\delta i}$. Let $R = R_1 \cap R_2 \cap \ldots \cap R_t$. Then R is an equivalence relation of finite index. We will prove that s R t implies $s R_L t$. This will mean that index $(R_L) \leq$ index (R). Since index (R) is finite, this will prove that index (R_L) is also finite.

Assume that s R t. Consider any $u \in S$ and any $\alpha \in [0, 1]$. Suppose $su \in L_\alpha$. We have to prove that $tu \in L_\alpha$. Now $\alpha \leq L(su) = \delta_j$ (say). Then $su \in L_{\delta j}$ which is a subset of L_α. By definition of R, we have $s R_j t$. Since R_j is right invariant, $su R_j tu$. Since $L_{\delta j} = \cup [v]_{\delta j}$, it follows that su belongs to one of the equivalence classes of R_j and hence tu also belongs to the same equivalence class. Hence tu $\in L_{\delta j}$ and since $L_{\delta j}$ is a subset of L_α, we have tu $\in L_\alpha$.

Proof of (iii) → (i)

We have to define a fuzzy automaton M such that L = L (M). For every element s $\in$ S, let [s] denote the equivalence class of s under the equivalence relation R_L.

Let Q = {[s] □ s $\in$ S}. Since R_L is of finite index, it follows that Q is a finite set. Define

I: Q → [0 , 1], f^*: Q x S x Q → [0 , 1] and F: Q → [0, 1] as follows.

I ([s]) = 1 if [s] = [e]

= 0 if [u] $\neq$ [st].

f^*([s], t, [u]) = 1 if [u] = [st]

= 0 if [u] $\neq$ [st]

F ([s]) = L (s).

We will first prove that F is well defined. For this, we have to prove that if [s] = [t], then L (s) = L (t). Assume that L (s) = β. We will prove that L (t) = β. Since [s] = [t], s R_L t so that L (s) = L (se) ≥ β only when L(t) = L (te) ≥ β. Since L(s) ≥ β, it follows that L[t] ≥ β.

Assume L [t] = γ > β. Take η = (β + γ) / 2. Clearly, β < η < γ = L[t]. Since s R_L t, L[t] > η implies that L[s] ≥ η > β. But this contradicts the fact that L(s) = β. Hence our assumption that L[t] > β is wrong. Since L[t] ≥ β, it follows that L[t] = β.

Take M = (Q, I, f^*, F). Then M is a fuzzy automaton and it remains to prove that L = L (M). For this, we have to prove that for all s $\in$ S, L (s) = L (M) (s).

We have

L (M) (s) = I o f_s^* o F

= $\wedge$\{ I ([t]) $\wedge$ (f^*_s o F) ([t])\} where I ([t]) > 0 and (f^*_s o F) ([t]) > 0

[t]

(f^*_s o F) ([t]) = $\wedge$\{ f^*_s ([t], [u]) $\wedge$ F([u]) \} where f^*_s ([t], [u]) > 0 and F([u]) > 0

[u]

= $\wedge$\{ f^* ([t], s, [u]) $\wedge$ F([u]) \} where f^*_s ([t], [u]) > 0 and F([u]) > 0

[u]

Note that f^*([t], s, [u]) = 1 if [ts] = [u] and 0 otherwise. Therefore, in the above expression f^*([t], s, [u]) = 1 only when [ts] = [u] . In all remaining cases (ie. whenever [ts] $\neq$ [u]) the term f^*([t], s, [u]) □ F([u]) becomes 0. Thus the above equation becomes

(f^*_s o F) ([t]) = F([u])

= L(ts) since F ([s]) = L(s).

Hence L (M) (s) = $\wedge$ { I([t]) □ (f^*_s o F) ([t]) }

[t]

Note that I ([t]) = 1 only when [t] = [e], I ([t]) = 0 whenever [t] $\neq$ [e].

Therefore, { I([t]) □ (f^*_s o F) ([t]) } = 0 whenever [t] $\neq$ [e] and

{ I([t]) □ (f^*_s o F) ([t]) } = (f^*_s o F) ([t]) when [t] = [e].

Thus the above equation becomes

$L(M)(s) = (f^*_s \text{ o } F)([t])$ where $[t] = [e]$.

$= L(ts)$ (by the above result)

$= L(es)$ (since $I[t] = 1$ when $[t] = [e]$ and R_L is a right invariant relation, $[ts] = [es]$)

$= L(s)$

Thus for all s all $s \in S$, $L(s) = L(M)(s)$. This proves that $L = L(M)$.

Example To Illustrate The Proof Of Theorem In Section 3

Let $\Sigma = \{0, 1\}$ and $S = \Sigma^*$, the set of all strings over the alphabet Σ. Consider the fuzzy automaton $M = (Q, f, I, F)$ where $Q = \{q_0, q_1, q_2\}$, f is the fuzzy subset $f: Q \times \Sigma \times Q \rightarrow [0, 1]$ defined as

$f(q_0, 0, q_0) = 0$	$f(q_0, 0, q_1) = 0.5$,	$f(q_0, 0, q_2) = 0.6$
$f(q_1, 0, q_0) = 0.0$,	$f(q_1, 0, q_1) = 0.0$,	$f(q_1, 0, q_2) = 0.7$
$f(q_2, 0, q_0) = 0$,	$f(q_2, 0, q_1) = 0$,	$f(q_2, 0, q_2) = 0.0$
$f(q_0, 1, q_0) = 0$,	$f(q_0, 1, q_1) = 0.3$,	$f(q_0, 1, q_2) = 0.4$
$f(q_1, 1, q_0) = 0$,	$f(q_1, 1, q_1) = 0$,	$f(q_1, 1, q_2) = 1$
$f(q_2, 1, q_0) = 0$,	$f(q_2, 1, q_1) = 0$,	$f(q_2, 1, q_2) = 0$

$I = \{q_0\}$ and F is the fuzzy subset of Q defined as $F(q_1) = 0.3$ and $F(q_2) = 0.7$.

It is found that $f^*(q, a, p) = f(q, a, p)$ for any $a \in A$, $q, p \in Q$.

For any string $w = sa$ of length two or more we will calculate $f^*(q_i, w, q_j)$ as follows:

$$f^*(q, sa, p) = \bigwedge_{r\in Q} [f^*(q, s, r) \wedge f(r, a, p)] \quad (s \in S, a \in A, q_i, q_j \in Q)$$

where $f^*(q, s, r) > 0$ and $f(r, a, p) > 0$

After computing f^*- matrix for a given string s, we will compute L(M)(s) as follows:

$$L(s) = I \text{ o } f_0^* \text{ o } F$$
$$= \bigwedge_{p\in Q} [I(p) \wedge (f_s^* \text{ o } F)(p)] \text{ where } I(p) > 0, (f_s^* \text{ o } F)(p) > 0.$$
$$= [I(q_0) \wedge (f_s^* \text{ o } F)(q_0)] \text{ where } (f_s^* \text{ o } F)(q_0) > 0.$$
$$= [(f_s^* \text{ o } F)(q_0)] \text{ where } (f_s^* \text{ o } F)(q_0) > 0.$$
$$= \wedge [F(r) \wedge f^*_s(q_0, r)] \text{ where } F(r) > 0.$$
$$= [F(q_1) \wedge f^*_s(q_0, q_1)] \wedge [F(q_2) \wedge f^*_s(q_0, q_2)].$$
$$= [0.3 \wedge f^*_s(q_0, q_1)] \wedge [0.7 \wedge f^*_s(q_0, q_2)].$$

Therefore, $L(s) = [0.3 \wedge f^*_s(q_0, q_1)] \wedge [0.7 \wedge f^*_s(q_0, q_2)]$ for any $s \in S$ where $f^*_s(q_0, q_1) > 0$ and $f^*_s(q_0, q_2) > 0$. (1)

Using (1) we calculate

$L(0) = [0.3 \wedge f^*_s(q_0, q_1)] \wedge [0.7 \wedge f^*_s(q_0, q_2)]$

$= [\ 0.3 \wedge f^{*}(q_0, 0, q_1)\ ] \wedge [\ 0.7 \wedge f^{*}(q_0, 0, q_2)\ ]$
$= [\ 0.3 \wedge 0.5\ ] \wedge [\ 0.7 \wedge 0.6\ ] = 0.3$
$L(1) = [\ 0.3 \wedge f^{*}(q_0, 1, q_1)\ ] \wedge [\ 0.7 \wedge f^{*}(q_0, 1, q_2)\ ]$
$= [\ 0.3 \wedge 0.3\ ] \wedge [\ 0.7 \wedge 0.4\ ] = 0.3$
$L(00) = \wedge [\ 0.7 \wedge f^{*}(q_0, 00, q_2)\ ]$ as $f^{*}(q_0, 00, q_1) = 0$.
$= \wedge [\ 0.7 \wedge 0.5\ ] = 0.5$
Similarly, we get L(01) = 0.5, L(10) = 0.3, L(11) = 0.3, L(011) = 0.4.
For all other strings $s \in S$, L (0) = 0.
Thus we have
L(0) = L(1) = L(10) = L(11) = 0.3
L(011) = 0.4
L(00) = L(01) = 0.5
The possible values of δ_i (after arranging them in nondecreasing order) are 0.3, 0.4, and 0.5.
Suppose $0 < \alpha \leq 0.3$
Let $D_\alpha(M) = M_\alpha$ denotes the nondeterministic automaton corresponding to α.
Then $I_\alpha = \{ q_0 \}$, $F_\alpha = \{ q_1, q_2 \}$, $d_\alpha(q_0, s) = \{ p \in Q / f_s^{*}(q_0, p) \geq 0.3 \}$
Now we calculate
$d_\alpha(q_0, 0) = \{ p \in Q$ ⍰ $f^{*}(q, 0, p) \geq 0.3 \} = \{ q_1, q_2 \}$
$d_\alpha(q_0, 1) = \{ p \in Q$ ⍰ $f^{*}(q, 1, p) \geq 0.3 \} = \{ q_1, q_2 \}$
$d_\alpha(q_0, 00) = \{ p \in Q$ ⍰ $f^{*}(q, 00, p) \geq 0.3 \} = \{ q_2 \}$
$d_\alpha(q_0, 01) = \{ p \in Q$ ⍰ $f^{*}(q, 01, p) \geq 0.3 \} = \{ q_2 \}$
$d_\alpha(q_0, 10) = \{ p \in Q$ ⍰ $f^{*}(q, 10, p) \geq 0.3 \} = \{ q_2 \}$
$d_\alpha(q_0, 11) = \{ p \in Q$ ⍰ $f^{*}(q, 11, p) \geq 0.3 \} = \{ q_2 \}$
$d_\alpha(q_0, 011) = \{ p \in Q$ ⍰ $f^{*}(q, 11, p) \geq 0.3 \} = \{ q_2 \}$
Now we calculate $L(D_\alpha(M))$ as follows
$L(D_\alpha(M)) = \{s \in S$ ⍰ there exists $q \in I_\alpha$ such that $(d_\alpha(q, s) \cap F_\alpha) \neq \phi\}$
$= \{s \in S$ ⍰ there exists $q \in I_{0.3}$ such that $(d_{0.3}(q, s) \cap F_{0.3}) \neq \phi\}$
Thus for any $s \in S$,
$L(D_{0.3}(M)) = \{s \in S$ ⍰ there exists $q \in I_{0.3}$ such that $(d_{0.3}(q, s) \cap F_{0.3}) \neq \phi\}$
$= \{0, 1, 00, 01, 10, 11, 011\}$
$L_{0.3} = \{0, 1, 00, 01, 10, 11\}$
Therefore, $\mathbf{L_{0.3} \subseteq L(D_{0.3}(M))}$
Furthermore, $[0]_{0.3} = \{0, 1\}$
$[00]_{0.3} = \{00, 01, 10, 11\}$
$\cup [s]_{0.3} = [0]_\alpha \cup [1]_\alpha = \{0, 1, 00, 01, 10, 11\} = L_{0.3}$
Suppose $0.3 < \alpha \leq 0.4$
Then $I_\alpha = \{ q_0 \}$, $F_\alpha = \{ q_2 \}$, $d_\alpha(q_0, s) = \{ p \in Q / f_s^{*}(q_0, p) \geq 0.3 \}$
Now we calculate
$d_\alpha(q_0, 0) = \{ p \in Q$ ⍰ $f^{*}(q, 0, p) \geq 0.4 \} = \{ q_2 \}$
$d_\alpha(q_0, 1) = \{ p \in Q$ ⍰ $f^{*}(q, 1, p) \geq 0.4 \} = \{ q_2 \}$
$d_\alpha(q_0, 00) = \{ p \in Q$ ⍰ $f^{*}(q, 00, p) \geq 0.4 \} = \{ q_2 \}$
$d_\alpha(q_0, 01) = \{ p \in Q$ ⍰ $f^{*}(q, 01, p) \geq 0.4 \} = \phi$

$d_\alpha(q_0, 10) = \{ p \in Q \mid f^*(q, 10, p) \geq 0.4 \} = \phi$
$d_\alpha(q_0, 11) = \{ p \in Q \mid f^*(q, 11, p) \geq 0.4 \} = \{ q_2 \}$
$d_\alpha(q_0, 011) = \{p \in Q \mid f^*(q, 011, p) \geq 0.3\} = \{ q_2 \}$
$L(D_\alpha(M)) = \{s \in S \mid \text{there exists } q \in I_\alpha \text{ such that } (d_\alpha(q, s) \cap F_\alpha) \neq \phi\}$
$= \{s \in S \mid \text{there exists } q \in I_{0.4} \text{ such that } (d_{0.4}(q, s) \cap F_{0.4}) \neq \phi\}$
Thus for any $s \in S$,
$L(D_{0.4}(M)) = \{s \in S \mid \text{there exists } q \in I_{04} \text{ such that } (d_{0.4}(q, s) \cap F_{0.4}) \neq \phi\}$
$= \{0, 1, 00, 01, 10, 11, 011\}$
$L_{0.4} = \{1, 011\}$
Therefore, $\mathbf{L_{0.4} \subseteq L(D_{0.4}(M))}$
Furthermore, $[0]_{0.4} = \{0, 011\}$
$\cup [s]_{0.4} = [0]_{0.4} = \{0, 011\} = \{0, 011\} = L_{0.4}$
Suppose $0.4 < \alpha \leq 0.5$
Then $I_\alpha = \{ q_0 \}$, $F_\alpha = \{ q_2 \}$, $d_\alpha(q_0, s) = \{ p \in Q \mid f_s^*(q_0, p) \geq 0.5 \}$
Now we calculate
$d_\alpha(q_0, 0) = \{ p \in Q \mid f^*(q, 0, p) \geq 0.5 \} = \{ q_2 \}$
$d_\alpha(q_0, 1) = \{ p \in Q \mid f^*(q, 1, p) \geq 0.5 \} = \phi$
$d_\alpha(q_0, 00) = \{ p \in Q \mid f^*(q, 00, p) \geq 0.5 \} = \{ q_2 \}$
$d_\alpha(q_0, 01) = \{ p \in Q \mid f^*(q, 01, p) \geq 0.5 \} = \{ q_2 \}$
$d_\alpha(q_0, 10) = \{ p \in Q \mid f^*(q, 10, p) \geq 0.5 \} = \phi$
$d_\alpha(q_0, 11) = \{ p \in Q \mid f^*(q, 11, p) \geq 0.5 \} = \phi$
$d_\alpha(q_0, 011) = \{p \in Q \mid f^*(q, 11, p) \geq 0.5 \} = \phi$
$L(D_{0.5}(M)) = \{s \in S \mid \text{there exists } q \in I_{0.5} \text{ such that } (d_{0.5}(q, s) \cap F_{0.5}) \neq \phi\}$
$= \{0, 00, 01\}$
$L_{0.5} = \{ 0, 00, 01 \}$
Therefore, $\mathbf{L_{0.5} = L(D_{0.5}(M))}$
Furthermore, $[0]_{0.5} = \{0, 00, 01\}$
$\cup [s]_{0.5} = [1]_{0.5.} = \{0, 00, 01\} = L_{0.5}$
For all $\alpha > 0.5$, then there exists no corresponding nondeterministic automaton and $L(D_\alpha(M)) = L_\alpha = \phi$.
When $\alpha = 0.3$, $L_{0.3} = \{0, 1, 00, 01, 10, 11\}$
$\alpha_L(s) = \alpha$ if $L(s) \geq \alpha$, 0 otherwise.
$0.3_L(0) = 0.3_L(1) = 0.3_L(10) = 0.3_L(11) = 0.3$
When $\alpha = 0.4$, $L_{0.4} = \{1, 011\}$
$0.4_L(011) = 0.4$
When $\alpha = 0.5$, $L_{0.5} = \{0, 00, 01\}$
$0.5_L(0) = 0.5_L(00) = 0.5_L(01) = 0.5$
$(\cup \alpha_L)(0) = \vee \alpha_L(0) = 0.3 \vee 0.5 = 0.3$
$(\cup \alpha_L)(1) = \vee \alpha_L(1) = 0.3$
$(\cup \alpha_L)(00) = \vee \alpha_L(00) = 0.5$
$(\cup \alpha_L)(01) = \vee \alpha_L(01) = 0.5$
$(\cup \alpha_L)(10) = \vee \alpha_L(10) = 0.3$
$(\cup \alpha_L)(11) = \vee \alpha_L(11) = 0.3$

$(\cup\, \alpha_L)(011) = \square\ \alpha_L(011) = 0.4$

$\cup\, \alpha_L = 0.3_L \cup 0.4_L \cup 0.5_L$

$= \{0,1, 00, 01, 10, 11, 011\}$

$= L$

This verifies $L = \cup\, \alpha_L$

Results and Conclusions

In this paper, Myhill Nerode theorem of finite automaton has been extended to fuzzy automaton where the composition considered is min-max composition. In min-min composition, it is found that L_α need is contained in L (D_α (M)). Anyway, we have been able to prove the analogue of Myhill Nerode Theorem for fuzzy automata even for min-min composition. It is found and proved that the Myhill Nerode Theorem holds good for fuzzy automata with mini-min composition with a condition that I (p), F (r) , f^*(p, r) be greater than zero for any string accepted by the automaton. The given example illustrates the proof of the theorem.

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