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Duality Theorems For Infinite Fourier Cosine And Sine Transforms

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Abstract

This paper presents two new properties called Duality Theorems, exclusively each for the Infinite Fourier Cosine and Fourier Sine Transforms respectively, which were hitherto not mentioned or given elsewhere in the literature. These two integral transforms which are derived from the Fourier Integral and the Infinite Fourier Transform are well known in the mathematical and engineering fraternity and are used in a variety of applications. The Infinite Fourier Cosine and Sine Transforms have their own interesting properties. In this paper, we present formal derivations of the Duality Theorems for these two transforms. The usage of the Duality Theorems help in finding the time – domain function from the Infinite Fourier Cosine and Sine Transforms' (frequency) domain and vice versa thereby reducing considerable labour and computation time involved in Integration, if a known Infinite Fourier Sine and Cosine Transform pair is known. The usages of the Duality Theorems have been exemplified with suitable illustrative examples.

Keywords: kernel, duality, transform, analysis, synthesis.

Introduction

The Fourier Transform was invented by Jean – Baptiste Joseph Fourier (1768 – 1830), a French mathematician and physicist during the early part of the 18th century. It is perhaps the most important Integral Transform that is studied and researched even now, next being the Laplace Transform and recently the Wavelet Transform [12].

The Fourier transform decomposes a time – domain function into frequencies that make it up. The Fourier transform of a time – domain function is a complex – valued function of frequency and hence, it corresponds to the frequency domain representation of the original signal. The term Fourier Transform refers to both the frequency domain representation and the mathematical operation that associates the frequency domain representation to a function of time [1]. The Infinite Fourier Transform and its inverse are derived from the famous Fourier Integral or the Fourier Integral formula or the Fourier Integral Expansion, which is in turn developed by the Trigonometric Fourier Series. It should be noted that while the Fourier Series hold for a periodic function, the Fourier Transform holds for an aperiodic or nonperiodic signals [2].

In order to get an expression for the Fourier Integral, the function $f(x)$ should satisfy certain criteria called Dirichlet's conditions: -

1. $f(x)$ should be absolutely integrable over a given period [3].
2. $f(x)$ must have a finite number of extrema in any given bounded interval, i.e. there must be a finite number of maxima and minima in the interval.
3. $f(x)$ must have a finite number of discontinuities in any given bounded interval, but the discontinuity cannot be infinite.
4. $f(x)$ must be bounded [4].

The last three conditions are satisfied if $f(x)$ is a function of bounded variation over a period. Based on the above statements, we assume that $f(x)$ satisfies Dirichlet's conditions in every interval $-c \leq x \leq c$, and that the integral $\int_{-\infty}^{\infty} |f(x)| dx < \infty$. Then, $f(x)$ is expressed in the form of a trigonometric Fourier Series in the interval $(-c, c)$ as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(\frac{n\pi x}{c}) + b_n \sin(\frac{n\pi x}{c})] \quad \ldots \quad (1)$$

where, a_0, a_n and b_n are the Fourier coefficients defined as [5]

$$a_0 = \frac{1}{c} \int_{-c}^{c} f(t) dt \quad \ldots \quad (2)$$

$$a_n = \frac{1}{c} \int_{-c}^{c} f(t) \cos(\frac{n\pi t}{c})\, dt, n = 0,1,2, \ldots \quad \ldots \quad (3)$$

$$b_n = \frac{1}{c} \int_{-c}^{c} f(t) \sin(\frac{n\pi t}{c})\, dt, n = 1,2,3, \ldots \quad \ldots \quad (4)$$

Substituting Eqs. (2), (3), and (4) in Eq. (1), and simplifying, we get,

$$f(x) = \frac{1}{2c} \int_{-c}^{c} f(t) dt + \frac{1}{c} \sum_{n=1}^{\infty} \int_{-c}^{c} f(t) [\cos(\frac{n\pi t}{c}) \cos(\frac{n\pi x}{c}) + \sin(\frac{n\pi t}{c}) \sin(\frac{n\pi x}{c})]\, dt$$

$$\Rightarrow f(x) = \frac{1}{2c} \int_{-c}^{c} f(t) dt + \frac{1}{c} \sum_{n=1}^{\infty} \int_{-c}^{c} f(t) \cos\{\frac{n\pi(t-x)}{c}\}\, dt$$

Since it is assumed that $f(x)$ is uniformly convergent in the closed interval having the range, $-c \leq x \leq c$, we get,

$$\Rightarrow f(x) = \frac{2}{c}\int_{-c}^{c} f(t)dt + \frac{1}{c}\int_{-c}^{c} f(t)[\sum_{n=1}^{\infty} cos\,\{\frac{n\pi(t-x)}{c}\}]\ dt$$

$$= \frac{2}{c}\int_{-c}^{c} f(t)[1 + \lim_{n\to\infty}\sum_{r=1}^{n} 2\,cos\,\{\frac{r\pi(t-x)}{c}\}]\,dt$$

$$\Rightarrow f(x) = \frac{1}{2c}\int_{-c}^{c} f(t)[1 + \lim_{n\to\infty}\sum_{r=-n}^{n} cos\,\{\frac{r\pi(t-x)}{c}\}]\,dt$$

$$\Rightarrow f(x) = \frac{1}{2c}\int_{-c}^{c} f(t)dt + \frac{1}{2\pi}\int_{-c}^{c} f(t)[\lim_{n\to\infty}\sum_{r=-n}^{n} \frac{1}{(\frac{c}{\pi})}\cos(\frac{r}{(\frac{c}{\pi})}(t-x))\ dt \quad \ldots (5)$$

Using the definition of integral as a limit of sum, we get,

$$f(x) = \frac{1}{2c}\int_{-c}^{c} f(t)\,dt + \frac{1}{2\pi}\int_{-c}^{c} f(t)[\int_{-\infty}^{\infty} \cos[s(t-x)]\ ds]dt \qquad \ldots \qquad (6)$$

Taking $c \to \infty$, and using the assumption that $f(x)$ is uniformly convergent in the closed interval $-c \leq x \leq c$, we get [6],

$$f(x) = 0 + \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)[\int_{-\infty}^{\infty} \cos[s(t-x)]\,ds]dt$$

Thus, if $-\infty < x < \infty$, we have the Fourier Integral given by

$$f(x) = \frac{1}{2\pi}\int_{s=-\infty}^{\infty}\int_{t=-\infty}^{\infty} f(t)\cos[s(t-x)]]\,dt\,ds \ldots \qquad (7)$$

Eq. (7) is the required Fourier Integral formula or the Fourier Integral Expansion formula.

$$f(x) = \frac{1}{2\pi}\int_{s=-\infty}^{\infty} \cos[s(t-x)]\ [\int_{t=-\infty}^{\infty} f(t)dt]ds$$

The function, $\cos[s(t-x)]$ is an even function. Hence, the above integral can be simplified and written as,

$$f(x) = \frac{1}{\pi}\int_{s=0}^{\infty}\int_{t=-\infty}^{\infty} f(t)\,\cos[s(t-x)]]\,dt\,ds \quad \ldots \qquad (8)$$

This is another form of Fourier Integral formula. We can put Eq. (8) in many forms. Expanding the $\cos[.]$ term in Eq. (8) using Euler's formula yields,

$$f(x) = \frac{1}{2\pi}\int_{s=0}^{\infty}\int_{t=-\infty}^{\infty} f(t)[e^{-is(t-x)} + e^{+is(t-x)}]\,dt\,ds$$

$$\Rightarrow f(x) = \frac{1}{2\pi}\int_{s=0}^{\infty}\int_{t=-\infty}^{\infty} f(t)e^{-is(t-x)}\,dt\,ds + \frac{1}{2\pi}\int_{s=0}^{\infty}\int_{t=-\infty}^{\infty} f(t)e^{+is(t-x)}\,dt\,ds (9)$$

Putting $s = -s'$ in the second integral of Eq. (9) and simplifying, we get,

$$f(x) = \frac{1}{2\pi}\int_{s=0}^{\infty}\int_{t=-\infty}^{\infty} f(t)e^{-is(t-x)}\,dt\,ds + \frac{1}{2\pi}\int_{s'=0}^{-\infty}\int_{t=-\infty}^{\infty} f(t)e^{-is'(t-x)}\,dt\,(-ds')$$

$$\Rightarrow f(x) = \frac{1}{2\pi}\int_{s=0}^{\infty}\int_{t=-\infty}^{\infty} f(t)e^{-is(t-x)}\,dt\,ds + \frac{1}{2\pi}\int_{s=-\infty}^{0}\int_{t=-\infty}^{\infty} f(t)e^{-is(t-x)}\,dt\,ds$$

Combining the above two integrals into one gives,

$$f(x) = \frac{1}{2\pi}\int_{s=-\infty}^{\infty}\int_{t=-\infty}^{\infty} f(t)e^{-is(t-x)}\,dt\,ds \qquad \ldots \qquad (10)$$

Eq. (10) is the exponential form of the Fourier Integral Formula. Now, substituting $s = -s''$ in Eq. (10) and simplifying, we get,

$$f(x) = \frac{1}{2\pi}\int_{s''=\infty}^{-\infty}\int_{t=-\infty}^{\infty} f(t)e^{+is''(t-x)}\,dt\,(-ds'')$$

$$f(x) = \frac{1}{2\pi}\int_{s=-\infty}^{\infty}\int_{t=-\infty}^{\infty} f(t)e^{+is(t-x)}\,dt\,ds \qquad \ldots \qquad (11)$$

Eq. (11) is another alternative form of the exponential form of the Fourier Integral Formula. From Eqs. (10) and (11), it is clear that the negative sign in the exponent, i.e., $-ist$, can be shifted to $+isx$, without affecting the value of the integral [7].

Furthermore, Eq. (11) can be written as

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{s=-\infty}^{\infty}[\frac{1}{\sqrt{2\pi}}\int_{t=-\infty}^{\infty} f(t)e^{ist}dt]e^{-isx}\,ds \ \ldots \qquad (12)$$

The inner integral is taken to be the Infinite Fourier Transform or simply the Fourier Transform denoted as $F[f(x)] = F(s)$. Then the outer integral becomes the Inverse Infinite Fourier Transform or the Inverse Fourier Transform. Hence, we can formally define the Complex or Infinite or Continuous – Time Fourier Transform of a function $f(x)$ as

$$F(s) = \mathfrak{I}[f(x)]] \triangleq \frac{1}{\sqrt{2\pi}}\int_{x=-\infty}^{\infty} f(x)e^{+isx}\,dx \qquad \ldots \qquad (13)$$

The term e^{+isx} is called the kernel of the Infinite Fourier Transform. Fourier Transform is called the Complex Fourier Transform because the nature of the kernel, e^{+isx} is by complex. Note that $\mathfrak{I}$ denotes the Infinite Fourier Transform operator. The variable s designates frequency – domain or the Fourier – domain variable.

The Inverse Fourier Transform of $F(s)$ is formally given by the expression,

$$f(x) = \mathfrak{I}^{-1}[F(s)] \triangleq \frac{1}{\sqrt{2\pi}}\int_{s=-\infty}^{\infty} F(s)e^{-isx}\,ds \qquad \ldots \qquad (14)$$

Eq. (13) is called the Analysis Equation or the Forward Transform of the Infinite Fourier Transform. Eq. (14) is called the Synthesis Equation or the Backward Transform of the Infinite Fourier Transform. Note that $\mathfrak{I}^{-1}$ signifies the Inverse Fourier Transform operator. The Inverse Fourier Transformation is also called Fourier Synthesis because this transformation synthesizes a frequency domain representation, and combines the contributions of all the different frequencies to recover the original function of time [8]. Then, we say that $f(x)$ and $F(s)$ form a Fourier Transform pair. Mathematically, this is represented as,

$$f(x) \overset{FT}{\leftrightarrow} F(s) \ \ldots \qquad (15)$$

Since $F(s)$ is complex, we can write it in terms of its real and imaginary parts.

$$F(s) = F_R(s) + iF_I(s)$$

Also, the magnitude of the Infinite Fourier Transform, $|F(s)|$ is an even function of s. The phase component of it is an odd function of s. These can be written mathematically as

$$|F(s)| = |F(-s)|$$

The magnitude or absolute value component of the Infinite Fourier Transform can be written in terms of the real and imaginary parts of $F(s)$ by the expression,

$$\circ\ |F(s)| = \sqrt{F_R^2(s) + F_I^2(s)}$$

The phase component of the Infinite Fourier Transform can be written in terms of the real and imaginary parts of $F(s)$ by the expression,

$$\Theta(s) = \tan^{-1}[\frac{F_I(s)}{F_R(s)}]$$

Thus, the magnitude component represents the amount of that frequency present in the original function, and the phase component represents the offset of the basic sinusoid in that frequency.

A. Infinite Fourier Cosine Transform: -

Expanding the $\cos[.]$ in Eq. (8), we get

$$f(x) = \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty f(t)\,[\cos st \cos sx + \sin st \sin sx]\, dt\, ds$$

Taking $A(s) = \int_{-\infty}^\infty f(t) \cos st\, dt$ and $B(s) = \int_{-\infty}^\infty f(t) \sin st\, dt$, we get,

$$f(x) = \frac{1}{\pi}\int_0^\infty [A(s)\cos sx + B(s)\sin sx] ds \qquad \ldots \qquad (16)$$

Let $f(t)$ be an even function of t so that $f(-t) = f(t)$. Then, we have $f(t)\cos st$ to be an even function of t and $f(t)\sin st$ to be an odd function of t respectively. Consequently, we get,

$A(s) = 2\int_0^\infty f(t)\cos st\, dt$ and $B(s) = 0$.

Using these values in Eq. (16), we get,

$$f(x) = \frac{1}{\pi}\int_0^\infty A(s)\cos sx\, ds = \frac{2}{\pi}\int_0^\infty \int_0^\infty f(t)\cos st \cos sx\, ds\, dt \ \ldots \qquad (17)$$

Eq. (17) is the Cosine form of the Fourier Integral Expansion formula. It can be written further as

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty \cos sx\ [\sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\cos st\, dt]\ ds \ \ldots \qquad (18)$$

The inner integral in Eq. (18) is taken to be the Infinite Fourier Cosine Transform and the outer integral is the Inverse Infinite Fourier Cosine Transform [9].

Thus, the Infinite Fourier Cosine Transform of a function $f(x)$ is formally defined as

$$F_c(s) = \Im_c[f(x)] \triangleq \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos(sx)\,dx \quad \dots \qquad (19)$$

Here, $\Im_c$ is called the Infinite Fourier Cosine Transform operator.

The Inverse Infinite Fourier Cosine Transform of a function $F_c(s)$ is defined as

$$f(x) = \Im_c^{-1}[F_c(s)] \triangleq \sqrt{\frac{2}{\pi}}\int_0^\infty F_c(s)\cos(sx)\,ds \quad \dots \qquad (20)$$

Eq. (19) is called the Analysis Equation or the Forward Transform of the Infinite Fourier Cosine Transform. Eq. (20) is called the Synthesis Equation or the Backward Transform of the Infinite Fourier Cosine Transform. $\Im_c^{-1}$ denotes the Inverse Infinite Fourier Cosine Transform operator. Then, we say that $f(x)$ and $F_c(s)$ form an Infinite Fourier Cosine Transform (IFCT) pair [10]. Mathematically, this is represented as,

$$f(x) \overset{IFCT}{\longleftrightarrow} F_c(s) \dots \qquad (21)$$

The Infinite Fourier Cosine Transform, unlike the Infinite Fourier Transform, is a real transform.

B. Infinite Fourier Sine Transform:-

From Eq. (16), we have,

$$f(x) = \frac{1}{\pi}\int_0^\infty [A(s)\cos sx + B(s)\sin sx]ds \quad \dots \qquad (22)$$

Let $f(t)$ be an odd function of t so that $f(-t) = -f(t)$. Then, we have $f(t)\cos st$ to be odd function of t and $f(t)\sin st$ to be an even function of t respectively. Consequently, we get,

$A(s) = 0$ and $B(s) = 2\int_0^\infty f(t)\sin st\ dt.$

Using these values in Eq. (22), and simplifying, we get,

$$f(x) = \frac{2}{\pi}\int_0^\infty\int_0^\infty f(t)\sin st\sin sx\ ds\ dt \quad \dots \qquad (23)$$

Eq. (23) is the Sine form of the Fourier Integral Expansion formula. It can be written further as

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty \sin sx\ \left[\sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\sin st\ dt\right]\ ds \quad \dots \qquad (24)$$

The inner integral in Eq. (24) is taken to be the Infinite Fourier Sine Transform and the outer integral is the Inverse Infinite Fourier Sine Transform [11].

Thus, the Infinite Fourier Sine Transform of a function $f(x)$ is formally defined as

$$F_s(s) = \Im_s[f(x)] \triangleq \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin(sx)\,dx \quad \dots \qquad (25)$$

Here, $\Im_s$ is called the Infinite Fourier Sine Transform operator.

The Inverse Infinite Fourier Sine Transform of a function $F_s(s)$ is defined as

$$f(x) = \Im_s^{-1}[F_s(s)] \triangleq \sqrt{\frac{2}{\pi}}\int_0^\infty F_s(s)\sin(sx)\,ds \quad \dots \qquad (26)$$

Eq. (25) is called the Analysis Equation or the Forward Transform of the Infinite Fourier Sine Transform. Eq. (26) is called the Synthesis Equation or the Backward Transform of the Infinite Fourier Sine Transform. $\Im_s^{-1}$ denotes the Inverse Infinite Fourier Sine Transform operator. Then, we say that $f(x)$ and $F_s(s)$ form an Infinite Fourier Sine Transform (IFST) pair. Mathematically, this is represented as,

$$f(x) \overset{IFST}{\longleftrightarrow} F_s(s) \ \dots \qquad (27)$$

The Infinite Fourier Sine Transform, unlike the Infinite Fourier Transform, is a real transform.

Duality Theorem For Infinite Fourier Cosine Transform

Statement: - If $f(x)$ and $F_c(s)$ form an Infinite Fourier Cosine Transform pair, i.e., if $F_c(s) = \Im_c[f(x)]$, then, $f_c(s) = \Im_c[F(x)]$. It means that i.e., $F(x)$ and $f_c(s)$ form an Infinite Fourier Cosine Transform pair.

Proof: - By definition, it follows that the Infinite Fourier Cosine Transform of a function $f(x)$ is given by

$$F_c(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\cos(sx)\,dx \quad \dots \qquad (28)$$

Likewise, the Inverse Infinite Fourier Cosine Transform of $F_c(s)$ is given by the relation,

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_c(s)\cos(sx)\,ds \quad \dots \qquad (29)$$

Replacing x by s and vice versa in the above equation, Eq. (29), we get,

$$f(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_c(x)\cos(sx)\,dx = \sqrt{\frac{2}{\pi}} \int_0^\infty F(x)\cos(sx)\,dx \qquad \dots \qquad (30)$$

where in the preceding integral, we have written $F_c(x) = F(x)$. Comparing Eq. (30) with Eqs. (19) and (20), we see that the RHS of it is nothing but the Infinite Fourier Cosine Transform of $F(x)$, with the time – and frequency – variables, i.e., x and s variables being interchanged respectively. Thus, we can formally write the Duality Theorem of the Infinite Fourier Cosine Transform in a compact form.

$$f_c(s) = \Im_c[F(x)] \ \dots \qquad (31)$$

Eq. (31) is the final step in the derivation of the Duality Theorem for Infinite Fourier Cosine Transform.

Duality Theorem For Infinite Fourier Sine Transform

Statement: - If $f(x)$ and $F_s(s)$ form an Infinite Fourier Sine Transform pair, i.e., if $F_s(s) = \Im_s[f(x)]$, then, $f_s(s) = \Im_s[F(x)]$. It means that i.e., $F(x)$ and $f_s(s)$ form an Infinite Fourier Sine Transform pair.

Proof: - By definition, it follows that the Infinite Fourier Sine Transform of a function $f(x)$ is given by

$$F_s(s) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin(sx)\,dx \quad \dots \tag{32}$$

Likewise, the Inverse Infinite Fourier Sine Transform of $F_s(s)$is given by the relation,

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty F_s(s)\sin(sx)\,ds \quad \dots \tag{33}$$

Replacing x by s and vice versa in the above equation, Eq. (33), we get,

$$f(s) = \sqrt{\frac{2}{\pi}}\int_0^\infty F_s(x)\sin(xs)\,dx = \sqrt{\frac{2}{\pi}}\int_0^\infty F(x)\sin(sx)\,dx \quad \dots \tag{34}$$

where in the preceding integral, we have written $F_s(x) = F(x)$. Comparing Eq. (34) with Eqs. (25) and (26), we see that the RHS of it is nothing but the Infinite Fourier Sine Transform of $F(x)$, with the time – and frequency – variables, i.e., x and s variables being interchanged respectively. Thus, we can formally write the Duality Theorem of the Infinite Fourier Sine Transform in a compact form.

$$f_s(s) = \mathfrak{I}_s[F(x)] \quad \dots \tag{35}$$

Eq. (35) is the final step in the derivation of the Duality Theorem for Infinite Fourier Sine Transform.

We shall verify the above theorem with three illustrative examples in the next section.

Illustrations of The Duality Theorems For Infinite Fourier Sine and Cosine Transforms With Suitable Examples

A. Duality Theorem for Infinite Fourier Cosine Transform:

Let us consider the computation of the Infinite Fourier Cosine Transform of the function,

$$f(x) = e^{-x^2} \quad \dots \tag{36}$$

Substituting the value of $f(x)$ in Eq. (19) yields

$$\varphi_c(s) = \mathfrak{I}_c[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x^2}\, cos\, sx\, dx \quad \dots \tag{37}$$

Differentiating Eq. (37) w.r.t s on both sides using Leibnitz's rule of differentiation under the integral sign, we get,

$$\varphi_c^{'}(s) = \frac{d}{ds}\left[\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x^2}\cos sx\, dx\right] = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x^2}\frac{\partial}{\partial s}\{\cos sx\}\, dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x^2}\{-x.\sin sx\}\, dx$$

Multiplying and dividing the RHS of the above integral by 2, we get,

$$\varphi_c^{'}(s) = \frac{1}{\sqrt{2\pi}}\int_0^\infty [e^{-x^2}\{-2x\}].\sin sx\, dx$$

Applying integration by parts, and noting that $\int e^{-x^2}.(-2x)dx = e^{-x^2}$, we get,

$$\varphi_c^{'}(s) = \frac{1}{\sqrt{2\pi}}[\sin sx\, e^{-x^2}]_0^\infty - s[\frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-x^2}.\cos sx\, dx]$$

Thus, we get, $\varphi_c^{'}(s) = -\frac{1}{2}s\varphi_c(s)$

$\Rightarrow \frac{\varphi_c^{'}(s)}{\varphi_c(s)} = -\frac{1}{2}s$ … (38)

Integrating both sides of Eq. (38) w.r.t. s, we get,

$\log_e \varphi_c(s) = -\frac{s^2}{4} + \log_e C_2$ … (39)

Here, $\log_e C_2$ is a constant of Integration.
Simplifying, we can write Eq. (39) as

$\varphi_c(s) = C_2 e^{-\frac{s^2}{4}}$ … (40)

If $s = 0$, Eq. (40) gives,

$C_2 = \varphi_c(0)$ … (41)

Also, from Eq. (37), $\varphi_c(0) = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x^2}dx$

From the theory of Gamma Functions, we know that $\int_0^\infty e^{-x^n}dx = \frac{1}{n}\Gamma(n)$ … (42)

Substituting $n = 2$ in Eq. (42), we get,

$\varphi_c(0) = \sqrt{\frac{2}{\pi}}\frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \sqrt{\frac{2}{\pi}}\frac{\sqrt{\pi}}{2} = \frac{1}{\sqrt{2}}$ … (43)

From Eqs. (41) and (43), it follows that

$C_2 = \varphi_c(0) = \frac{1}{\sqrt{2}}$ … (44)

Substituting Eq. (44) in Eq. (40), we get,

$\varphi_c(s) = \mathfrak{I}_c\left[e^{-x^2}\right] = \frac{1}{\sqrt{2}}e^{-\frac{s^2}{4}}$ … (45)

Next, consider the computation of the Inverse Infinite Fourier Cosine Transform of

$$\varphi_c(s) = e^{-s^2} \qquad \ldots \qquad (46)$$

$$f(x) = \mathfrak{I}_c^{-1}[\varphi_c(s)] = \mathfrak{I}_c^{-1}[e^{-s^2}] = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-s^2}\, cos\, sx\, ds \qquad \ldots \qquad (47)$$

Differentiating Eq. (47) w.r.t x on both sides using Leibnitz’s rule of differentiation under the integral sign, we get,

$$f'(x) = \frac{d}{dx}\left[\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-s^2}\cos sx\, ds\right] = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-s^2}\,\frac{\partial}{\partial x}\{\cos sx\}\, ds$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-s^2}\,\{-s.\sin sx\}\, ds$$

Multiplying and dividing the RHS of the above integral by 2, we get,

$$f'(x) = \frac{1}{\sqrt{2\pi}}\int_0^\infty [e^{-s^2}\{-2s\}].\sin sx\, ds$$

Applying integration by parts, and noting that $\int[e^{-s^2}.(-2s)ds = e^{-s^2}$, we get,

$$f'(x) = \frac{1}{\sqrt{2\pi}}[\sin sx\, e^{-s^2}]_0^\infty - x[\frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-s^2}.\cos sx\, ds]$$

Thus, we get,

$$\frac{f'(x)}{f(x)} = -\frac{1}{2}x \qquad \ldots \qquad (48)$$

Integrating both sides of Eq. (48) w.r.t. x, we get,

$$\log_e f(x) = -\frac{x^2}{4} + \log_e C_3 \qquad \ldots \qquad (49)$$

Here, $\log_e C_3$ is a constant of Integration.
Simplifying, we can write Eq. (49) as

$$f(x) = C_3 e^{-\frac{x^2}{4}} \qquad \ldots \qquad (50)$$

If $x = 0$, Eq. (50) gives,

$$C_3 = f(0) \quad \ldots \qquad (51)$$

Also, from Eq. (47), $f(0) = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-s^2} cos\, 0s\, ds = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-s^2} ds \qquad \ldots \qquad (52)$

From the theory of Gamma Functions, we know that $\int_0^\infty e^{-\alpha^n} d\alpha = \frac{1}{n}\Gamma(n) \ldots$ (53)

Substituting $n = 2$ in Eq. (6) gives,

$$f(0) = \sqrt{\frac{2}{\pi}}\frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \sqrt{\frac{2}{\pi}}\frac{\sqrt{\pi}}{2} = \frac{1}{\sqrt{2}} \qquad \ldots \qquad (54)$$

From Eqs. (51) and (54), we get,

$$C_3 = f(0) = \frac{1}{\sqrt{2}} \qquad \ldots \qquad (55)$$

Substituting Eq. (55) in Eq. (50), we get,

$$f(x) = \frac{1}{\sqrt{2}} e^{-\frac{x^2}{4}} \qquad \ldots \qquad (56)$$

Next, we apply the Duality Property just derived for finding the Inverse Infinite Fourier Cosine Transform of

$$\varphi_c(s) = e^{-s^2} \qquad \ldots \qquad (57)$$

It has been established from Eq. (45) that the Infinite Fourier Cosine Transform of $f(x) = e^{-x^2}$ is

$$\varphi_c(s) = \frac{1}{\sqrt{2}} e^{-\frac{s^2}{4}} \qquad \ldots \qquad (58)$$

That is,

$$e^{-x^2} \overset{IFCT}{\longleftrightarrow} \frac{1}{\sqrt{2}} e^{-\frac{s^2}{4}} \qquad \ldots \qquad (59)$$

Now, replacing x by s and vice versa in the Eq. (59) yields the following expression,

$$e^{-s^2} \overset{IFCT}{\longleftrightarrow} \frac{1}{\sqrt{2}} e^{-\frac{x^2}{4}} \qquad \ldots \qquad (60)$$

That is,

$$f(x) = \Im_c^{-1}[\varphi_c(s)] = \Im_c^{-1}[e^{-s^2}] = \frac{1}{\sqrt{2}} e^{-\frac{x^2}{4}} \qquad \ldots \qquad (61)$$

Eq. (61) is exactly the same as Eq. (56) with the time – and frequency – domain variables interchanged on the L.H.S. and the same expression on the R.H.S. This verifies our Duality Theorem for the Infinite Fourier Cosine Transform.

B. Duality Theorem for Infinite Fourier Sine Transform: -

Let us find the Infinite Fourier Sine Transform of

$$f(x) = \frac{e^{-ax}}{x}, a > 0 \qquad \ldots \qquad (62)$$

Let $\varphi_s(s) = \Im_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \, dx = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-ax}}{x} \sin sx \, dx \ldots$ (63)

Differentiating Eq. (63) w.r.t s using Leibnitz's rule of differentiation under the integral sign, we get,

$$\varphi_s'(s) = \frac{d}{ds}\left[\sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-ax}}{x} \sin sx \, dx\right] = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-ax}}{x} \frac{\partial}{\partial s}(\sin sx) \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \cos sx \, dx$$

$$\Rightarrow \varphi_s^{'}(s) = \sqrt{\frac{2}{\pi}}\,[\frac{e^{-ax}\{-a\cos sx + s\sin sx\}}{(-a)^2+s^2}]_{x=0}^{\infty}$$

Simplifying, we get,

$$\varphi_s^{'}(s) = \sqrt{\frac{2}{\pi}}\cdot\frac{a}{s^2+a^2} \qquad \ldots \qquad (64)$$

Integrating Eq. (64) both sides w.r.t. s, we get,

$$\varphi_s(s) = \sqrt{\frac{2}{\pi}}\tan^{-1}\left(\frac{s}{a}\right) + T \qquad \ldots \qquad (65)$$

Note that T is a constant of Integration in Eq. (65).
If $s = 0$, then Eq. (65) gives,

$$\varphi_s(0) = \sqrt{\frac{2}{\pi}}\tan^{-1}\left(\frac{0}{a}\right) + K = K$$

$$\Rightarrow C = \varphi_s(0) \qquad \ldots \qquad (66)$$

But, from Eq. (63), if $s = 0$, we have,

$$\varphi_s(0) = \sqrt{\frac{2}{\pi}}\int_0^{\infty}\frac{e^{-ax}}{x}\sin 0x\,dx = 0 \ldots \qquad (67)$$

Thus, from Eqs. (66) and (67), we get,

$$C = \varphi_s(0) = 0 \qquad \ldots \qquad (68)$$

Substituting Eq. (68) in Eq. (66), we get,

$$\varphi_s(s) = F_s[\frac{e^{-ax}}{x}] = \sqrt{\frac{2}{\pi}}\tan^{-1}\left(\frac{s}{a}\right) \ldots \qquad (69)$$

Next, we shall find the Inverse Infinite Fourier Sine Transform of

$$\varphi_s(s) = \frac{e^{-as}}{s}, a > 0 \qquad \ldots \qquad (70)$$

Thus, using the Inverse Infinite Fourier Sine Transform formula, we get,

$$f(x) = \Im_s^{-1}[\varphi_s(s)] = \sqrt{\frac{2}{\pi}}\int_0^{\infty}\varphi_s(s)\ \sin sx\,ds = \sqrt{\frac{2}{\pi}}\int_0^{\infty}\frac{e^{-as}}{s}\sin sx\,ds \quad \ldots \qquad (71)$$

Differentiating Eq. (71) w.r.t x using Leibnitz's rule of differentiation under the integral sign, we get,

$$f^{'}(x) = \frac{d}{dx}\left[\sqrt{\frac{2}{\pi}}\int_0^{\infty}\frac{e^{-as}}{s}\sin sx\,ds\right] = \sqrt{\frac{2}{\pi}}\int_0^{\infty}e^{-as}\cos sx\,ds$$

$$\Rightarrow f^{'}(x) = \sqrt{\frac{2}{\pi}}\,[\frac{e^{-as}\{-a\cos sx + x\sin sx\}}{(-a)^2+x^2}]_{x=0}^{\infty}$$

Simplifying, we get,

$$f'(x) = \sqrt{\frac{2}{\pi}} \cdot \frac{a}{x^2+a^2} \quad \ldots \tag{72}$$

Integrating Eq. (72) both sides w.r.t. x, we get,

$$f(x) = \sqrt{\frac{2}{\pi}} \tan^{-1}\left(\frac{x}{a}\right) + T_1 \quad \ldots \tag{73}$$

Note that T_1 is a constant of Integration in Eq. (73).
If $x = 0$, then Eq. (73) gives,

$$f(0) = \sqrt{\frac{2}{\pi}} \tan^{-1}\left(\frac{0}{a}\right) + T_1 = T_1$$

$$\Rightarrow T_1 = f(0) \quad \ldots \tag{74}$$

But, from Eq. (74), if $x = 0$, we have,

$$f(0) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{e^{-as}}{s} \sin 0s \, ds = 0 \quad \ldots \tag{75}$$

Thus, from Eqs. (74) and (75), it follows that

$$T_1 = f(0) = 0 \quad \ldots \tag{76}$$

Substituting Eq. (76) in Eq. (74), we get,

$$f(x) = \sqrt{\frac{2}{\pi}} \tan^{-1}\left(\frac{x}{a}\right) \quad \ldots \tag{77}$$

Next, we apply the Duality Property just derived for finding the Inverse Infinite Fourier Sine Transform of

$$\varphi_s(s) = \frac{e^{-as}}{s}, a > 0 \quad \ldots \tag{78}$$

It has been established from Eq. (45) that the Infinite Fourier Sine Transform of $f(x) = \frac{e^{-ax}}{x}$ is

$$\varphi_s(s) = F_s[\frac{e^{-ax}}{x}] = \sqrt{\frac{2}{\pi}} \tan^{-1}\left(\frac{s}{a}\right) \quad \ldots \tag{79}$$

That is,

$$\frac{e^{-ax}}{x} \overset{IFST}{\longleftrightarrow} \sqrt{\frac{2}{\pi}} \tan^{-1}\left(\frac{s}{a}\right) \quad \ldots \tag{80}$$

Now, replacing x by s and vice versa in the Eq. (80) yields the following expression,

$$\frac{e^{-as}}{s} \overset{IFST}{\longleftrightarrow} \sqrt{\frac{2}{\pi}} \tan^{-1}\left(\frac{x}{a}\right) \quad \ldots \tag{81}$$

That is,

$$f(x) = \mathfrak{F}_s^{-1}[\varphi_s(s)] = \mathfrak{F}_s^{-1}[\frac{e^{-as}}{s}] = \sqrt{\frac{2}{\pi}}\tan^{-1}\left(\frac{x}{a}\right) \dots \qquad (82)$$

Eq. (82) is exactly the same as Eq. (77) with the time – and frequency – domain variables interchanged on the L.H.S. and the same expression on the R.H.S. This verifies our Duality Theorem for the Infinite Fourier Sine Transform.

Conclusions

This paper gives a new property for the Infinite Fourier Cosine and Sine Transforms which has been not been mentioned and derived until now in the literature, although the Duality Theorem for the Infinite Fourier Transform exists and is being used. Given functions whose Infinite Fourier Cosine Transform and Infinite Fourier Sine Transform pairs are known, by using the corresponding but simple Duality Property of the corresponding transforms, we can find the relative Inverse Transforms of the frequency – domain functions whose shape resembles the time – domain function (whose Infinite Fourier Cosine Transform and Infinite Fourier Sine Transform exists). Thus, by doing so, the actual labour of integration involved in finding the Inverse Transform is eradicated thereby saving computation cost and time, which proves the usage of this duality property. The Duality property for these two transforms can be used effectively in applications such as Signal Processing, Communications, Image Processing, and Mechanics.

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