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Non-Resonant And Resonant Planar Oscillation Of The Satellite

Ayub Khan[1] **and Neeti Goel**[2]

[1]*Department of Mathematics, Zakir Hussain Delhi College,*
University of Delhi, Delhi 110023, India
[2]*Department of Mathematics, University of Delhi, Delhi 110007, India*
Email: neeti_2606@hotmail.com

Abstract

The rotational motion of an artificial Earth Satellite has been investigated for the non-resonant and resonant cases. The computational draw of resonant curves portrays the presence of strange attractors.

Keywords: Hamiltonian, Resonance, Chaos.

1. Introduction

The satellite with a magnetic stabilization have been discussed by various researchers because it has a great significance in the space research. The results reported here are mainly confined to the investigation of the resonant and non-resonant cases. Our numerical study emphatically assures the existence of chaotic behaviour.

The equation of motion in the present problem (Fishell[1] and Beletsky[2]) is

$$\frac{d^2\eta}{dv^2}+\alpha\sqrt{1+3\sin^2 v}\,\sin\eta=\frac{6\sin 2v}{(1+3\sin^2 v)^2} \qquad (1)$$

where η is the angle between the magnetic moment vector M of the satellite and the geomagnetic field B, v is the angle between the plane of the Earth's equator and the radius vector of the satellite and α is a dimensionless constant that depends on the magnetic moment of the satellite and the moment of inertia of the satellite relative to the principal axis that is perpendicular to the plane of the orbit.

Assuming $e=\frac{1}{(1+3\sin^2 v)^2}$, $\alpha=\omega^2 e^{1/4}$.

Equation (1) is reduced to

$$\frac{d^2\eta}{dv^2}+\omega^2\sin\eta=6e\sin 2v \tag{2}$$

Adding $\omega^2\eta$ to both sides of eq. (2), we get

$$\frac{d^2\eta}{dv^2}+\omega^2\eta=6e\sin 2v+\omega^2(\eta-\sin\eta) \tag{3}$$

In equation (3), the non-linearty $(\eta-\sin\eta)$ is assumed to be sufficiently weak and hence it may also be assumed to be of order of e. Therefore taking $\omega^2=\beta e$, we get

$$\eta''+\omega^2\eta=e[6\sin 2v+\beta(\eta-\sin\eta)]. \tag{4}$$

For $e=0$, the generating solution of the zeroth order is given by

$$\eta=a\cos\psi,\quad \Psi=\omega v+\Psi^x$$

where amplitude 'a' and the phase 'ψ^x' are constants to be determined by the initial conditions. The solution of the eq. (4) is obtained in the form

$$\eta=a\cos\Psi+eu_1(a,\Psi,v)+e^2u_2(a,\Psi,v)+\ldots \tag{5}$$

where the amplitude 'a' and the phase 'ψ' are determined by the differential equations:

$$\frac{da}{dv}=eA_1(a)+e^2A_2(a)+\ldots \tag{6}$$

$$\frac{d\Psi}{dv}=\omega+eB_1(a)+e^2B_2(a)+\ldots \tag{7}$$

From (5), we calculate η', η''. Substituting them in (4) and ignoring the terms containing fractional powers of e, we get

$$\begin{aligned}
&e\left[\frac{\partial^2u_1}{\partial v_1}+\omega^2\frac{\partial^2u_1}{\partial\Psi_1}-2A_1\omega\sin\Psi-2aB_1\omega\cos\Psi+2\omega\frac{\partial^2u_1}{\partial v\partial\Psi}\right]\\
&+e^2\left[\omega^2\frac{\partial^2u_2}{\partial\Psi^2}+2\omega\frac{\partial^2u_2}{\partial\Psi\partial v}+\frac{\partial^2u_2}{\partial v^2}-\left(aB_1-A_1\frac{\partial A_1}{\partial a}\right)\cos\Psi\right.\\
&-\left(aA_1\frac{\partial B_1}{\partial a}+2A_1B_1\right)\sin\Psi+2\omega B_1\frac{\partial^2u_1}{\partial\Psi^2}+2\omega A_1\frac{\partial^2u_1}{\partial\Psi\partial a}+2B_1\frac{\partial^2u_1}{\partial\Psi\partial v}\\
&\left.+2A_1\frac{\partial^2u_1}{\partial a\partial v}-2\omega A_2\sin\Psi-2\omega aB_2\cos\Psi\right]+\omega^2(eu_1+e^2u_2)+54u_1e^2\sin^2 2v\\
&=6e\sin 2v+\beta e[a\cos\Psi+eu_1]-\beta e\sin(a\cos\Psi+eu_1+\ldots)
\end{aligned} \tag{8}$$

Equating like powers of e, we get

$$\begin{aligned}
&\omega^2\frac{\partial^2u_1}{\partial\Psi^2}+2\omega\frac{\partial^2u_1}{\partial\Psi\partial v}+\frac{\partial^2u_1}{\partial v^2}-2\omega A_1\sin\Psi-2a\omega B_1\cos\Psi+\omega^2u_1\\
&=6\sin 2v+a\beta\cos\Psi-\beta\sin(a\cos\Psi).
\end{aligned} \tag{9}$$

Equating like powers of e^2, we get

$$\omega^2 \frac{\partial^2 u_2}{\partial \Psi^2} + 2\omega \frac{\partial^2 u_2}{\partial \Psi \partial v} + \frac{\partial^2 u_2}{\partial v^2} - \left(aB_1 - A_1 \frac{\partial A_1}{\partial a} \right) \cos \Psi - \left(A_1 a \frac{\partial B_1}{\partial a} + 2A_1 B_1 \right) \sin \Psi$$

$$+2\omega B_1 \frac{\partial^2 u_1}{\partial \Psi^2} + 2A_1 \frac{\partial^2 u_1}{\partial \Psi \partial v} + 2B_1 \frac{\partial^2 u_1}{\partial \Psi \partial v} + 2\omega A_1 \frac{\partial^2 u_1}{\partial a \partial \Psi} - 2a\omega B_2 \cos \Psi - 2\omega A_2 \sin \Psi$$

$$+\omega^2 u_2 + 54 u_1 \sin^2 2v$$

$$= \beta(1 - \cos(a \cos \Psi)) u_1 . \tag{10}$$

Using Fourier Expansions given by

$$\sin(a \cos \Psi) = 2 \sum_{k=0}^{\infty} (-1)^k J_{2k+1}(a) \cos(2k+1)\Psi$$

$$\cos(a \cos \Psi) = J_0(a) + 2 \sum_{k=0}^{\infty} (-1)^k J_{2k}(a) \cos 2k\Psi$$

where J_k, $k = 0,1,2$ stands for Bessel functions. Equating the coefficients of $\cos\psi$ and $\sin\psi$ to zero in equation (9) so that $u_1(a,\psi,v)$ should not contain the resonant terms, we get

$$A_1(a) = 0$$

$$B_1(a) = \frac{\beta[2J_1(a) - a]}{2a\omega}$$

Further, substituting $A_1(a)$ and $B_1(a)$ in (9), we get

$$\omega^2 \frac{\partial^2 u_1}{\partial \Psi^2} + 2\omega \frac{\partial^2 u_1}{\partial \Psi \partial u} + \frac{\partial^2 u_1}{\partial v^2} + \omega^2 u_1 = 6 \sin 2v - 2\beta \sum_{k=1}^{\infty} (-1)^k J_{2k+1}(a) \cos(2k+1)\Psi \tag{11}$$

Integrating the above equation we get

$$u_1 = \frac{6 \sin 2v}{\omega^2 - 4} + \frac{\beta}{2\omega^2} \sum_{k=1}^{\infty} (-1)^k \frac{J_{2k+1}(a) \cos(2k+1)\Psi}{k(k+1)}$$

$$\frac{\partial u_1}{\partial \Psi} = \frac{-\beta}{2\omega^2} \sum_{k=1}^{\infty} (-1)^k \frac{J_{2k+1}(a)(2k+1) \sin(2k+1)\Psi}{k(k+1)}$$

$$\frac{\partial u_1}{\partial v} = \frac{12 \cos 2v}{\omega^2 - 4}$$

Substituting the values of A_1, B_1, $\frac{\partial u_1}{\partial \Psi}$, $\frac{\partial u_1}{\partial v}$ and u_1 in (10) and then equating the coefficients of $\cos\psi$ and $\sin\Psi$ to zero, to avoid the resonant terms, we get

$$A_2 = 0$$

$$B_2 = \frac{3\beta^2 J_0(a) J_3(a)}{8a\omega^2} - \frac{3\beta^2 J_3(a)}{8a\omega^3} - \frac{\beta[2J_1(a) - a]}{4a\omega^2}$$

Thus in the 1st approximation the solution is

$$\eta = a \cos \Psi ; \tag{12}$$

$$\frac{da}{dv} = 0 ; \tag{13}$$

$$\frac{d\Psi}{dv}=\omega+\frac{\omega}{2a}[2J_1(a)-a]. \tag{14}$$

In the 2$^{\text{nd}}$ approximation, the solution is

$$\eta=a\cos\Psi+e\left[\frac{6\sin 2v}{\omega^2-4}+\frac{\beta}{2\omega^2}\sum_{k=1}^{\infty}(-1)^k\frac{J_{2k+1}(a)\cos(2k+1)\Psi}{k(k+1)}\right] \tag{15}$$

$$\frac{da}{dv}=0; \tag{16}$$

$$\frac{d\Psi}{dv}=\omega+\frac{\omega}{2a}[2J_1(a)-a]+\frac{3\omega J_0(a)J_3(a)}{8a}-\frac{3\omega J_3(a)}{8a}-\frac{e(2J_1(a)-a)}{4a} \tag{17}$$

Equation (15) gives us the main resonance at $\omega=\pm 2$.

We have calculated that the system experiences resonance behaviour when $\omega=\pm 2$ upto 2$^{\text{nd}}$ order of approximation. We now proceed to study the resonance at $\omega=\pm 2$.

For $e=0$, the generating solutions are

$$\eta=a\cos\Psi$$

$$\Psi=\frac{v}{k}+\theta \tag{18}$$

where, amplitude 'a' and the phase 'θ' are determined by the following equations:

$$\frac{da}{dv}=eA_1(a,\theta); \tag{19}$$

$$\frac{d\theta}{dv}=\omega-\frac{1}{k}+eB_1(a,\theta); \tag{20}$$

$$\frac{d\psi}{dv}=\frac{1}{k}+\frac{d\theta}{dv}; \tag{21}$$

$$=\omega+eB_1(a,\theta)$$

where $A_1(a,\theta)$ and $B_1(a,\theta)$ are particular solutions periodic with respect to θ. Using (18), (19), (20) and (21), we find $\frac{d\eta}{dv}$ and $\frac{d^2\eta}{dv^2}$ and substituting them in equation (4) we get

$$e\left[\left\{\left(\omega-\frac{1}{k}\right)\frac{\partial A_1}{\partial\theta}-2B_1a\omega\right\}\cos\psi-\left\{a\left(\omega-\frac{1}{k}\right)\frac{\partial B_1}{\partial\theta}+2\omega A_1\right\}\sin\psi\right]$$

$$+e^2\left[\left\{A_1\frac{\partial A_1}{\partial a}+B_1\frac{\partial A_1}{\partial\theta}-aB_1\right\}\cos\psi-\left\{a\left(A_1\frac{\partial B_1}{\partial a}+B_1\frac{\partial B_1}{\partial a}\right)+2A_1B_2\right\}\sin\psi\right]$$

$$=6e\sin 2v+e\beta a\cos\psi-e\beta\sin(a\cos\psi) \tag{22}$$

Comparing the coefficient of e,

$$\left\{\left(\omega-\frac{1}{k}\right)\frac{\partial A_1}{\partial\theta}-2B_1a\omega\right\}\cos\psi-\left\{a\left(\omega-\frac{1}{k}\right)\frac{\partial B_1}{\partial\theta}+2\omega A_1\right\}\sin\psi$$

$$=6e\sin 2v+\beta a\cos\psi-\beta\sin(a\cos\psi)$$

$$= 6\sin 2v + \beta a\cos\psi - \beta\left[2\sum_{k=0}^{\infty}(-1)^k J_{2k+1}(a)\cos(2k+1)\psi\right]$$

$$= 6\sin 2(k\psi - k\theta) + \beta(a - 2J_1(a))\cos\psi - 2\beta\sum_{k=1}^{\infty}(-1)^k J_{2k+1}(a)\cos(2k+1)\psi \tag{23}$$

Comparing the coefficient of $\cos\psi$ and $\sin\psi$, we get the following cases:

Case a. When $k = \frac{1}{2}$

$$(\omega - 2)\frac{\partial A_1}{\partial\theta} - 2B_1 a\omega = -6\sin\theta + \beta[a - 2J_1(a)], \tag{24}$$

$$a(\omega - 2)\frac{\partial B_1}{\partial\theta} + 2\omega A_1 = -6\cos\theta. \tag{25}$$

Solving equations (24) and (25) we get

$$A_1 = -\frac{6}{\omega + 2}\cos\theta;$$

$$B_1 = \frac{6}{\omega + 2}\sin\theta - \frac{\beta(a - 2J_1(a))}{2\omega a}$$

Case b. When $k = -\frac{1}{2}$

$$\left(\left(\omega + 2\right)\frac{\partial A_1}{\partial\theta} - 2B_1 a\omega\right)\cos\psi - \left\{a\left(\omega + 2\right)\frac{\partial B_1}{\partial\theta} + 2\omega A_1\right\}\sin\psi$$

$$= 6\sin 2(k\psi - k\theta) + \beta(a - 2J_1(a))\cos\psi$$

i.e.

$$\left(\omega + 2\right)\frac{\partial A_1}{\partial\theta} - 2B_1 a\omega = 6\sin\theta + \beta(a - 2J_1(a)), \tag{26}$$

$$a\left(\omega + 2\right)\frac{\partial B_1}{\partial\theta} + 2\omega A_1 = 6\cos\theta. \tag{27}$$

Solving equations (26) and (27), we get

$$A_1 = \frac{6\cos\theta}{(\omega - 2)};$$

$$B_1 = -\frac{6\sin\theta}{a(\omega - 2)} - \frac{\beta\{a - 2J_1(a)\}}{2a\omega}.$$

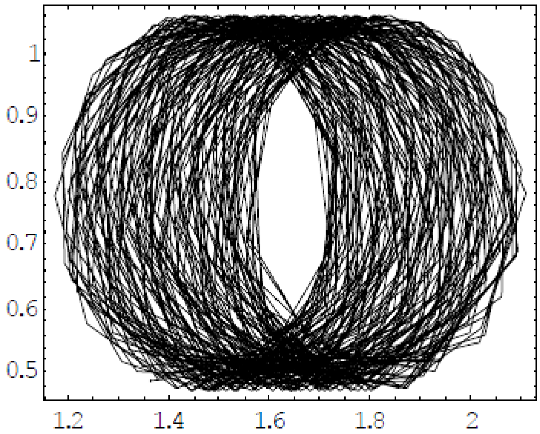

Figure 1

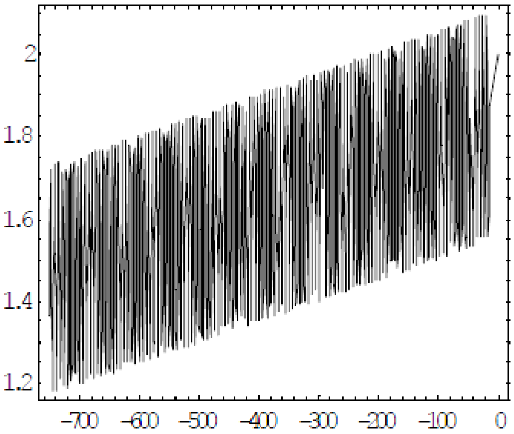

Figure 2

2. RESULTS AND DISCUSSIONS

During the analysis, we observe that for $e = 0.497$, $\omega = 2.0$, $\beta = 5.5$, $k = 4.0$. Fig. 1, the resonance curve plotted between ψ and a and Fig. 2, the resonance curve plotted between θ and ψ indicates interestingly the presence of piano like strange attracters near resonance.

3. CONCLUSION

In the computational studies analytically we have seen that the amplitude remains constant upto the second order of approximation. Also we have estimated the resonant solutions.

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