

Fuzzy Retrial Queueing System With Coxian – 2 Vacation

Dr. A. Viswanathan

*Professor & Dean, Department of Mathematics,
SNS College of Technology, Coimbatore, Tamil Nadu, 641035, India,
viswa_mathematics@gmail.com*

Dr K Udaya Chandrika

*Professor, Department of Mathematics
Avinashilingam Institute for Home Science and Higher Education for Women,
Coimbatore, Tamil Nadu, 641043, India
udayakchandrika@gmail.com*

*** J Ebenesar Anna Bagyam**

*Assistant Professor, Department of Mathematics
SNS College of Technology, Coimbatore, Tamil Nadu, 641035, India,
Corresponding Author ebenesar.j@gmail.com

ABSTRACT

Retrial queueing system with two phases of vacation (Coxian – 2 vacation) under Bernoulli scheduled is considered. Assuming the arrival rate, retrial rate, service rate and vacation rates are fuzzy members. The membership functions of the system characteristics are constructed. The α – cut approach is used to transform a fuzzy retrial queue into a family of conventional crisp retrial queues. By means of the membership functions of the system characteristics, a set of parametric non-linear programs is developed to describe the family of crisp retrial queues. A numerical example is presented to illustrate the validity of the proposed approach.

KEY WORDS Fuzzy Queue, Coxian - 2 Vacation and Bernoulli Vacation

1. INTRODUCTION

Queueing systems in which arriving customers who find all servers occupied may retry for service after a period of time are called Retrial queues. Retrial queueing models are mainly motivated by applications to telephone switching systems,

telecommunication networks and computer systems packet switch networks, shared bus local area networks operating under the Carrier-Sense Multiple Access Protocol and collision avoidance star local area networks.

In recent years queues with server vacation have emerged as an important area of research due to their various applications in production systems, communication systems, computer networks and etc. Some comprehensive studies on the recent results for a variety of vacation models can be found in [3], [4], [5] and [9]. Within the context of traditional queueing theory, the inter arrival times and service times are required to follow certain probability distributions. However, in many practical applications, the arrival pattern and service pattern are more suitably described by linguistic terms such as fast, slow, or moderate, rather than by probability distributions. Thus, fuzzy retrial queues would be potentially much more useful and realistic than the commonly used crisp retrial queues. Using Zadeh's extension principle [10,11] fuzzy queueing models have analysed by the researchers like Li and Lee [7], Negi and Lee [8], Kao et al. [6], Chen [1, 2].

In the above papers, the work on fuzzy queueing model is focused on ordinary queues with one or two fuzzy variables. In this paper, we develop an approach that provides system characteristics for the retrial queues with phase type vacation having five fuzzy variables. Through α - cuts and Zadeh's extension principle, the fuzzy retrial queue with phase type vacations is transformed to a family of crisp retrial queues with phase type vacations. As α varies, the family of crisp retrial queues is described and solved using parametric Non - Linear Programming (NLP). The NLP solutions completely and successfully yield the membership functions of the system characteristics.

2. THE MATHEMATICAL MODEL

Consider a single server retrial queue in which customers arrive at the system according to a Poisson process with rate λ . If an arriving customer finds the server idle, the customer obtains service immediately. Otherwise, if the server is found busy or on vacation, the arriving primary customer joins a retrial orbit of finite capacity in order to seek service after a random amount of time. The retrial time is assumed to be exponentially distributed with rate η . The service time is also exponentially distributed with mean $1/\mu$. The vacation period has two heterogeneous phases. Phase 2 follows phase 1 vacation. At each service completion epoch, the server may opt for a phase 1 vacation with probability p or else continue to be in the system with complementary probability $1-p$. After phase 1 vacation the server may take phase 2 vacation with probability q or return back to the system with probability $1-q$. Vacation times are exponentially distributed with rates β_i , $i = 1, 2$.

The stability condition is $\lambda(\eta + \lambda)(1/\eta\mu + (p + pq)/(\eta\beta_1)) < 1$

Then the expected number of customer in the orbit is in steady state is

$$L_q = \{ \lambda [1 - (\eta / (\eta + \lambda))] / \mu + \lambda [p\lambda / \beta_1 + pq\lambda / \beta_2] / \mu + [1 - (\eta / (\eta + \lambda))] [p\lambda / \beta_1 + pq\lambda / \beta_2] + pq \lambda^2 / (\beta_1 \beta_2) + \lambda^2 [1 / \mu^2 + p / \beta_1^2 + pq / \beta_2^2] \} / \{ \eta / (\eta + \lambda) - \lambda / \mu - p\lambda / \beta_1 - pq\lambda / \beta_2 \} \tag{1}$$

and the expected number of customer in the system is given by

$$L_s = \lambda / \mu + L_q \tag{2}$$

3. PROBLEM FORMULATION

To extend the applicability of the retrial queue with phase type vacation, we allow for fuzzy specification of system parameters. Suppose the arrival rate λ , retrial rate η , service rate μ and vacation rates β_i are approximately known and can be represented by the fuzzy sets $\bar{\lambda}$, $\bar{\mu}$, $\bar{\eta}$ and $\bar{\beta}_i$ respectively. Let $\phi_{\bar{\lambda}}(x)$, $\phi_{\bar{\eta}}(y)$, $\phi_{\bar{\mu}}(s)$ and $\phi_{\bar{\beta}_i}(v_i)$ denote the membership functions of $\bar{\lambda}$, $\bar{\eta}$, $\bar{\mu}$ and $\bar{\beta}_i$ respectively. We then have the following fuzzy sets:

$$\bar{\lambda} = \{ (x, \phi_{\bar{\lambda}}(x)) / x \in X \} \tag{3a}$$

$$\bar{\eta} = \{ (y, \phi_{\bar{\eta}}(y)) / y \in Y \} \tag{3b}$$

$$\bar{\mu} = \{ (s, \phi_{\bar{\mu}}(s)) / s \in S \} \tag{3c}$$

$$\bar{\beta}_1 = \{ (v_1, \phi_{\bar{\beta}_1}(v_1)) / v_1 \in V_1 \} \tag{3d}$$

$$\bar{\beta}_2 = \{ (v_2, \phi_{\bar{\beta}_2}(v_2)) / v_2 \in V_2 \} \tag{3e}$$

where X, Y, S, V₁ and V₂ are the crisp universal sets of the arrival, retrial, service, phase1 vacation and phase 2 vacation rates respectively. Using Zadeh’s approach, the α – cuts of $\bar{\lambda}$, $\bar{\eta}$, $\bar{\mu}$, $\bar{\beta}_1$ and $\bar{\beta}_2$ as crisp intervals are defined as follows:

$$\lambda(\alpha) = [x_\alpha^L, x_\alpha^U] = \left[\min_{x \in X} \{ x / \phi_{\bar{\lambda}}(x) \geq \alpha \}, \max_{x \in X} \{ x / \phi_{\bar{\lambda}}(x) \geq \alpha \} \right] \tag{4a}$$

$$\eta(\alpha) = [y_\alpha^L, y_\alpha^U] = \left[\min_{y \in Y} \{ y / \phi_{\bar{\eta}}(y) \geq \alpha \}, \max_{y \in Y} \{ y / \phi_{\bar{\eta}}(y) \geq \alpha \} \right] \tag{4b}$$

$$\mu(\alpha) = [s_{\alpha}^L, s_{\alpha}^U] = \left[\min_{s \in S} \{s / \phi_{\bar{\mu}}(s) \geq \alpha\}, \max_{s \in S} \{s / \phi_{\bar{\mu}}(s) \geq \alpha\} \right] \quad (4c)$$

$$\beta_1(\alpha) = [v_{1\alpha}^L, v_{1\alpha}^U] = \left[\min_{v_1 \in V_1} \{v_1 / \phi_{\bar{\beta}_1}(v_1) \geq \alpha\}, \max_{v_1 \in V_1} \{v_1 / \phi_{\bar{\beta}_1}(v_1) \geq \alpha\} \right] \quad (4d)$$

$$\beta_2(\alpha) = [v_{2\alpha}^L, v_{2\alpha}^U] = \left[\min_{v_2 \in V_2} \{v_2 / \phi_{\bar{\beta}_2}(v_2) \geq \alpha\}, \max_{v_2 \in V_2} \{v_2 / \phi_{\bar{\beta}_2}(v_2) \geq \alpha\} \right] \quad (4e)$$

Where $\lambda(\alpha)$, $\eta(\alpha)$, $\mu(\alpha)$, $\beta_1(\alpha)$ and $\beta_2(\alpha)$ are crisp sets. Using α -cuts, the arrival time, retrial time, service time and vacation times can be represented by different levels of confidence intervals. Consequently, a fuzzy queue can be reduced to a family of crisp queues with different α -level sets $\{\lambda(\alpha) / 0 < \alpha \leq 1\}$, $\{\eta(\alpha) / 0 < \alpha \leq 1\}$, $\{\mu(\alpha) / 0 < \alpha \leq 1\}$, $\{\beta_1(\alpha) / 0 < \alpha \leq 1\}$ and $\{\beta_2(\alpha) / 0 < \alpha \leq 1\}$. These two sets form nested structures for expressing the relationship between ordinary sets and fuzzy sets. Let the confidence intervals of the fuzzy sets $\bar{\lambda}$, $\bar{\eta}$, $\bar{\mu}$, $\bar{\beta}_1$ and $\bar{\beta}_2$ are $[L_{\lambda(\alpha)}, U_{\lambda(\alpha)}]$, $[L_{\eta(\alpha)}, U_{\eta(\alpha)}]$, $[L_{\mu(\alpha)}, U_{\mu(\alpha)}]$, $[L_{\beta_1(\alpha)}, U_{\beta_1(\alpha)}]$ and $[L_{\beta_2(\alpha)}, U_{\beta_2(\alpha)}]$ respectively. When arrival time, retrial time, service time and vacation times are fuzzy numbers, based on Zadeh's extension principle, the membership function of the performance measure $f(x, y, s, v_1, v_2)$ is defined as

$$\phi_{f(\bar{\lambda}, \bar{\eta}, \bar{\mu}, \bar{\beta}_1, \bar{\beta}_2)}(z) = \sup_{\Omega} \min\{\phi_{\bar{\lambda}}(x), \phi_{\bar{\eta}}(x), \phi_{\bar{\mu}}(x), \phi_{\bar{\beta}_1}(x), \phi_{\bar{\beta}_2}(x) \mid z = f(x, y, s, v_1, v_2)\}$$

where the supremum is taken over the set

$$\Omega = \{x \in X, y \in Y, s \in S, v_1 \in V_1, v_2 \in V_2 \mid x(y+x)(1/ys + (p+pq)/(yv_1)) < 1\}$$

Our approach is to construct the membership function $\phi_{f(\bar{\lambda}, \bar{\eta}, \bar{\mu}, \bar{\beta}_1, \bar{\beta}_2)}(z)$ which is based on deriving the α -cuts of $\phi_{f(\bar{\lambda}, \bar{\eta}, \bar{\mu}, \bar{\beta}_1, \bar{\beta}_2)}(z)$. The corresponding, Parametric programming technique for finding lower and upper bounds of the α -cut of $\phi_{f(\bar{\lambda}, \bar{\eta}, \bar{\mu}, \bar{\beta}_1, \bar{\beta}_2)}(z)$ are

$$L_{\phi(\alpha)} = \min_{\Omega} f(x, y, s, v_1, v_2)$$

$$\text{s.t.} \quad L_{\lambda(\alpha)} \leq x \leq U_{\lambda(\alpha)}, \quad L_{\eta(\alpha)} \leq y \leq U_{\eta(\alpha)}, \quad L_{\mu(\alpha)} \leq s \leq U_{\mu(\alpha)}, \\ L_{\beta_1(\alpha)} \leq v_1 \leq U_{\beta_1(\alpha)} \text{ and } L_{\beta_2(\alpha)} \leq v_2 \leq U_{\beta_2(\alpha)}$$

$$\begin{aligned}
 U_{\phi(\alpha)} &= \max_{\Omega} f(x, y, s, v_1, v_2) \\
 \text{s.t.} \quad & L_{\lambda(\alpha)} \leq x \leq U_{\lambda(\alpha)}, \quad L_{\eta(\alpha)} \leq y \leq U_{\eta(\alpha)}, \quad L_{\mu(\alpha)} \leq s \leq U_{\mu(\alpha)}, \\
 & L_{\beta_1(\alpha)} \leq v_1 \leq U_{\beta_1(\alpha)} \text{ and } L_{\beta_2(\alpha)} \leq v_2 \leq U_{\beta_2(\alpha)}
 \end{aligned}$$

If both $L_{\phi(\alpha)}$ and $U_{\phi(\alpha)}$ are invertible with respect to α . Then a left shape function $L(z) = L_{\phi(\alpha)}^{-1}$ and a right shape function $R(z) = U_{\phi(\alpha)}^{-1}$ can be derived, from which the membership function $\phi_{f(\bar{\lambda}, \bar{\eta}, \bar{\mu}, \bar{\beta}_1, \bar{\beta}_2)}(z)$ is constructed:

$$\phi_{f(\bar{\lambda}, \bar{\eta}, \bar{\mu}, \bar{\beta}_1, \bar{\beta}_2)}(z) = \begin{cases} L(z), & z_1 \leq z \leq z_2 \\ 1, & z_2 \leq z \leq z_3 \\ R(z), & z_3 \leq z \leq z_4 \end{cases}$$

where $z_1 = L_{\phi(\alpha)}|_{\alpha=0}$, $z_2 = L_{\phi(\alpha)}|_{\alpha=1}$, $z_3 = U_{\phi(\alpha)}|_{\alpha=1}$ and $z_4 = U_{\phi(\alpha)}|_{\alpha=0}$

4. NUMERICAL EXAMPLE

Suppose the arrival rate, retrial rate, service rate and phase type vacation rates are trapezoidal fuzzy numbers represented by $\bar{\lambda} = [3, 4, 5, 6]$, $\bar{\eta} = [34, 36, 38, 40]$, $\bar{\mu} = [18, 19, 20, 21]$, $\bar{\beta}_1 = [13, 14, 15, 16]$ and $\bar{\beta}_2 = [15, 16, 17, 18]$. Assume that $p = q = 0.5$. First it is easy to find that $[x_{\alpha}^L, x_{\alpha}^U] = [6 - z, z + 3]$, $[y_{\alpha}^L, y_{\alpha}^U] = [40 - z, 34 + 2z]$, $[s_{\alpha}^L, s_{\alpha}^U] = [21 - z, 18 + z]$, $[v_{1\alpha}^L, v_{1\alpha}^U] = [16 - z, 13 + z]$ and $[v_{2\alpha}^L, v_{2\alpha}^U] = [18 - z, 15 + z]$. Next it is obvious that when $x = x_{\alpha}^U$, $y = y_{\alpha}^L$, $s = s_{\alpha}^L$, $v_1 = v_{1\alpha}^L$ and $v_2 = v_{2\alpha}^L$, the number of customers attain its maximum value and when $x = x_{\alpha}^L$, $y = y_{\alpha}^U$, $s = s_{\alpha}^U$, $v_1 = v_{1\alpha}^U$ and $v_2 = v_{2\alpha}^U$, the number of customers attain its minimum value.

4.1. The fuzzy mean number of customer in the orbit L_q

According to equation (8), the α – cuts of L_q are

$$\begin{aligned}
 \mathcal{L}_{q, \alpha}^{\mathcal{L}} &= \frac{2(3+a)^2(-31982652+8048619a-801621a^2+39397a^3-952a^4+9a^5)}{3(-21+a)(-18+a)(-16+a)(226068-88695a+9399a^2-377a^3+5a^4)} \\
 \mathcal{L}_{q, \alpha}^{\mathcal{U}} &= \frac{2(-6+a)^2(14062590+4203441a+496026a^2+28783a^3+817a^4+9a^5)}{3(13+a)(15+a)(18+a)(34800+41940a+6276a^2+317a^3+5a^4)}
 \end{aligned}$$

With the help of MATLAB[®] 7.0, the inverse functions of $L(z) = \mu_{L_q}^{\bar{L}}$ and $R(z) = \mu_{L_q}^{\underline{L}}$ exist, yielding the membership function.

$$\phi_{L_q}^{\bar{L}}(z) = \begin{cases} L(z), & \frac{888407}{6329904} \leq z \leq \frac{82324}{233325} \\ 1, & \frac{82324}{233325} \leq z \leq \frac{234895825}{266014896} \\ R(z), & \frac{234895825}{266014896} \leq z \leq \frac{156251}{56550} \end{cases}$$

4.2 The fuzzy mean number of customer in System L_s

By the same argument and equation (2), the α - cuts of L_s are

$$\mu_{L_s}^{\bar{L}} = \frac{(3+a)(-18438984+3122802a-163596a^2+955a^3+146a^4-3a^5)}{3(-18+a)(-16+a)(226068-88695a+9399a^2-377a^3+5a^4)}$$

$$\mu_{L_s}^{\underline{L}} = \frac{(-6+a)(10506060+2181564a+147927a^2+2437a^3-101a^4-3a^5)}{3(13+a)(15+a)(34800+41940a+6276a^2+317a^3+5a^4)}$$

The membership function is

$$\phi_{L_s}^{\bar{L}}(z) = \begin{cases} L(z), & \frac{256097}{904272} \leq z \leq \frac{128989}{233325} \\ 1, & \frac{128989}{233325} \leq z \leq \frac{16047355}{14000784} \\ R(z), & \frac{16047355}{14000784} \leq z \leq \frac{58367}{18850} \end{cases}$$

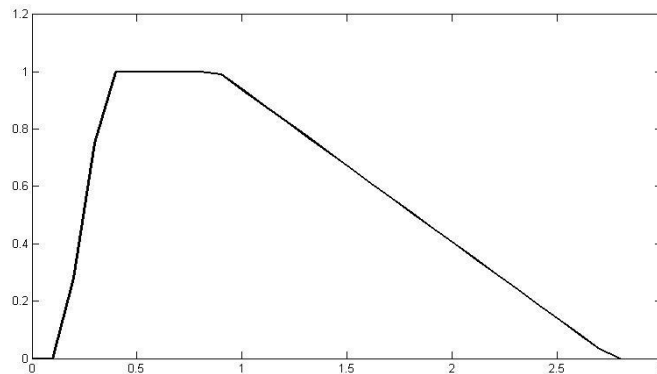


Figure 1 The membership function for $\phi_{L_q}^{\bar{L}}(z)$

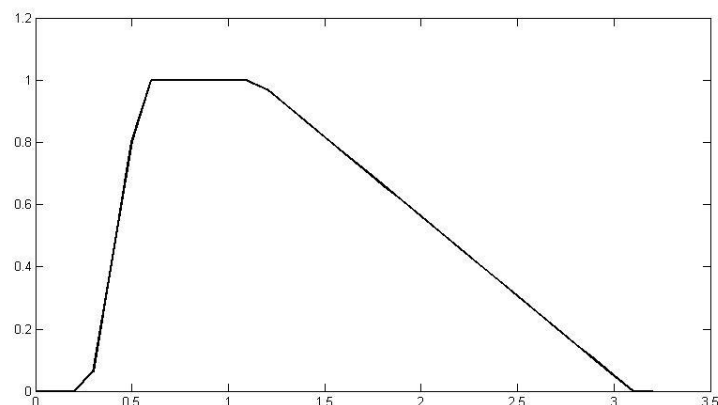


Figure 2 The membership function for $\phi_{L_s}^- (z)$

Next, we perform α – cuts of fuzzy mean number of customer in the orbit and fuzzy mean number of customer in the system at different possibility α levels are presented in table 1.

Table 1: α cuts of L_q and L_s

α	$L_{q,\alpha}^L$	$L_{q,\alpha}^D$	$L_{s,\alpha}^L$	$L_{s,\alpha}^D$
0	0.1404	2.7631	0.2832	3.0964
0.1000	0.1546	2.4043	0.3029	2.7303
0.2000	0.1700	2.1098	0.3239	2.4285
0.3000	0.1868	1.8641	0.3462	2.1756
0.4000	0.2049	1.6565	0.3700	1.9609
0.5000	0.2247	1.4792	0.3954	1.7765
0.6000	0.2462	1.3262	0.4226	1.6165
0.7000	0.2695	1.1932	0.4518	1.4766
0.8000	0.2949	1.0767	0.4830	1.3533
0.9000	0.3226	0.9740	0.5166	1.2438
1.0000	0.3528	0.8830	0.5528	1.1462

From the table we find that for fuzzy mean number of customer in the orbit L_q , the value of L_q at $\alpha = 1$ is $[0.3528, 0.8830]$, indicating that it is definitely possible that the number customers in the queue falls between 0.3528 and 0.8830. Moreover, the range of orbit length at $\alpha = 0$ is $[0.1404, 2.7631]$, indicating that the number of customers in the orbit will never exceed 2.7631 or fall below 0.1404 and fuzzy mean number of customer in the system L_s , the value of L_s at $\alpha = 1$ is $[0.5528, 1.1462]$, indicating that it is definitely possible that the number customers in the system falls between 0.5528 and 1.1462, the range of system length at $\alpha = 0$ is $[0.2832, 3.0964]$,

indicating that the number of customers in the system will never exceed 3.0964 or fall below 0.2832. The above information will be very useful for designing a queueing system, which involves one or a combination of several decisions, such as efficiency of the server and system capacity.

Another point to note is the correspondence between the membership functions of the arrival rate, retrial rate, service rate, vacation rates and that of the performance measures. Referring figure 1 and 2, when all the fuzzy numbers are trapezoidal fuzzy numbers, membership function of the correspondence performance measures $\phi_{L_q}^-(z)$ and $\phi_{L_s}^-(z)$ considered in this paper also look like trapezoidal, although they can be nonlinear.

REFERENCES

1. Chen, S.P., 2005, "Parametric Nonlinear Programming Approach to Fuzzy Queues with Bulk Service", *Eur. J. Oper. Res.*, 163, pp. 434-444.
2. Chen, S.P., 2004, "Parametric Non-Linear Programming for Analyzing Fuzzy Queues with Finite Capacity", *Eur. J. Oper. Res.*, 157, pp. 429-438.
3. Choudhury, G., and Ke, J.C., 2012, "A Batch Arrival Retrial Queue with General Retrial Times under Bernoulli Vacation Schedule for Unreliable Server and Delaying Repair", *Applied Mathematical Modelling*, 36, pp. 255-269.
4. Ebenesar Anna Bagyam, J., and Udaya Chandrika, K., 2012, "Performance Analysis of Single Server Retrial Queueing System with Bernoulli and Phase Type Vacations", *International Journal of Computer Applications*, 54 (7), pp. 30-35.
5. Ebenesar Anna Bagyam, J., and Udaya Chandrika, K., 2010, "Single Server Retrial Queueing System with Two Different Vacation Policies", *Int. J. Contemp. Math. Sciences*, 5(32), pp. 1591-1598.
6. Kao, C., Li, C.C., and Chen, S.P., 1999, "Parametric Programming to the Analysis of Fuzzy Queues", *Fuzzy set Syst.*, 107, pp. 93-100.
7. Li, R. J., and Lee, E.S., 1989, "Analysis of Fuzzy Queues", *Comput. Math .Appl.*, 17, pp.1143-1147.
8. Negi, D.S., and Lee, E.S., 1992, "Analysis and Simulation of Fuzzy Queue", *Fuzzy set Syst.*, 46, pp. 321-330.
9. Sumitha, D., Ebenesar Anna Bagyam, J., Udaya Chandrika, K. and Prakash Rani, K., 2012, "Bulk Arrival Two Phase Retrial Queueing System with Orbital Search, Impatient Customers and Different Types of Server Vacations", *International Research Journal*, 1(3), pp. 977-987.
10. Zadeh, L.A., 1978, "Fuzzy Sets as a Basis for a Theory of Possibility", *Fuzzy set syst.*, 1, pp. 3-28.
11. Zimmermann, H. J., 2001, "Fuzzy Set Theory and its Applications", 4th Ed., Kluwer Academic, Boston.