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$Zc-$ Lindelof Spaces in General Topology

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Abstract

In this paper, we introduce the concept of Zc –Lindelof using ωZc-open sets. Some properties and theorems using ωZc-open sets through Zc –open sets are also discussed.

Keywords: Z –open ,Zc –open sets, ωZc-open, ωZc-closed, Zc- open cover, Zc –Lindelof.

1. Introduction:

A topological space X is said to be Lindelof, or have the Lindelof property, if every open cover of X has a countable subcover. The Lindelof property was introduced by Alexandroff and Urysohn in 1929, the term 'Lindelof ' referring back to Lindelof 's result that any family of open subsets of Euclidean space has a countable sub-family with the same union. The Lindelöf property is a weakening of the more commonly used notion of compactness which requires the existence of a finite subcover.

2. Preliminaries:

Throughout this paper (X,τ) represents a topological space on which no separation axiom is assumed unless otherwise stated. Let $A \subseteq (X,\tau)$,then $cl(A)$ and $int(A)$ denotes the closure of A and the interior of A respectively.

Definition 2.1 [1] : A subset A of a space X is said to be

i) Z-open set if $A \subseteq cl(\delta\text{-}int(A)) \cup int(cl(A))$,

ii) Z-closed set if $int(\delta\text{-}cl(A)) \cap cl(int(A)) \subseteq A$. The family of all Z-open (resp. Z-closed sets) subsets of a space (X,τ) will be denoted by $ZO(X)$ (resp., $ZC(X)$).

Definition 2.2 [2]: A subset A of a space X is Zc-open`if for each $x \in A \in ZO(X)$,there exists a closed set F such that $x \in F \subset A$. A subset A of a space X is Zc-closed if $X-A$ is Zc-open. The family of all Zc-open (resp. Zc-closed) subsets of a topological space (X,τ) is denoted by $ZcO\ (X,\tau)$ or $ZcO(X)$ (resp. $ZcC\ (X,\tau)$ or $ZcC(X)$).

Example 2.3 [2]: Consider X={a,b,c,d} with $\tau = \{\emptyset, X, \{a\},\{b\},\{c\},\{a,b\},\{b,c\},\{a,c\},\{a,b,c\}\}$.

Then the family of closed sets are $\{\emptyset, X, \{b,c,d\},\{a,b,d\},\{a,c,d\},\{c,d\},\{a,d\},\{b,d\},\{d\}\}$.

The family of Z-open sets are : $ZO(X)=\{\emptyset, X, \{a\},\{b\},\{c\},\{a,b\},\{b,c\},\{a,c\},\{a,d\},\{b,d\},\{c,d\}, \{a,b,c\},\{a,c,d\},\{a,b,d\}, \{b,c,d\}\}$

The family of Zc-open sets are : $ZcO(X)=\{ \emptyset, X, \{a,d\},\{b,d\},\{c,d\},\{a,b,d\},\{b,c,d\},\{a,c,d\}\}$.

The family of Zc-closed sets are: $ZcC(X)=\{ \emptyset, X, \{b,c\},\{a,c\},\{a,b\},\{c\},\{a\},\{b\}\}$.

Proposition 2.4 [2]: (i) A subset A of a space X is Zc-open if and only if A is Z-open and it is the union of closed sets. That is where A is Z-open and F_α is closed sets for each α.

(ii) A subset A of a space X is Zc-closed if and only if A is Z-closed and it is an intersection of open sets. ■

Remark 2.5 [2]: It is clear from the definition of Zc-open(resp. Zc-closed) sets, that every Zc-open(resp. Zc-closed) subset of a space X is Z-open, but the converse is not true in general as shown in example 2.3 where $\{a\},\{b\},\{c\}$ belongs to $ZO(X)$ whereas $\{a\},\{b\},\{c\}$ does not belongs to $ZcO(X)$ and $\{a,d\},\{b,d\}$ belongs to $ZO(X)$ whereas $\{a\},\{b\},\{c\}$ does not belongs to $ZcC(X)$.

Definition 2.6 [3] : Let (X,τ) be a topological space. Then

(i) *Zc-interior* of A is union of all Zc-open sets contained in A and is denoted by $Zc\text{-}Int(A)$.

(ii) *Zc-closure* of A is the intersection of all Zc-closed sets containing A and is denoted by $Zc\text{-}cl(A)$.

3. $\omega Z-$open sets

Definition 3.1: A subset W of a space is ω-open iff for each $x \in W$, there exists $U \in \tau$ such that $x \in U$ and $U - W$ is countable.

Definition 3.2: A point $x \in X$ is called a condensation point of A if for each $U \in \tau$ with $x \in U$, the set $U \cap A$ is countable A is said to be ω –closed if it contains all its condensation points. The complement of an ω –closed set is ω –open.

Definition 3.3: Let (X,τ) be a topological space. Then

(i) the union of all open sets contained in A is called the *ω-interior* of A and is denoted by $int_{\omega}(A)$.

(ii) the intersection of all ω −closed sets containing A is called the ω −*closure* of A and is denoted by $cl_{\omega}(A)$.The family of all ω −open subsets of (X,τ) will be denoted by τ_{ω} or $\omega O(X)$,forms a topology on X finer than τ.

Definition 3.4: A subset A of a space X is said to be ωZ −open if for every $x\in A$, there exists an Z-open subset$U_x \subseteq X$ containing x such that $U_x - A$ is countable. The complement of an subset is said to be ωZ− closed.

Example 3.5: Let $X = \{a,b,c\}$and $\tau = \{X,\emptyset,\{a\},\{b\},\{a,b\}\}$.$ZO(X)$ are $\{X,\emptyset,\{a\},\{b\},\{a,b\},\{b,c\},\{c,a\}\}$.Then $\{c\}$ is ωZ –open since X is a countable set and it is not Z –open.

Lemma 3.6: A subset of a space X is ωZ –open iff for every $x\in A$, there exists a Z –open subset U containing x and a countable subset C such that $U-C\subseteq A$.

Proof: Let A be ωZ –open and $x\in A$.By definition there exists an Z –open subset U_x containing x such that $(U_x - A)$ is countable. Let $C = U_x - A = U_x \cap (X-A)$.Then $U_x - C \subseteq A$. Conversely , let $x\in A$. Then there exists a Z –open subset U_x containing x and a countable subset C such that$U_x - C\subseteq C$ and $U_x - A$ is countable set.

Theorem 3.7: Let X be a space and $C\subseteq X$. If C is ωZ –closed then $C\subseteq A\cup B$ for some Z-closed subset A and a countable subset B.

Proof: If C is ωZ –closed,then $X-C$ is ωZ –open and hence for every $x\in X-C$, there exists a Z-open set U containing x and a countable set B such that $U-B \subseteq X-C$.Thus $C \subseteq X-(U-B) = X-(U\cap(X-B)) = (X-U)\cup B$.Let $A = X-U$.Then A is Z-closed such that $C\subseteq A\cup B$.

4. *ωZc* –open sets

Definition 4.1: A subset A of a space X is said to be ωZc-open if for every $x\in A$, there exists a Zc-open subset $U_x \subseteq X$ containing x such that $U_x - A$ is countable. The complement of an ωZc −open subset is said to be ωZc− closed.

Example 4.2: Let $X = \{a,b,c\}$and $\tau = \{X,\emptyset,\{a\},\{b\},\{a,b\}\}$. $ZcO(X)$ are $\{X,\emptyset,\{a,b\},\{b,c\}\}$.Then $\{a\},\{b\},\{c\}$ is ωZc –open since X is countable but they are not Zc –open sets.

Lemma 4.3: Let (X,τ) be a topological space. The intersection of an clopen set and a Zc-open set is Zc-open.

Proposition 4.4: The intersection of an ωZc-open set and an ω −open set is ωZc-open.

Proof: Let A be an ωZc-open set and B an ω −open set in a space X. Let x be any point of $A\cap B$.Since A is ωZc-open, there exists a Zc-open set U_A containing x such that $(U_A - A)$ is countable. Since B is ω −open, there exists an open set U_B containing x such that $(U_B - B)$ is countable. By lemma 4.3,$U_A \cap U_B$ is a Zc-open set containing x. Also, $(U_A\cap U_B) - A\cap B = (U_A\cap U_B)\cap[(X-A)\cup(X-B)] = [U_A\cap U_B\cap(X-A)]\cup[U_A\cap U_B\cap(X-B)]\subseteq(U_A\cap(X-A))\cup(U_B\cap(X-B))$.Since $(U_A\cap(X-A))\cup(U_B\cap(X-B))$ is countable, we get $(U_A\cap U_B) - A\cap B$, as a countable set. Thus $A\cap B$ is ωZc-open.

Result 4.5: The intersection of two ωZc-open sets need not be ωZc-open.

Proposition 4.6: The union of any family of ωZc-open sets is ωZc-open.

Proof: If $\{A_\alpha : \alpha\in\Lambda\}$ is a collection of ωZc-open subsets of X, then for every $x\in\bigcup_{\alpha\in\Lambda}A_\alpha$, $x\in A_\beta$for some $\beta\in\Lambda$.Hence there exists an Zc-open subset U of X containing x such that $U-A_\beta$is countable. Now as $U-(\bigcup_{\alpha\in\Lambda}A_\alpha)\subseteq U-A_\beta$ and thus $U-(\bigcup_{\alpha\in\Lambda}A_\alpha)$ is countable. Therefore $\bigcup_{\alpha\in\Lambda}A_\alpha$ is ωZc-open.

Lemma 4.7: A subset A of a space X is ωZc-open iff for every $x\in A$,there exists an Zc-open subset U containing x and a countable subset C such that $U-C\subseteq A$.

Proof: It follows from Lemma 3.6.

Theorem 4.8: Let X be a space and $C\subseteq X$ .If C is ωZc-closed, then $C\subseteq K\cup B$ for some Zc-closed subset K and a countable subset B.

Proof: Follows from Theorem 4.7

Theorem 4.9: If each non-empty Zc-open set of a space X is uncountable, then $Zc\text{-}cl(A) = \omega Zc\text{-}cl(A)$ for each clopen set A of X.

Proof: Clearly $\omega Zc\text{-}cl(A)\subseteq Zc\text{-}cl(A)$.On the other hand, let $x\in Zc\text{-}cl(A)$ and B be an ωZc-open subset containing x. Then by lemma 4.7, there exists a Zc-open set V containing x and a countable set C such that $V-C\subseteq B$.Thus $(V-C)\cap A\subseteq B\cap A$ and so $(V\cap A)-C\subseteq B\cap A$.Since $x\in V$ and $x\in Zc\text{-}cl(A)$, $V\cap A\neq\emptyset$ and $V\cap A$ is Zc-open. Since V is Zc-open and A is open. By hypothesis, each non-empty Zc-open set of a space is uncountable and hence $(V\cap A)-C$ is also uncountable. Thus $B\cap A$ is uncountable. Therefore, $B\cap A\neq\emptyset$ which means that $x\in\omega Zc\text{-}cl(A)$.

Corollary 4.10: If each non-empty Zc –open set of a space X is uncountable, then $Zc\text{-}Int(A) = \omega Zc\text{-}Int(A)$ for each closed set A of X.

Theorem 4.11: Every ωZc-open is ωZ −open . ***Proof:*** Let A be an ωZc-open, then for each $x \in A$, there exists Zc –open set U_x containing x such that $U_x - A$ is countable. Since every Zc –open set is Z –open, A is ωZ –open.

Lemma 4.12: Every Zc –open is ωZc-open .

5. *Zc* –Lindelof Space

Definition 5.1: A space X is said to be Zc –Lindelof if every Zc –open cover of X has a countable subcover.

Definition 5.2: A subset A of X is said to Zc –Lindelof relative to X if every cover of A by Zc –open sets of X has a countable subcover.

Theorem 5.3: For any space X, the following are equivalent.

(a) X is Zc –Lindelof

(b) Every ωZc-open cover of X has a countable subcover.

Proof: (a)⇒(b) Let $\{G_\alpha: \alpha \in \Lambda\}$be any two ωZc-open cover of X. For each $x \in X$, there exists $\alpha(x) \in \Lambda$ such that $x \in G_{\alpha(x)}$. Since $G_{\alpha(x)}$ is ωZc-open, there exists Zc –open set $V_{\alpha(x)}$ such that $x \in V_{\alpha(x)}$ and $V_{\alpha(x)} - G_{\alpha(x)}$ is countable.The family of $\{V_{\alpha(x)}: x \in X\}$is a Zc –open cover of X. Since X is Zc –Lindelof there exists a countable subset say $\alpha(x_1),\alpha(x_2),\ldots\ldots\alpha(x_n),\ldots\ldots$such that $X = \{V_{\alpha(x_i)}: i \in N\}$.So, $X = \bigcup_{i \in N}\{(V_{\alpha(x_i)} - G_{\alpha(x_i)}) \cup G_{\alpha(x_i)}\}=(V_{\alpha(x_i)} - G_{\alpha(x_i)})\cup(\bigcup_{i \in N} G_{\alpha(x_i)})$. For each $\alpha(x_i)$,$V_{\alpha(x_i)} - G_{\alpha(x_i)}$ is a countable set and there exists a countable subset $\Lambda_{\alpha(x_i)}$ of Λ such that $V_{\alpha(x_i)} - G_{\alpha(x_i)} \subseteq \bigcup\{G_\alpha: \alpha \in \Lambda_{\alpha(x_i)}\}$. Hence we observe that $X \subseteq (\bigcup_{i \in N}(\bigcup\{G_\alpha: \alpha \in \Lambda_{\alpha(x_i)}\}))\cup(\bigcup_{i \in N} G_{\alpha(x_i)})$.Thus (a)⇒(b).

(b) ⇒(a)

Let $\{G_\alpha: \alpha \in \Lambda\}$ be any Zc –open cover of X. We claim: X is Zc –Lindelof. Every Zc –open is ωZc-open, by lemma 4.12.Also by theorem 5.3 $\{G_\alpha: \alpha \in \Lambda\}$is ωZc-open cover of X has a countable subcover. Hence X is Zc –Lindelof.

Proposition 5.4: Every ωZc-closed subset of a Zc –Lindelof space of X is Zc –Lindelof relative to X.

Proof: Let A be an ωZc-closed subset of X. Let $\{G_\alpha: \alpha \in \Lambda\}$ be a cover of A by Zc –open set of X. Now for each $x \in A^c$, there is a Zc –open set V_x such that $V_x \cap A$ is countable. Since X is Zc –Lindelof and the collection $\{G_\alpha: \alpha \in \Lambda\} \cup \{V_x: x \in A^c\}$ is a Zc –open cover of X ,there exists a countable subcover $\{G_{\alpha i}: i \in \mathrm{N}\} \cup \{V_{xi}: i \in \mathrm{N}\}$.Since $\bigcup_{i \in N}(V_{xi} \cap A)$ is countable, so for each $x_j \in \bigcup(V_{xi} \cap A)$, there is $G_{\alpha(xj)} \in \{G_\alpha: \alpha \in \Lambda\}$such that $x_j \in G_{\alpha(xj)}$and $j \in N$. Hence $\{G_{\alpha i}: i \in \mathrm{N}\} \cup \{G_{\alpha(xj)}: j \in N\}$ is a countable subcover of $\{G_\alpha: \alpha \in \Lambda\}$ and it covers A. Hence A is Zc –Lindelof relative to X.

Proposition 5.5: If X is a space such that every Zc –open subset of X is a Zc –Lindelof relative to X, then every subset is Zc –Lindelof relative to X.

Proof: Let A be an arbitrary subset of X and let $\{U_i: i \in \mathrm{I}\}$be a cover of A by Zc –open set.Then the family $\{U_i: i \in \mathrm{I}\}$is a Zc –open cover of the Zc –open set $\bigcup\{U_i: i \in \mathrm{I}\}$.By our assumption there is a countable subfamily $\{U_{ij}: j \in \mathrm{I}\}$ which covers $\bigcup\{U_i: i \in \mathrm{I}\}$.This subfamily is also a cover of the set A.

Theorem 5.6: A space X is Zc –Lindelof if and only if for every collection of Zc –closed sets with countable intersection property $\bigcap_{\alpha \in \Lambda} F_\alpha \neq \emptyset$.

Proof: Let X is Zc –Lindelof.$\{F_\alpha: \alpha \in \Lambda\}$ be a collection of Zc –closed sets with countable intersection property, $\bigcap_{\alpha \in \Lambda} F_\alpha \neq \emptyset$.Suppose that $\bigcap_{\alpha \in \Lambda} F_\alpha = \emptyset$.Then $X = \bigcup_{\alpha \in \Lambda} {F_\alpha}^c$ where ${F_\alpha}^c$ is Zc –open set for each $\alpha \in \Lambda$.Hence $\{{F_\alpha}^c: \alpha \in \Lambda\}$is a Zc –open cover of X which is Zc –Lindelof, there exist countably many members$\alpha_1,\alpha_2,\ldots\alpha_n,\ldots$ such that $X = \bigcup_{i \in \mathrm{N}} {F_{\alpha i}}^c = (\bigcap_{i \in \mathrm{N}} F_{\alpha i})^c, \bigcap_{i \in \mathrm{N}} F_{\alpha i} = F_{\alpha 1} \cap \ldots \cap F_{\alpha n} \cap \ldots = \emptyset$ a contradiction to our assumption that $\{F_\alpha: \alpha \in \Lambda\}$ has a countable intersection property. Hence $\bigcap_{\alpha \in \Lambda} F_\alpha \neq \emptyset$. Conversely, let every collection of Zc –closed subset of X with the countable intersection property,$\bigcap_{\alpha \in \Lambda} F_\alpha \neq \emptyset$.Suppose that X is not Zc –Lindelof, then there exist Zc –open cover $\{G_\alpha: \alpha \in \Lambda\}$of X has no countable subcover $\{G_{\alpha 1},G_{\alpha 2},\ldots G_{\alpha n},\ldots\}$,thus $X = G_{\alpha 1} \cup G_{\alpha 2} \ldots \cup G_{\alpha n},\ldots$. Then $(\bigcup_{i \in N} G_{\alpha i})^c = \bigcap_{i \in \mathrm{N}} {G_{\alpha i}}^c \neq \emptyset$. But $\{{G_\alpha}^c: \alpha \in \Lambda\}$ be a collection of Zc –closed set of X with countable intersection property by assumption. Then $\bigcap_{\alpha \in \Lambda} {G_\alpha}^c \neq \emptyset$, $(\bigcup_{\alpha \in \Lambda} G_\alpha)^c \neq \emptyset$ which is a contradiction that G is a Zc –open cover of X, which have a countable subcover and hence X is Zc –Lindelof.

References

[1] EL-Magharabi, A. I., & Mubarki, A. M. (2011). Z-open sets and Z-continuity in topological spaces. International Journal of Mathematical Archive (IJMA), 2(10), 1819-1827.

[2] RM.Sivagama Sundari, A.P.Dhanabalan, Zc-open sets and Zs-open sets in topological spaces, *ijsr.*,3(5),(2016).

[3] RM.SivagamaSundari, A.P.Dhanabalan,On Zc Separation Axioms and On s*g.Zc*–closed , *Zc.g*–closed sets, ISRJ, vol.6, Issue -8,(2016).