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Adomian Decomposition Approach to Solve the Simple Harmonic Quantum Oscillator

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Abstract

The simple harmonic quantum oscillator problem has been solved using two methods namely, the algebraic method where the raising and lowering operators have been used, and the Frobenius method. Here, we will adopt the Adomian decomposition method to solve this problem and to derive the Hermite polynomials in much easier way than the above mentioned methods.

Keywords: Harmonic Oscillator, Adomian Decomposition, Hermite Polynomial.

INTRODUCTION.

The problem of the simple harmonic oscillator is one of the oldest problems in classical and quantum mechanics[1, 2]. In this problem we will solve the time independent Schrodinger equation with potential depends on the square of the displacement. There are two methods used to solve this problem. The first method is the algebraic method that depends on raising and lowering operators where the quantization rules of oscillator comes naturally[2]. It is very efficient way once we have introduced the problem to the second quantization formalism and the Boson system is discussed. The second method to solve this problem is by using the Frobenius method[3] and adopt the Taylor series expansion. This is very lengthy way and one needs to know how to terminate the series solution and get the physical acceptable polynomial solution. In fact the dependence of the potential on the square of the displacement makes the solution of the time independent Schrodinger equation difficult. We have to capture the physical solution rather than the mathematical solution where the mathematical solution is divergent and has no physical significance[2].

In this work we will adopt the Adomian decomposition method[4]. In the last two decades, the Adomian decomposition method has been used widely in solving ordinary and partial differential equations[5-8], Sturm-Liouville problems[9], physical problems[10-14], and nonlinear and stochastic problems[5-7]. Even though the Adomian decomposition method provide a series solution, however it provides a more rapid and realistic solution[15]. We will show that the power of this method is analytically a straight forward process and we can get the Hermite polynomials one by one. Moreover, this method gives the right sign of the Hermite polynomials. It is much easier, more attractable and understandable than the Frobenius method.

This paper is organized by writing the time independent Schrodinger equation for the simple harmonic oscillator. Then a brief treatment to convert the problem to Hermite differential equation. After that we will adopt the Adomian decomposition method. Since the equation is second ordinary differential equation we leave the two arbitrary constants through all our calculations and show that the Hermite polynomials come naturally with the correct sign. Finally, we will give a discussion and a brief conclusion.

THEORY

The time independent Schrodinger's equation for a simple harmonic oscillator is given by:

$$\frac{-\hbar^2}{2m}\frac{d^2\psi}{dx^2}+\frac{1}{2}m\omega^2x^2\psi=E\psi \tag{1}$$

It contains potential depends on the square of the position x which makes it difficult to solve. It is clear that $\psi(\xi)\sim e^{\pm\xi^2/2}$ where $\xi=\sqrt{\frac{m\omega}{\hbar}}x$ is an acceptable mathematical asymptotic solutions of the above equation which transform the above equation to the Hermite like equation. Since we are dealing with a physical problem, we will ignore the positive exponent solution which diverges at $\pm\infty$. Those will reduce the above equation into the following:

$$\frac{d^2U}{d\xi^2}-2\xi\frac{dU}{d\xi}+2\left(\frac{E}{\hbar\omega}-\frac{1}{2}\right)U=0\ , \tag{2}$$

where $\psi(\xi)=U(\xi)e^{-\xi^2/2}$.

Now let us define $\lambda=\left(\frac{E}{\hbar\omega}-\frac{1}{2}\right)$, and change the dummy ξ to x. This will reduce equation (2) to the following form

$$\frac{d^2U}{dx^2}-2x\frac{dU}{dx}+2\lambda U=0 \tag{3}$$

We will solve this eigenvalue problem using the Adomian decomposition technique. Consider the operator $L=\frac{d^2}{dx^2}$. Applying this operator to equation to get:

$$LU=2x\frac{dU}{dx}-2\lambda U \tag{4}$$

Now consider the integral operator $L^{-1}(\cdot)=\iint(\cdot)dxdx$ and apply it from left to equation (4) i.e.,

$$L^{-1}LU=L^{-1}\left(2x\frac{dU}{dx}-2\lambda U\right)$$

Then, we get the following result

$$U=C_o+C_1x+L^{-1}\left(2x\frac{dU}{dx}-2\lambda U\right) \tag{5}$$

Where C_0 and C_1 are arbitrary constants. Following the Adomian method, the general solution for U:

$$U = \sum_{n=0}^{\infty} U_n \tag{6}$$

Now equate the coefficients to get:

$$U_o = C_o + C_1 x$$
$$U_{n+1} = L^{-1}\left(2x\frac{dU_n}{dx} - 2\lambda U_n\right); \quad n \geq 0 \tag{7}$$

By calculating the first few terms of equation (7), one can write the general solution in terms of C_o and C_1 (see the appendix for the detailed calculations and the calculations of other terms) as

$$U_\lambda = C_o\left[1 + \sum_{n=1}^{\infty}\frac{(-2)^n}{(2n)!}x^{2n}\prod_{i=1}^{n}\big(\lambda - (2i-2)\big)\right] + C_1\left[x - \sum_{n=1}^{\infty}\frac{2^n}{(2n+1)!}x^{2n+1}\prod_{i=1}^{n}\big(\lambda - (2i-1)\big)\right] \tag{8}$$

Equation (8) gives the Hermite Polynomial to any order. To terminate the series solution, it is clear that λ has to be an integer. If this integer is even then $C_1 = 0$, otherwise $C_0 = 0$. As an example for λ = 0, 2 equation (8) gives, respectively $U_0 = C_0 \sim H_0$ and $U_2 = C_0(1-2x^2) \sim H_2$. The constants C_0 and C_1 can be chosen to normalize. Since $\lambda = \left(\frac{E}{\hbar\omega} - \frac{1}{2}\right)$, and this term should be an integer, we reach to the well-known energy formula for the harmonic oscillator, i.e., $E = \hbar\omega\left(n + \frac{1}{2}\right)$.

Finally, we were able to solve the time independent Schrodinger equation by adopting the Adomian decomposition method. Moreover, this work can be extend to include anharmonic terms of the potential without lying on the perturbation theories. The above treatment can be considered another method of solving the Harmonic oscillator problem. It is more efficient and shorter compared to the Taylor series where points have to be ordinary and the algebraic method that depends on Lie algebra treatment[16].

CONCLUSIONS

Using the Adomian decomposition approach we were able to solve the time independent Schrodinger equation for the simple harmonic quantum oscillator problem. We get all Hermite polynomials in a simple manner with the right sign. Moreover, this simple method can be extended to include anharmonic potential terms.

Appendix:

In this appendix we will show some of the major calculation used above.

The following steps show how we calculate U_1 up to U_5 which can be easily followed**.** Then in the last step we showed the general form of *U*.

$$U_1 = L^{-1}\left(2x\frac{dU_o}{dx} - 2\lambda U_o\right)$$
$$= L^{-1}(2C_1(1-\lambda)x - 2\lambda C_o)$$
$$= \iint (2C_1(1-\lambda)x - 2\lambda C_o)dxdx$$
$$U_1 = -\frac{2\lambda C_o}{2}x^2 + \frac{2C_1(1-\lambda)}{2\cdot 3}x^3$$
$$U_2 = L^{-1}\left(2x\frac{dU_1}{dx} - 2\lambda U_1\right)$$
$$= L^{-1}\left(\frac{2^2C_o(\lambda-0)(\lambda-2)}{2}x^2 + \frac{2^2C_1(1-\lambda)(3-\lambda)}{2\cdot 3}x^3\right)$$
$$U_2 = \frac{2^2C_o(\lambda-0)(\lambda-2)}{2\cdot 3\cdot 4}x^4 + \frac{2^2C_1(1-\lambda)(3-\lambda)}{2\cdot 3\cdot 4\cdot 5}x^5$$
$$U_3 = L^{-1}\left(2x\frac{dU_2}{dx} - 2\lambda U_2\right)$$
$$U_3 = \frac{-2^3C_o(\lambda-0)(\lambda-2)(\lambda-4)}{2\cdot 3\cdot 4\cdot 5\cdot 6}x^6 + \frac{2^3C_1(1-\lambda)(3-\lambda)(5-\lambda)}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}x^7$$

$$U_3 = \frac{-2^3 C_o(\lambda-0)(\lambda-2)(\lambda-4)}{6!}x^6 + \frac{2^3 C_1(1-\lambda)(3-\lambda)(5-\lambda)}{7!}x^7$$

$$U_4 = L^{-1}\left(2x\frac{dU_3}{dx} - 2\lambda U_3\right)$$

$$U_4 = \frac{2^4 C_o(\lambda-0)(\lambda-2)(\lambda-4)(\lambda-6)}{8!}x^8 + \frac{2^4 C_1(1-\lambda)(3-\lambda)(5-\lambda)(7-\lambda)}{9!}x^9$$

$$U_5 = L^{-1}\left(2x\frac{dU_4}{dx} - 2\lambda U_4\right)$$

$$U_5 = \frac{-2^5 C_o(\lambda-0)(\lambda-2)(\lambda-4)(\lambda-6)(\lambda-8)}{10!}x^{10} + \frac{2^5 C_1(1-\lambda)(3-\lambda)(5-\lambda)(7-\lambda)(9-\lambda)}{11!}x^{11}$$

$$U_5 = \frac{-2^5 C_o(\lambda-0)(\lambda-2)(\lambda-4)(\lambda-6)(\lambda-8)}{10!}x^{10} - \frac{2^5 C_1(\lambda-1)(\lambda-3)(\lambda-5)(\lambda-7)(\lambda-9)}{11!}x^{11}$$

$$U_5 = \frac{(-2)^n C_o}{10!}x^{2n}\prod_{i=1}^{5}\big(\lambda-(2i-2)\big) - \frac{2^n C_1}{11!}x^{2n+1}\prod_{i=1}^{5}\big(\lambda-(2i-1)\big)$$

$$\vdots$$

$$U_n = \frac{(-2)^n C_o}{(2n)!}x^{2n}\prod_{i=1}^{n}\big(\lambda-(2i-2)\big) - \frac{2^n C_1}{(2n+1)!}x^{2n+1}\prod_{i=1}^{n}\big(\lambda-(2i-1)\big)$$

$$U = U_o + \sum_{n=1}^{\infty} U_n = C_o + \sum_{n=1}^{\infty}\frac{(-2)^n C_o}{(2n)!}x^{2n}\prod_{i=1}^{n}\big(\lambda-(2i-2)\big) + C_1 x - \sum_{n=1}^{\infty}\frac{2^n C_1}{(2n+1)!}x^{2n+1}\prod_{i=1}^{n}\big(\lambda-(2i-1)\big)$$

$$U = C_o\left[1 + \sum_{n=1}^{\infty}\frac{(-2)^n}{(2n)!}x^{2n}\prod_{i=1}^{n}\big(\lambda-(2i-2)\big)\right] + C_1\left[x - \sum_{n=1}^{\infty}\frac{2^n}{(2n+1)!}x^{2n+1}\prod_{i=1}^{n}\big(\lambda-(2i-1)\big)\right]$$

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