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Accurate Neighborhood Resolving Sets of a Graph

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Abstract

Let G be a simple connected graph. A subset S of $V(G)$ is called a neighborhood set (n-set) of G if $G = \cup_{v \in S}\langle N[v]\rangle$, where $N[v]$ denotes the closed neighborhood of the vertex v in G. Further for an ordered subset $S = \{v_1, v_2, \ldots, v_k\}$ of $V(G)$ and a vertex u of G, we associate a vector $\Gamma(u) = (d(u, s_1), d(u, s_2), \ldots, d(u, v_k))$ with respect to S, where $d(u, v)$ denotes the distance between u and v in G. A subset S of $V(G)$ is said to be a resolving set (r-set) of G if $\Gamma(u) \neq \Gamma(v)$, for all $u, v \in V(G) - S$, $u \neq v$. A neighborhood set of G which is also a resolving set is called a neighborhood resolving set (nr-set) of G. An nr-set S of G is called an accurate neighborhood resolving set (anr-set) of G if $\bar{S}$ has no nr-set of G of cardinality of S. The purpose of this paper is to compute minimum cardinality of nr-sets and anr-sets of certain graphs and their certain derived graphs.

Key words: neighborhood set, resolving set, neighborhood resolving set, accurate neighborhood resolving set.

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1. Introduction

All the graphs considered here are non trivial, undirected, finite, connected and simple. We use the standard terminology, the terms not defined here may be found in [3, 1]. Let $G(V, E)$ be a graph. For a vertex $v \in V$, $N(v)$ denotes the set of all vertices of G which are adjacent to v and $N[v] = N(v) \cup \{v\}$. A subset S of V is called a neighborhood set or n-set of G if $G = \cup_{v \in S}\langle N[v]\rangle$, where for a subset S of V, $\langle S\rangle$ denotes the subgraph of G induced by the set S. An nr-set S is called minimal if no proper subset of S is an nr-set. The minimum cardinality of a minimal n-set is called the neighborhood number of G and is denoted by $ln(G)$. The concept of neighborhood number for a graph was first introduced by E. Sampathkumar et al. [6].

A subset S of V is called a resolving set or an r-set of G if for each pair $u, v \in V - S, u \neq v$, there is a vertex w in S such that $d(u, w) \neq d(v, w)$. The minimum cardinality of a minimal n-set is called the resolving number of G and is denoted by $lr(G)$. The concept of resolving number for a graph was first introduced by F. Harary and R.A. Melter [4] and independently by P.J. Slater [9].

A subset S of V is called a neighborhood resolving set or nr-set of G if S is both neighborhood and resolving set of G. The minimum cardinality of a minimal nr-set is called the neighborhood resolving number of G and is denoted by $lnr(G)$. An nr set S of G is called an accurate neighborhood resolving set or anr-set of G if $\bar{S}$ has no nr-set of G of cardinality of S. The minimum cardinality of a minimal anr-set is called the accurate neighborhood resolving number of G and is denoted by $lnr_a(G)$.

For a graph $G(V, E)$ with $V = \{v_1, v_2, \ldots, v_n\}$, consider the corresponding set $V' = \{u_1, u_2, \ldots, u_n\}$, then the Mycielski graph $M(G)$ of G is the with $V(M(G)) = V \bigcup V' \bigcup \{w\}$ and $E(M(G)) = E(G) \bigcup \{v_i u_j, : v_i v_j \in E(G), 1 \leq i, j \leq n\} \bigcup \{u_i w : 1 \leq i \leq n\}$

While finding an n-set S for a graph G, we say that a vertex $v \in V$ covers an edge $e = xy$, $x, y \in V$ if $e \in \langle N[v]\rangle$ and we see that S is an n-set of G if each edge of G is covered by some vertex of S.

Throughout this paper P_n denotes a path on n vertices with vertex set $V = \{v_1, v_2, \ldots, v_n\}$, where for $1 \leq i \leq n-1, v_i \sim v_{i+1}$. C_n denotes a cycle on n vertices with vertex set $V = \{v_1, v_2, \ldots, v_n\}$, where for $1 \leq i \leq n-1, v_i \sim v_{i+1}$ and $v_1 \sim v_n$. $W_{1,n}$ denotes the wheel graph with $V = \{v, v_1, v_2, \ldots, v_n\}$, where v is the central vertex and $v_1, v_2, \ldots, v_n$ are the rim vertices of the wheel.

Remark 1.1. *For any graph G, as every anr-set is also an nr-set, we have $lnr_a(G) \geq lnr(G)$.*

Remark 1.2. *For $P_n, n \geq 3$, any 2-element subset of vertices is a resolving set.*

Remark 1.3. *For any graph G with n vertices and satisfying a property p, if p-number $p(G) = m$ where $m \geq \lceil \frac{n+1}{2} \rceil$ then the accurate p-number $p_a(G) = m$.*

Proof. Clearly $p_a(G) \geq m$. Conversely, let S be a p-set of G with $|S| = m$ where $m \geq \lceil \frac{n+1}{2} \rceil$. Then $|\bar{S}| = n - m = n - \lceil \frac{n+1}{2} \rceil = \lceil \frac{n-1}{2} \rceil < \lceil \frac{n+1}{2} \rceil \leq m$. Thus $\bar{S}$ can not contain a subset of cardinality m. Hence S is an $accurate$ p-set of G so that $p_a(G) \leq m$, proving $p_a(G) = m$. □

Theorem 1.4 ((E. Sampathkumar, Prabha S, Neeralagi [6]). *A set S of vertices of a graph G is an n-set if and only if every line of $\langle V(G) - S\rangle$ belongs to a triangle one of whose vertices belong to S.*

Remark 1.5. *If G is a triangle free graph, then by Theorem 1.4 a set S is an n-set of G if and only if for each edge $e = v_i v_j$ of G either $v_i \in S$ or $v_j \in S$.*

Theorem 1.6 (E. Sampathkumar, Prabha S, Neeralagi [7]). *For any positive integer n,* $ln(P_n) = \lfloor \frac{n}{2} \rfloor$

Theorem 1.7 (B. Sooryanarayana, A. S. Suma [10]). *For any positive integer n,*

$$lnr(P_n) = \begin{cases} \lceil \frac{n}{2} \rceil, & for\ n \leq 3 \\ \lfloor \frac{n}{2} \rfloor, & for\ n \geq 4 \end{cases}$$

Theorem 1.8 (E. Sampathkumar, Prabha S, Neeralagi [6]). *For any integer* $n \geq 3$, $ln(C_n) = \lceil \frac{n}{2} \rceil$.

Theorem 1.9 (B. Sooryanarayana, A. S. Suma [10]). *For each integer* $i \geq 3$, *every* i *element subset* S *of vertices of a cycle* C_n *is always an* r*-set.*

Theorem 1.10 (B. Sooryanarayana, A. S. Suma [10]). *For any integer* $n \geq 3$,

$$lnr(C_n) = \begin{cases} 3, & for\ n = 4 \\ \lceil \frac{n}{2} \rceil, & otherwise \end{cases}$$

Theorem 1.11 (B. Shanmukha, B. Sooryanarayana, K.S.Harinath [8]). *If* $W_{1,n}$ *is the wheel graph for* $n \geq 3$, *then*

1. $lr(W_{1,3}) = lr(W_{1,6}) = 3$
2. $lr(W_{1,4}) = lr(W_{1,5}) = 2$
3. $lr(W_{1,x+5k}) = 3 + 2k$, *if* $x = 7, 8$ *and* $= 4 + 2k$, *if* $x = 9, 10, 11$.

Theorem 1.12 (Peter S. Buczkowski, Gary chartrand,Christopher Poisson, Ping Zhang [5]). *If* $n \geq 7$, *then* $lr(W_{1,n}) = \lfloor \frac{2(n+1)}{5} \rfloor$.

2. nr-sets of a Paths and Cycles

Theorem 2.1. *For any positive integer n,*

$$lnr_a(P_n) = \begin{cases} \lceil \frac{n}{2} \rceil, & for\ n = 1, 3 \\ 2, & for\ n = 2 \\ \lfloor \frac{n}{2} \rfloor, & for\ n \geq 4 \end{cases}$$

Proof. Let $G = P_n$. For $n = 1$ the result is obvious.

For $n = 2$, by Theorem 1.7, $lnr(G) = 1$ and $S_1 = \{v_1\}, S_2 = \{v_2\}$ are the only $nr - sets$ of G of cardinality 1, but their complements are also $nr - sets$ of G of cardinality 1 so that we get $lnr_a(G) = 2$.

For $n = 3$, by Theorem 1.7, $lnr(G) = 2$ and by Remark 1.3 it follows that $lnr_a(G) = 2$.

Let $n \geq 4$. By Remark 1.1 and Theorem 1.7, it is clear that $lnr_a(G) \geq \lfloor \frac{n}{2} \rfloor$. We prove the reverse inequality. We note that G is triangle free and Remark 1.5 is applicable in G.

Case 1: n is odd.

Let $n = 2k + 1$. Let $S = \{v_2, v_4, \ldots, v_{n-1}\}$. Then $|S| = k$. As $\langle V(G) - S \rangle$ contains no edge, from Theorem 1.4 , S is a neighborhood set of G. Also as $|S| \geq 2$, from Remark 1.2, it is also a resolving set of G. Hence S is an nr-set of G. Now, $\bar{S} = \{v_1, v_3, \ldots, v_n\}$ and $|\bar{S}| = k + 1$. We show that no k element subset of $\bar{S}$ is an nr-set of G. Let $T \subset \bar{S}$ such that $|T| = k$. Then for some odd $i, 1 \leq i \leq n, v_i \notin T$. But then for $i = 1$, edge v_1v_2 is not covered by vertices of T as $v_1, v_2 \notin T$ and for $i = n$, edge $v_{n-1}v_n$ is not covered by vertices of T as $v_{n-1}, v_n \notin T$. For $i, 1 < i < n$, edges $v_{i-1}v_i$ and v_iv_{i+1} are not covered by vertices of T as $i - 1$ and $i + 1$ are even, $v_{i-1}, v_{i+1} \notin T$ also $v_i \notin T$. Thus T can not be an n-set of G and hence not an nr-set of G. So S is an anr-set of G and $|S| = k = \lfloor \frac{n}{2} \rfloor$. Hence we get $lnr_a(G) \leq \lfloor \frac{n}{2} \rfloor$ so that $lnr_a(G) = \lfloor \frac{n}{2} \rfloor$ in this case.

Case 2: n is even.

Let $n = 2k$. Let $S = \{v_2, v_3, v_5 \ldots, v_{n-1}\}$. Then $|S| = k$. As $\langle V(G) - S \rangle$ contains no edge, from Theorem 1.4 , S is a neighborhood set of G. Also as $|S| \geq 2$, from Remark 1.2, it is also a resolving set of G. Hence S is an nr-set of G. Now, $\bar{S} = \{v_1, v_4, v_6, \ldots, v_n\}$ and $|\bar{S}| = k$. We show that $\bar{S}$ is not an nr-set of G. As both $v_2, v_3 \notin \bar{S}$, the edge v_2v_3 is not covered by the vertices of $\bar{S}$, so that it is not an n-set and hence not an nr-set of G. Hence S is an anr-set of G. Thus, $lnr_a(G) = k = \lfloor \frac{n}{2} \rfloor$ in this case. □

Theorem 2.2. *For any integer* $n \geq 3$, $lnr_a(C_n) = \lceil \frac{n+1}{2} \rceil$.

Proof. As $lnr(C_4) = 3$ by Theorem 1.10, we have $lnr_a(C_4) = 3$ by Remark 1.3. Let $n \neq 4$. By Remark 1.1 and Theorem 1.10, we must have $lnr_a(G) \geq \lceil \frac{n}{2} \rceil$.

Case 1: n is even.

If n is even, then $\lceil \frac{n}{2} \rceil = \frac{n}{2}$ and hence $n - \frac{n}{2} = \frac{n}{2}$. We note that $\{v_1, v_3, \ldots, v_{n-1}\}$ and $\{v_2, v_4, \ldots, v_n\}$ are the only two nr-sets of C_n of cardinality $\frac{n}{2}$. Therefore if S is an nr-set of C_n of cardinality $\frac{n}{2}$, then $\bar{S}$ is also an nr-set of C_n of cardinality $\frac{n}{2}$. Hence we must have $lnr_a(C_n) \geq \frac{n}{2} + 1$. Conversely, $\{v_1, v_3, \ldots, v_{n-1}, v_n\}$ is an nr-set of C_n with $|S| = \frac{n}{2} + 1$. Then $|\bar{S}| = \frac{n}{2} - 1 < |S|$, so that S is an anr-set of C_n. Thus, $lnr_a(C_n) = \frac{n}{2} + 1 = \lceil \frac{n+1}{2} \rceil$ in this case.

Case 2: n is odd.

If n is odd, then $\lceil \frac{n}{2} \rceil = \lceil \frac{n+1}{2} \rceil$. Hence by Remark 1.3 , $lnr_a(C_n) = \lceil \frac{n+1}{2} \rceil$, thus proving the theorem. □

3. nr-sets of Mycielski graph of Paths and Cycles

Theorem 3.1. *For any integer* $n \geq 2$,

$$ln(M(P_n)) = 2\lfloor \frac{n}{2} \rfloor + 1.$$

Proof. Let $G = M(P_n)$. Let $V(G) = \{v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n, w\}$ where $V(P_n) = \{v_1, v_2, \ldots, v_n\}$. We note that as G is a triangle free graph, by Remark 1.5, for any edge uv of G, $uv \in \langle N(p) \rangle$ for any vertex $p \in V(G)$ implies that $u = p$ or $v = p$.

Let S be a neighborhood set of G. We note that to cover the edges u_iw, $1 \leq i \leq n$ either $w \in S$ or $u_i \in S$, for all i, $1 \leq i \leq n$.

Case 1: $w \notin S$.

To cover the edges $u_i w, 1 \leq i \leq n$, as $w \notin S$, we must have $u_i \in S$, for all $i, 1 \leq i \leq n$. These vertices u_i, $for\ all\ i, 1 \leq i \leq n$ also cover the edges $u_i v_{i+1}, v_i u_{i+1}, 1 \leq i \leq n-1$. But then to cover the edges of P_n from Theorem 1.7, $\lfloor \frac{n}{2} \rfloor$ vertices of $V(P_n)$ must belong to S, so that $|S| \geq n + \lfloor \frac{n}{2} \rfloor$.

Case 2: $w \in S$.

In this case all the edges $u_i w, 1 \leq i \leq n$ are covered by w. Consider the four vertices u_i, u_{i+1}, v_i, v_{i+1}, $1 \leq i \leq n-1$. To cover the edge $u_i v_{i+1}$ in G, we must have $u_i \in S$ or $v_{i+1} \in S$. Also to cover the edge $v_i u_{i+1}$ in G, we must have $v_i \in S$ or $u_{i+1} \in S$. So to cover the edges $u_i v_{i+1}$ and $v_i u_{i+1}, 1 \leq i \leq n-1$, we must have $2\lfloor \frac{2n}{4} \rfloor = 2\lfloor \frac{n}{2} \rfloor$ vertices from $\{v_i : 1 \leq i \leq n\} \cup \{u_i : 1 \leq i \leq n\}$ in S. Hence we must have $|S| \geq 2\lfloor \frac{n}{2} \rfloor + 1$.

By comparing the above two cases as $1 + 2\lfloor \frac{n}{2} \rfloor \leq n + \lfloor \frac{n}{2} \rfloor$, for $n \geq 2$, we must have $ln(G) \geq 2\lfloor \frac{n}{2} \rfloor + 1$.

We now show that the set

$$S = \begin{cases} \{w, u_2, v_2, u_4, v_4, \ldots, u_{n-1}, v_{n-1}\}, & \text{if } n \text{ is odd} \\ \{w, u_2, v_2, u_4, v_4, \ldots, u_n, v_n\}, & \text{if } n \text{ is even} \end{cases}$$

with $|S| = 2\lfloor \frac{n}{2} \rfloor + 1$ is an n-set of G.

All the edges $wu_i, 1 \leq i \leq n$ are covered by the vertex w in S. The edges $u_i v_{i+1}$ and $v_i u_{i+1}, 1 \leq i \leq n-1$ are covered by the vertices u_i and v_i if i is even and u_{i+1} and $v_i + 1$ if i is odd. Also the edges $v_i v_{i+1}, 1 \leq i \leq n-1$ are covered by v_i if i is even and by v_{i+1} if i is odd. So S is a n-set of G. As $|S| = 2\lfloor \frac{n}{2} \rfloor + 1$, we must have $ln(G) \leq 2\lfloor \frac{n}{2} \rfloor + 1$. Hence we get $ln(G) = 2\lfloor \frac{n}{2} \rfloor + 1$. □

Corollary 3.2. *For any integer* $n \geq 2$

$$lnr(M(P_n)) = \begin{cases} 5, & for\ n = 3 \\ 2\lfloor \frac{n}{2} \rfloor + 1, & otherwise \end{cases}$$

Proof. For $n = 2$, as $M(P_2) \simeq C_5$, $lnr(M(P_2)) = lnr(C_5) = 3$ by Theorem 1.10.

Let $G = M(P_n)$. Let $n = 3$. We see that $d(u_1, x) = d(u_3, x)$ and $d(v_1, x) = d(v_3, x)$ for each $x \in V(G)$. Hence if T is an n-set of G then we must have either $u_1 \in T\ or\ u_3 \in T$ and $v_1 \in T\ or\ v_3 \in T$. If $u_1 \in T$ and $v_1 \in T$ (or $u_1 \in T$ and $v_3 \in T$) then T to be an n-set of G, we must have either $u_3, v_3, w \in T\ or\ u_2, u_3, v_3 \in T$ (either $v_1, u_3, w \in T\ or\ u_2, u_3, v_1 \in T$) so that T is an nr-set of G and $|T| = 5$ in every case (other cases follow by symmetry) and hence $lnr(G) = 5$.

Let $n > 3$. We show that the set

$$S = \begin{cases} \{w, u_2, v_2, u_4, v_4, \ldots, u_{n-1}, v_{n-1}\}, & \text{if } n \text{ is odd} \\ \{w, u_2, v_2, u_4, v_4, \ldots, u_n, v_n\}, & \text{if } n \text{ is even} \end{cases}$$

in Theorem 3.1 also resolves G by showing that for each pair $x, y \in V - S$, $x \neq y$, there is a vertex z in S such that $d(z, x) \neq d(z, y)$. Let $x, y \notin S$. We consider the different cases.

If $x = v_i$, $y = u_j, 1 \leq i, j \leq n$, then $d(w, x) = 3$ where as $d(w, y) = 2$.

If $x = v_1(or\ u_1)$, $y = v_3(or\ u_3)$, then $d(v_4, x) = 3$ where as $d(v_4, y) = 1$.

If $x = v_1(or\ u_1)$, $y = v_j(or\ u_j)$, $3 < j \leq n$, then $d(v_2, x) = 1$ where as $d(v_2, y) > 1$.

If $x = v_i(or\ u_i)$, $y = v_j(or\ u_j), 1 < i, j \leq n$, (w.l.g we may assume that $i < j$), i, j are odd ($(i-1)$ is even) then $d(v_{i-1}, x) = 1$ where as $d(v_{i-1}, y) > 1$.

Thus S is an nr-set of G and $|S| = 1 + 2\lfloor \frac{n}{2} \rfloor$, we must have $lnr(G) = 2\lfloor \frac{n}{2} \rfloor + 1$. □

Corollary 3.3. *For any integer* $n \geq 2$

$$lnr_a(M(P_n)) = \begin{cases} 5, & for\ n = 3 \\ 2\lfloor \frac{n}{2} \rfloor + 1, & otherwise \end{cases}$$

Proof. By Theorem 3.2 $lnr(M(P_2)) = 3$, $lnr(M(P_3)) = 5$. So by Remark 1.3 we have $lnr_a(M(P_2)) = 3$, $lnr_a(M(P_3)) = 5$.

For $n \geq 4$, let $G = M(P_n)$. By the proof of Corollary 3.2 we see that if S is a minimal nr-set of G with $|S| = 2\lfloor \frac{n}{2} \rfloor + 1$ then w must belong to S. If T is any nr-set of G with $w \notin T$ then $|T| \geq n + \lfloor \frac{n}{2} \rfloor$. As $2\lfloor \frac{n}{2} \rfloor + 1 < n + \lfloor \frac{n}{2} \rfloor$, for $n \geq 3$, $\bar{S}$ can not have an nr-set of G of cardinality $2\lfloor \frac{n}{2} \rfloor + 1$ (Infact no nr-set of G in $\bar{S}$), proving the corollary. □

4. nr-sets of a Wheel graph

Theorem 4.1. *For any integer* $n \geq 3$*,*

$$lnr(W_{1,n}) = \begin{cases} 3, & for\ n = 3,\ 4,\ 5,\ 6 \\ \lfloor \frac{2(n+1)}{5} \rfloor + 1, & for\ n \geq 7 \end{cases}$$

Proof. Let $G = W_{1,n}$. Let $V(G) = \{v, v_1, v_2, \ldots, v_n\}$, where v is the central vertex and $v_1, v_2, \ldots, v_n$ are the rim vertices of the wheel $W_{1,n}$.

The set $S = \begin{cases} \{v_1, v_2, v\}, & \text{if } n = 3 \\ \{v_1, v_3, v_5\}, & \text{if } n = 6 \end{cases}$ is an nr-set of G. As from Theorem 1.11, $lr(G) = 3$, we must have $lnr(G) = 3$.

For $n = 4$, 5, as in 1.11, $lr(G) = 2$ and those resolving vertices must be adjacent rim vertices. But we note that a set of 2 adjacent rim vertices can not form an n-set of G. Hence $lnr(G) \geq 3$. Note that the set $S = \{v_1, v_2, v\}$ is an nr-set of G, so that $lnr(G) = 3$.

Let $n \geq 7$. If T is any nr-set of G such that $v \notin T$, then as T is an n-set of G, it must contain alternate rim vertices, so that $|T| \geq \lceil \frac{n}{2} \rceil$. We have $lr(G) = \lfloor \frac{2(n+1)}{5} \rfloor$ by Theorem 1.12. Let S be a minimal r-set of G containing $\lfloor \frac{2(n+1)}{5} \rfloor$ vertices. As $d(v, v_i) = 1$ for all i, $1 \leq i \leq n$, v does not resolve any other vertex of G, so we have $v \notin S$. But, as $\lceil \frac{n}{2} \rceil > \lfloor \frac{2(n+1)}{5} \rfloor$, for $n \geq 7$, we have S can not be an n-set of G. Hence we note that $S \cup \{v\}$ is a minimal nr-set of G containing v, as $\{v\}$ is an n-set of G. So if v belongs to a minimal nr-set of G, then its cardinality must be $\lfloor \frac{2(n+1)}{5} \rfloor + 1$ and if v does not belong to an nr-set of G, then its cardinality must be greater than or equal to $\lceil \frac{n}{2} \rceil$. Hence, as $\lceil \frac{n}{2} \rceil \geq \lfloor \frac{2(n+1)}{5} \rfloor + 1$, for $n \geq 7$, we must have, $lnr(G) = \lfloor \frac{2(n+1)}{5} \rfloor + 1$. □

Corollary 4.2. *For any integer $n \geq 3$,*

$$lnr_a(W_{1,n}) = \begin{cases} 3, & for\ n = 3,\ 4,\ 5 \\ 4, & for\ n = 6,\ 7 \\ 5, & for\ n = 8,\ 9,\ 10 \\ \lfloor \frac{2(n+1)}{5} \rfloor + 1, & for\ n \geq 11 \end{cases}$$

Proof. Let $G = W_{1,n}$.

For $n = 3, 4$, by Theorem 4.1, $lnr(G) = 3$. So by Remark 1.3, $lnr_a(G) = 3$.

For $n = 5$, the set $S = \{v_1, v_2, v\}$ is an nr-set of G and $\bar{S} = \{v_3, v_4, v_5\}$ can not contain an n-set and hence an nr-set of G, as there is no vertex in $\bar{S}$ to cover the edge $e = v_1v_2$ in G. So S is an anr-set of G with $|S| = 3$. As by Theorem 4.1, $lnr(G) = 3$, we have $lnr_a(G) = 3$.

For $n = 6$, by Theorem 4.1, $lnr(G) = 3$. But we note that the 3 vertices in an nr-set of G must be the alternate rim vertices of G. But then the remaining 3 rim vertices in its complement is also an nr-set of G, so that $lnr_a(G) \geq 4$. But then the set $S = \{v, v_1, v_3, v_5, \}$ is an nr-set of G and $|\bar{S}| = 3$. Hence S is an anr-set of G. Thus $lnr_a(G) = 4$.

For $n = 7$, by the same argument as in for $n = 5$, we can show that $lnr_a(G) = 4$ by taking the set $S = \{v, v_1, v_5, v_7\}$. (Note that there is no vertex in $\bar{S}$ to cover the edge $e = v_7v_1$ in G.)

For $n = 8$, by Theorem 4.1, $lnr(G) = 4$. Let S be an nr-set of G with $|S| = 4$. We note that if $v \notin S$, then S must contain the alternate rim vertices of G. But then the remaining 4 rim vertices will be in $\bar{S}$ and they form an nr-set of G of cardinality 4. So S can not be an anr-set of G. If $v \in S$ then the remaining 3 verices are to be chosen from the rim vertices. We can choose them by making gaps of 1,1,3 only. (here gap refers for the distance between these vertices in C_n as $W_{1,n} = C_n + \{v\}$) (note that making gaps of 1,2,2 doesn't give an n-set of G). But in that case $\bar{S}$ contains an nr-set of G of cardinality 4 (2 end vertices in the gap 3 and each vertex from the two 1 gaps), so that S can not be an anr-set of G. Hence $lnr_a(G) \geq 5$. But then the set $S = \{v, v_1, v_4, v_6, v_8\}$ is an nr-set of G and $|\bar{S}| = 4$. Hence S is an anr-set of G. Thus $lnr_a(G) = 5$.

Again, for $n = 9$, by the same argument as in for $n = 5$, we can show that $lnr_a(G) = 5$ by taking the set $S = \{v, v_1, v_4, v_6, v_9\}$. (Note that there is no vertex in $\bar{S}$ to cover the edge $e = v_9v_1$ in G).

For $n = 10$, the set $S = \{v,\ v_1,\ v_4,\ v_6,\ v_9\}$ is an nr-set of G and $\bar{S} = \{v_2,\ v_3,\ v_5,\ v_7,\ v_8,\ v_{10}\}$. We note that a minimal nr-set of G in $\bar{S}$ is itself and $|\bar{S}| = 6$. So S is an anr-set of G with $|S| = 5$. As by Theorem 4.1, $lnr(G) = 5$, we have $lnr_a(G) = 5$.

For $n = 12$, by the same argument as in for $n = 10$, we can show that $lnr_a(G) = 6$ by taking the set $S = \{v,\ v_1,\ v_4,\ v_6,\ v_9,\ v_{11}\}$.

Also, for $n = 14$, by the same argument as in for $n = 5$, we can show that $lnr_a(G) = 7$ by taking the set $S = \{v, v_1, v_4, v_6, v_9, v_{11}, v_{14}\}$. (Note that there is no vertex in $\bar{S}$ to cover the edge $e = v_{14}v_1$ in G).

For $n = 11, 13$ and $n \geq 15$, we note that $\lceil \frac{n}{2} \rceil > \lfloor \frac{2(n+1)}{5} \rfloor + 1$. As argued in the proof of Theorem 4.1,we have if v belongs to a minimal nr-set of G, then its cardinality must be $\lfloor \frac{2(n+1)}{5} \rfloor + 1$ and if v does not belong to an nr-set of G, then its cardinality must be greater than or equal to $\lceil \frac{n}{2} \rceil$. So if S is a minimal nr-set of G containing v with cardinality $\lfloor \frac{2(n+1)}{5} \rfloor + 1$, then $v \notin \bar{S}$. So $\bar{S}$ can not contain an nr-set of G of cardinality $\lfloor \frac{2(n+1)}{5} \rfloor + 1$. Hence S is an anr-set of G. As by Theorem 4.1, $lnr(G) = \lfloor \frac{2(n+1)}{5} \rfloor + 1$, we have $lnr_a(G) = \lfloor \frac{2(n+1)}{5} \rfloor + 1$. □

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