Solitons and Exact Solutions for a Nonlinear (2+1)dim PDE

G.Bharathi 1, S.Padmasekaran 2.
1Department of Mathematics, Sethupathy Govt.Arts College, Ramanathapuram, Tamil Nadu, India.
2Department of Mathematics, Periyar University, Salem, Tamil Nadu, India.

Abstract
The exact solutions of a (2+1) dimensional nonlinear partial differential equations are obtained by Homogeneous Balance Method and used it to determine the solitary wave solutions. The homogeneous balance (HB) method has drawn lots of interests in seeking the solitary wave solution and other kinds of solutions. Wang showed the HB method is powerful for finding analytic solitary wave solutions of PDE. In this paper, the solitary wave solution of nonlinear PDE has been obtained by this method.

Keywords: Nonlinear equation, Homogeneous balance method, solitary wave solution.

1. INTRODUCTION
The homogeneous balance method (HBM) is introduced by Mingliang Wang [1, 2] to obtain solitary wave solutions of variant Boussinesq equations. Yubin Zhou, Zhibin Li [3], Lei Yang, Zhengang Zhu and Yinghai Wang [4] proposed a generalized homogeneous balance method and used it to determine the solitary wave solutions of the Boussinesq equation. Homogeneous Balance method yields special exact solutions and solitons for nonlinear partial differential equations Biao Li, Yong Chen and Hongqing Zhang [5], Feng [6], Lei Yang, Liu Jianbin, Yang Kongqing [7].

Homogeneous Balance Method is easy to apply and always yield a special exact solutions of nonlinear Partial Differential Equations. Wang showed the Homogeneous Balance Method is powerful for finding analytic solitary wave solutions of Partial Differential Equations. The idea is the highest nonlinear term partially balanced with the highest derivative term. In this paper, the solitary wave solution of nonlinear Partial Differential Equations has been obtained by this method. The exact solutions of (2+1) dimensional Partial Differential Equation is exponential functions besides the solitary wave solution. Based on the backlund transformations, exact solutions are also obtained.

In the present paper we consider the (2+1) dimensional partial differential equation

\[ u_t + u^p u_x + \alpha u + \beta u^q - u_{xx} + \gamma u_{xxx} + u_{yy} = 0. \]  

(1)

Specially we take \( p = 2 \), \( q = 2 \) and \( p = 2 \), \( q = 3 \), then (1) reduces to

\[ u_t + u^2 u_x + \alpha u + \beta u^2 - u_{xx} + \gamma u_{xxx} + u_{yy} = 0. \]  

(2)

\[ u_t + u^2 u_x + \alpha u + \beta u^3 - u_{xx} + \gamma u_{xxx} + u_{yy} = 0. \]  

(3)

Suppose a partial differential equation, in two variables

\[ P(u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, \ldots) = 0, \]  

(4)

where \( P \) is in general a polynomial function of its arguments, the subscripts denote the partial derivatives. A function \( f = f(w) \) of one variable only so that a suitable linear combination of the following functions,

\[ 1, f(w), f(w)^2, f(w)^3, f(w)^{xx}, f(w)^{xt}, f(w)^{tt}, \ldots, \]  

(5)

is actually a solution of (4). Here we will show how to find \( f(w) \), the quasisolution \( w = w(x,t) \) and a suitable linear combination of the functions in (5). This method of looking for special solutions of a nonlinear equation in mathematical physics is called the homogeneous balance method which consists of four steps:

First step: Choose a suitable linear combination of the functions (5), maybe its coefficients to be determined, so that the highest nonlinear terms and the highest order partial derivarive terms in the given equation are both transformed into the polynomials with a highest equality degree in partial derivatives of \( w(x,t) \) in spite of \( f(w) \) and its various derivatives. “ The highest equality degree” here is quite essential.

Applying first step to (2) we assume that

\[ u = \frac{\partial^{m+n} f(w)}{\partial x^m \partial t^n} + \text{all partial derivative terms with lower than (m + n) order of } f(w), \]  

(6)

\[ u = f(m+n) w_x^m w_t^n + \text{all terms with lower than (m + n) degree in various partial derivatives of } w(x,t), \]  

(7)

where \( m \geq 0, n \geq 0 \) are integers to be determined. The nonlinear term in (2) is transformed into

\[ u^2 u_x = f(m+n+3p+1) w_x^m w_t^n + \text{all terms with lower than } 2(m + n) \text{degree in various partial derivatives of } w(x,t). \]  

(8)

The highest order partial derivative term in equation (2) is transformed into

\[ u_{xxx} = f(m+n+p+3) w_x^{m+3} w_t^n w_y^p + \text{all terms with lower than (m + n + 3) degree in various partial derivatives of } w(x,t). \]  

(9)

Requiring the highest degrees in partial derivatives of \( w(x,t) \) in (8) and (refaq6) are equal (i.e. nonlinear and supersive effects are partially balanced) Yields

\[ 3m + 1 = m + 3, \quad 3n = n, \quad 3p = p \]  

(10)
which has a non-negative integer solution: \( m = 1, n = 0, p = 0 \) therefore we can choose the linear combination as follows,

\[
  u = af_x + b, \tag{11}
\]

Second step: Substituting the linear combination chosen in the first step into (4), collecting all terms with the highest degree of derivatives of \( u(x,t) \) and setting its coefficient to zero (we call that making a partial balance between the highest nonlinear terms and highest order partial derivative terms in (4)), we obtain an ordinary differential equation for \( f \) and then solve it, in most cases \( f(w) \) is a logarithm function in the nonlinear and highest order partial derivative terms, the ODE and its solution are in (5). Third step: Starting from the ODE and its solution obtained in the second step can be replaced by the expression obtained in the second step in (4), we call that making a partial balance between the highest nonlinear terms and the highest order partial derivative terms in (4). we obtain a set of nonlinear algebraic equations for some constants to be determined. Substituting the exponential function assumed into each \( k \) these homogeneous equation in partial derivatives of \( u(x,t) \). We obtain a set of nonlinear algebraic equations for some constants to be determined. If there exists a solution for these nonlinear algebraic equations, then \( w(x,t) \) and the coefficient of the linear combination chosen in the first step can be determined.

Fourth step: Substituting \( f(w) \) and \( w(x,t) \) as well as some constants obtained in the second and third steps into the combination chosen in the first step, after doing some calculations, we obtain an exact solution of (4).

Outline of this Paper - In section 2, analysis of the Homogeneous balance method and the exact solutions of (2) and in section 3, the same solution of (3) are obtained, and in section 4, results and discussions.

2. EXPONENTIAL SOLUTIONS OF EQUATION (2) In order that the highest nonlinear term and the highest derivative term \( u^2 u_x, \gamma u_{xxx} \) are balanced, we suppose that the solution of (2) is of the form

\[
  u(x,t,y) = af'w_x + b,
  \quad w = w(x,t,y), \tag{12}
\]

where the functions \( f \) and \( w \) as well as the constants \( a \) and \( b \) are to be determined. If follows from (12) that

\[
  u_t = af''w_tw_x + af'w_xt, \tag{13}
\]

\[
  u^2 u_x = af'' f'_w w_x^2 + ab^2 f'' w_x^2 + 2a^2 b f' w_x + a^3 f''^2 w_x^2 + ab^2 f' w_{xx} + 2a^2 b f''^2 w_x w_{xx}, \tag{14}
\]

\[
  \alpha u = \alpha [af'w_x + b], \tag{15}
\]

\[
  \beta w^2 = \beta [a^2 f'' w_x^2 + b^2 + 2ab f' w_x], \tag{16}
\]

\[
  -u_{xx} = -[af''' w_x^3 + 3af'' w_x w_{xx} + af' w_{xxx}], \tag{17}
\]

\[
  \gamma u_{xxx} = \gamma [af'' w_x^3 + 6af''' w_x^2 w_{xx} + 3af'' w_x^2 + 4af'' w_x w_{xx} + af' w_{xxx}], \tag{18}
\]

\[
  u_{yy} = af'' w_x^2 w_x + af'' w_y w_x + 2af'' w_x w_y + af' w_{xyy}. \tag{19}
\]

First collecting the terms with \( w_x^2 \) in (14) and (18) and setting its coefficient to zero, we obtain

\[
  \gamma f'' + a^2 f''' f'' = 0, \tag{21}
\]

Then the solution of (21) is

\[
  f = A \log w, \tag{22}
\]

where

\[
  A^2 = \frac{6\gamma}{a^3}. \tag{23}
\]

The above transformation yields

\[
  f' = c_1 w + c_2,
  \quad f'' = c_1,
\]

thereby

\[
  f'' = -A f'',
  \quad f''' = \frac{A^2}{2} f'''. \tag{24}
\]
Using the above results equations (13)-(19) reduces to

\[
\begin{align*}
  u_t & = a f''' w_t w_x + a f' w_{xt}, \\
  u_x & = a f''' f'' w_x + a f'' f' w_x + 2 a b f' w_x + \frac{Aa^3}{2} f''' w_x^2 w_{xx} + a b^2 f' w_x \\
  & \quad - 2 A a^2 b f'' w_x w_{xx}, \\
  \alpha u & = \alpha [a f' w_x + b], \\
  \beta u^2 & = \beta [ - A a^2 f'' w_x^2 + b^2 + 2 a b f' w_x] , \\
  - u_{xx} & = - (a f'' w_x^3 + 3 a f'' w_x w_{xx} + a f' w_{xxx}) , \\
  \gamma u_{xxx} & = \gamma \left[ a f''' w_x^4 + 6 a f''' w_x^2 w_{xx} + 3 a f' w_{xxx} + 4 a f'' w_x w_{xxx} + a f' w_{xxx}\right] , \\
  u_{yy} & = a f''' w_y^2 w_x + a f''' w_y w_{xx} + a f' w_{xyy}. 
\end{align*}
\]

Next substituting (25)-(31) into the left hand side of (2) and using (24), we obtain

\[
\begin{align*}
  u_t + u_x^2 + \alpha u + \beta u^2 - u_{xx} + \gamma u_{xxx} + u_{yy} & = a f''' w_t w_x + a f' w_{xt} + \left[ a b^2 f'' w_x^2 + 2 a b f' w_x \right] \\
  & \quad + \frac{Aa^3}{2} f''' w_x^2 w_{xx} + a b^2 f' w_x - 2 A a^2 b f'' w_x w_{xx} + \alpha [a f' w_x + b] \\
  & \quad + \beta [- A a^2 f'' w_x^2 + b^2 + 2 a b f' w_x] - [a f'' w_x^3 + 3 a f'' w_x w_{xx} + a f' w_{xxx}] \\
  & \quad + a f' w_{xxx} + \gamma \left[ + 6 a f''' w_x^2 w_{xx} + 3 a f'' w_x^3 + a f' w_{xxx} w_{xx} + a f'' w_x w_{xxx} + a f' w_{xxx} + a f'' w_y^2 w_x + a f' w_y w_{xx} + 2 a f' w_{xyy} \right]. 
\end{align*}
\]

Setting the coefficients of \( f''', f'', f', f \) and \( f^0 \) to zero respectively, we obtain the following equations for the determination of \( w(x, t, y) \)

\[
\begin{align*}
  a w_t w_x + a b^2 w_x^2 - 2 A a^2 b w_x w_{xx} - \beta A a^2 w_x^2 - 3 a w_x w_{xx} + 6 a \gamma w_{xx}^2 \\
  & + 4 a \gamma w_x w_{xxx} + a w_{yy} w_x + 2 a w_{xyy} & = 0, \\
  a w_{xt} + a b^2 w_{xx} + 2 a b w_x^2 + a w_x + 2 a b \beta w_x - a w_{xx} + a \gamma w_{xx} + a w_{xyy} & = 0, \\
  \alpha b + \beta b^2 & = 0.
\end{align*}
\]

The solution of equations (34)-(37) is of the form

\[
\begin{align*}
  w(x, t) & = w_0 + e^{k_1 x + k_2 t + k_3 y + k_4},
\end{align*}
\]

where \( w_0, k_4 \) are arbitrary constants and \( k_1, k_2, k_3 \) are constants to be determined by the following equations

\[
\begin{align*}
  & a A a^3 - 2 A a^2 b k_1^2 - \beta A a^2 k_1^2 - 3 a k_1^3 + 6 a \gamma k_1^4 + a k_2^2 k_1 & = 0, \\
  & a k_2^2 k_1 - 2 A a^2 b k_1^4 - 2 a b^2 k_1^2 - \beta A a^2 k_1^3 - 3 a k_1^2 + 6 a \gamma k_1^4 + a k_2 k_3^2 + 2 a k_1 k_3^2 & = 0, \\
  & a k_1 k_2 + a b^2 k_1^2 + 2 a b^2 k_1 + a a k_1 + 2 a b \beta k_1 - a k_1^3 + a \gamma k_1^4 + a k_1 k_3^2 & = 0, \\
  & \alpha b + \beta b^2 & = 0.
\end{align*}
\]

Substituting (38) and (22) in (12), we obtain an exact solution of equation (2)

\[
\begin{align*}
  u(x, t, y) & = \frac{a A a k_1}{w_0 + e^{k_1 x + k_2 t + k_3 y + k_4}} \left[ e^{k_1 x + k_2 t + k_3 y + k_4} + b. \right]
\end{align*}
\]
3. EXPONENTIAL SOLUTIONS OF EQUATION (3)

Again the highest nonlinear term and the highest derivative term \( u^2 u_x \cdot \gamma u_{xxx} \) are balanced, we suppose that the solution of (3) is of the form

\[
    u(x, t, y) = a f' w_x + b,
\]

\[
    w = w(x, t, y),
\]

(44)

where the functions \( f \) and \( w \) as well as the constants \( a \) and \( b \) are to be determined. If follows from (44) that

\[
    u_t = a f'' w_x w_x + a f' w_{xt},
\]

(45)

\[
    u^2 u_x = a f'' f' w_x^2 + ab f'' w_x^2 + 2a^2 b f' w_x + a^3 f'' w^2 w_{xx} + ab^2 f' w_{xx} + 2a^2 b f' w_{xx},
\]

(46)

\[
    \alpha u = \alpha [ a f' w_x + b ],
\]

(47)

\[
    \beta u^3 = \beta \left[ A^2 a f'' w_x w_x + 3a f' w_{xx} + a f' w_{xxx} \right],
\]

(48)

\[
    -u_{xx} = - \left[ a f'' w_x + 3a f' w_{xx} + a f' w_{xxx} \right],
\]

(49)

\[
    \gamma u_{xxx} = \gamma \left[ A^2 a f'' w_x w_x + 3a f' w_{xx} + 4a f'' w_x w_{xx} + 4a f'' w_{xx} + a f' w_{xxx} \right],
\]

(50)

\[
    u_{yy} = a f'' w_y w_x + a f' w_{yy} w_x + 2a f'' w_{yy} w_y + a f' w_{xyy},
\]

(51)

(52)

First collecting the terms with \( w_x^2 \) in (46) and (50) and setting its coefficient to zero, we obtain

\[
    \gamma f^{(IV)} + a^2 f'' f'^2 = 0.
\]

(53)

Then the solution of (53) is

\[
    f = A \log w,
\]

(54)

where

\[
    A^2 = -\frac{6 \gamma}{a^3}.
\]

(55)

The above transformation yields

\[
    f' = c_1 w + c_2,
\]

(56)

\[
    f'' = c_1,
\]

(57)

thereby

\[
    f'^2 = -A f' f'',
\]

(58)

\[
    f''^2 = \frac{A^2}{2} f''.
\]

(59)

Using the above results equations (45)-(19) reduces to

\[
    u_t = a f'' w_x w_x + a f' w_{xt},
\]

(60)

\[
    u^2 u_x = a f'' f' w_x^2 + ab^2 f'' w_x^2 + 2a^2 b f' w_x + \frac{Aa^3}{2} f'' w_x w_x + ab^2 f' w_{xx} + 2a^2 b f' w_{xx},
\]

(61)

\[
    \alpha u = \alpha [ a f' w_x + b ],
\]

(62)

\[
    \beta u^3 = \beta \left[ A^2 a f'' w_x w_x + 3a f' w_{xx} + a f' w_{xxx} \right],
\]

(63)

\[
    -u_{xx} = - \left[ a f'' w_x + 3a f' w_{xx} + a f' w_{xxx} \right],
\]

(64)

\[
    \gamma u_{xxx} = \gamma \left[ A^2 a f'' w_x w_x + 3a f' w_{xx} + 4a f'' w_x w_{xx} + 4a f'' w_{xx} + a f' w_{xxx} \right],
\]

(65)

\[
    u_{yy} = a f'' w_y w_x + a f' w_{yy} w_x + 2a f'' w_{yy} w_y + a f' w_{xyy},
\]

(66)

\[
    u_{y} = a f'' w_x w_x + a f' w_{xx},
\]

(67)
Next substituting (57)-(63) into the left hand side of (3) and using (56), we obtain
\[
\begin{align*}
    u_t + u^2 u_x + \alpha u + \beta u^3 - u_{xxx} + \gamma u_{xxx} + u_{yy} &= af'' w_1 w_x + af' w_{xt} + [ab^2 f'' w_x^2 + 2a^2 b f' w_x] \\
        &+ \frac{Aa^3}{2} f''' w_{xx} + ab^2 f' w_{xx} - 2Aa^2 b f'' w_x w_x + \alpha [af' w_x + b] \\
        &+ \beta \left( \frac{A^2 a^3}{2} f'm'_w x + b^3 - 3Aa^2 b f'' w_x + 3ab^2 f' w_x \right) - [af'' w_x] \\
        &+ 3a f'' w_x w_x + af' w_{xx} + \gamma \left[ +6a f''' w_x w_x + 3a f'' w_x \right] \\
        &+ 4a f'' w_{xx} w_x + af' w_{xx} + af'' w^2 + af' w_y w_x + 2a f''' w_x w_y \\
        &+ af' w_{xyy}.
\end{align*}
\]
(65)

Setting the coefficients of \( f''', f'', f', f \) and \( f'^0 \) to zero respectively, we obtain the following equations for the determination of \( w(x,t,y) \)
\[
\begin{align*}
    \frac{Aa^3}{2} w_x^2 w_x - aw_x + \frac{\beta A^2 a^3}{2} w_x^3 + 6a \gamma w_x^2 w_x + aw_y^2 w_x &= 0, \\
    aw_{xx} + ab^2 w_x^2 - 2Aa^2 b w_x w_x - 3Aa^2 b \beta w_x^2 - 3aw_x w_x + 6a \gamma w_x^2 w_x \\
        &+ 4a \gamma w_x w_x + aw_{yy} w_x + 2aw_{xy} w_y &= 0, \\
    aw_{xt} + ab^2 w_x + 2a^2 b w_x + \alpha w_x + 3ab^2 \beta w_x - aw_{xxx} + aw_{xx} + aw_{yy} &= 0, \\
    ab + \beta b^3 &= 0.
\end{align*}
\]
(66)-(69)

The solution of equations (66)-(69) is of the form
\[
w(x,t) = w_0 + e^{k_1 x + k_2 t + k_3 y + k_4},
\]
(70)
where \( w_0, k_1 \) are arbitrary constants and \( k_1, k_2, k_3 \) are constants to be determined by the following equations
\[
\begin{align*}
    \frac{Aa^3}{2} k_4^2 - ak_4^3 + \frac{\beta A^2 a^3}{2} k_4^3 + 6a \gamma k_4^2 + ak_4 + 2ak_1 k_3 &= 0, \\
    ak_2 k_3 + ab^2 k_4^2 - 2Aa^2 b k_4^3 - 3Aa^2 b \beta k_4^2 - 3ak_4^3 + 6a \gamma k_4^2 + 4a \gamma k_1 + ak_1 k_3 + 2ak_1 k_3 &= 0, \\
    ak_1 k_2 + ab^2 k_1^2 + 2a^2 b k_1 + \alpha k_1 + 3ab^2 \beta k_1 - ak_1^3 + 2a \gamma k_1^3 + 3ak_1 k_3 &= 0, \\
    ab + \beta b^3 &= 0.
\end{align*}
\]
(71)-(74)

Substituting (70) and (54) in (44), we obtain a exact solution of equation (3)
\[
u(x,y,t) = \frac{aA k_1}{w_0 + e^{k_1 x + k_2 t + k_3 y + k_4}} \left[ e^{k_1 x + k_2 t + k_3 y + k_4} + b \right].
\]
(75)

4. RESULTS AND CONCLUSIONS

In this paper, we apply the Homogeneous balance method to the equations (2) and (3), successfully obtained special solutions to that equations. Also use of the Homogeneous balance method, special exact solutions of many other typical nonlinear equations. The exact solution of (2) and (3) is
\[
u(x,y,t) = \frac{aA k_1}{w_0 + e^{k_1 x + k_2 t + k_3 y + k_4}} \left[ e^{k_1 x + k_2 t + k_3 y + k_4} + b \right].
\]
(76)

The exact solution of (2) and (3) is in terms of exponential functions.

REFERENCES


