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Mathematics of Similarity in Fluid Dynamics

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Abstract

This research paper has been prepared to put focus on the concept of similarity which is very important for mathematical science as well as in engineering science specially in fluid dynamics. Since we know that the concept of similarity is based on three conditions. These conditions are geometrical similarity, dynamical similarity and kinematical similarity. We test the relative strength of speeds, forces or lengths with the help of non-dimensional number. Non-dimensional numbers are backbone for similarity concept. In this paper we have given emphasis on the need of non-dimensional number and its mathematical derivation. As we know that, a number of new inventions have been derived from the nature with the help of similarity concept. So this research paper also encourages the new researcher to find out new works by using similarity concept from the nature, as the nature is an infinite source of hidden new research and also put them in mathematical form. We have also made a clear concept on the application of Buckingham Pi theorem, in fluid dynamics, in this paper, to find out non-dimensional numbers. Such deeds surely will enhance the interest of new researchers in field of non-dimensional numbers in fluid dynamics.

Keywords: Similarity, Geometrical similarity, Dynamic similarity, Kinematic similarity, Non-dimensional number, Fluid dynamics, Buckingham Pi theorem.

INTRODUCTION

The concept of similarity is a very important to study any complex problem. It can be applied to any branch of natural science as well as mathematical science. As we know that the nature, in which we live is very vast. A large number of new research concepts have been derived from nature. These researches are very beneficial for the welfare of human kind on this planet. So, this research paper puts focus on the concept of similarity in fluid dynamics. As fluid dynamics is collection of natural science, mathematics science as well as engineering science. Similarity is an important application of non-dimensional numbers. We use the relationship between the model and the prototypes. For example we have made the shape of boat by taking the prototype as fish.

There are three types of similarity [1] in fluid dynamics which are used by engineering science as well as by mathematical science.

(i) Geometric similarity

(ii) Kinematic similarity

(iii) Dynamic similarity

(i) Geometric Similarity:

$(\frac{L}{B})_{model} = (\frac{L}{B})_{prototype}$

The ratio of length of prototype becomes equal to the ratio of corresponding length of model.

(ii) Kinetic Similarity:

$(\frac{V_V}{V_H})\ model = (\frac{U_V}{U_H})\ prototype$

$(\frac{f_v}{f_H})\ model = (\frac{f_v}{f_H})\ prototype$

Such type of similarity fulfils the condition of stream line flow of fluid. The ratio of velocity or acceleration of prototype becomes equal to the ratio of velocity or acceleration of model.

(iii) **Dynamic Similarity:** $\left(\frac{F'_H}{F'_V}\right)_{model} = \left(\frac{F_H}{F_V}\right)_{prototype}$

Here ratio of forces of prototype to that of corresponding forces of model are taken equal. If the model and the prototype satisfy the conditions of geometrical, kinetic and dynamic similarity, they are said to be completely similar. The similarity of two flows depends on the equality of the dimensionless parameters for one flow to the corresponding parameters of the other. We can also say that if these non-dimensional parameters are taken same for any two flows, they are geometrically and dynamically similar even though the velocities at corresponding points may be different.

2. Some common non-dimensional numbers in fluid dynamics and their area of application:

Non – Dimensional Number	Definition	Significance	Application
Reynolds Number [2]	$\rho V L/\mu$	Inertial force/Viscous force	All branches of FluidDynamics
Froude Number [3] & [4]	V^2/gl	Inertial force/Viscous force	Free surface flows
Mach Number [5]	V/α	Flow speed/Speed of sound	Compressible flow
Prandtl Number [6]	$\mu\, C_p/k$	Viscous dissipation/Conduction	Heat Transfer
Ratio of Specific Heats K. [7]	C_p/C_v	Enthalpy/Internal Energy	Compressible flow
Roughness Ratio [17]	h/L	Height of Roughness/Body Length	Turbulent flow
Pressure coefficient C_p [8]	$p\text{-}p_\infty/\frac{1}{2}\rho V^2$	Static Pressure/Dynamic Pressure	Aerodynamics, Hydrodynamics
Drag coefficient C_D [9]	$D/\frac{1}{2}\rho V^2 A$	Drag/Dynamic Force	Aerodynamics, Hydrodynamics
Lift Coefficient C_L [10]	$L/\frac{1}{2}\rho V A^2$	Lift /Dynamic Force	Aerodynamics, Hydrodynamics
Skin Friction, C_f [11]	$\eta wall/\frac{1}{2}\rho V^2 A$	Wall Shear Stress/Dynamic Force	Boundary Layer Flow
Peclet Number P_r [12]	$C_p\mu/k$	Heat transfer by conclusion/Heat transfer by conclusion	Heat Transfer
Nusselt Number (Nu) [13]	Hl/k	Total heat transfer/Conduction heat transfer	Heat Transfer

3. Importance of Non-Dimensional Numbers:

In Fluid Dynamics the use non-dimensional numbers are very important. As a non– dimensional number gives a way to compare relatively the strength of forces, lengths, speeds or accelerations of a model to that of a prototype.

For example, if match number of the prototype fluid be,

$M_p = (V/C)_{prototype} > 1$ and $M_m = (V/C)_{model} > 1$, then fluid flow of model will be similar to the given prototype.

Thus the relative strength of model with respect to corresponding prototype is analysed, with the help of non-dimensional number.

4. Theorem: (Buckingham Pi Theorem): [14]

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For given a physical equation,

Φ (Q_1, Q_2, Q_3,..........,Q_n) = 0, where Q's are dimensional physical quantities pertinent to the physical phenomenon, there can be (n-m) dimensionless $\prod$ quantities, those describe the

Same phenomenon as,

$f(\prod_1, \prod_2, \prod_3, \ldots\ldots \prod_{n-m}) = 0$. The ($n$-$m$) dimensional group are often called $\prod$ groups.

5. Application of Buckingham Pi theorem:

We consider a drag force about a sphere in fluid here.

Let us suppose that F, D, V, ρ and μ be independent variables. Out of these variables three D, V and ρ are repeating variables. Therefore non – repeating variable are 5 – 3 = 2, and hence non – dimensional numbers are $\prod_1$, $\prod_2$.

Now, our problem will be

$$\prod_1 = \prod_1 (D, V, \rho, F) \quad (1)$$

$$\prod_2 = \prod_2 (D, V, \rho, \mu) \quad (2)$$

Let us suppose,

$$\prod_1 = D^a V^b \rho^c F \quad (3)$$

$$\prod_2 = D^e V^f \rho^g \mu \quad (4)$$

Variable	Dimension
Force F (Drag Force)	MLT – 2
Diameter D	L
Velocity V	LT – 1
Density ρ	ML – 3
Viscosity μ	ML– 1 T – 1

Now, we put these dimensions in equations (3) and (4).

∏1 = (L) a (LT – 1) b (ML – 3)c (MLT – 2) ∏2 = (L)e (LT – 1)f (ML – 3)g (ML- 1 T – 2) or,

$\prod_1$ = La+b-3c+1 T – b – 2 M c + 1

$\prod_2$ = L e+f – 3 g-1 T – f – 1 M g + 1

As we have taken $\prod_1$ and $\prod_2$ as non–dimensional quantities so, $a + b - 3c + 1 = 0$, $-b - 2 = 0$, $c + 1 = 0$, $e + f - 3g - 1 = 0$, $-f - 1 = 0$, $g + 1 = 0$

After solving these equations we get, $a = -2$, $b = -2$, $c = -1$ $e = -1$, $f = -1$, $g = -1$ therefore non-dimensional numbers are,

$\prod_1 = F/\rho V^2 D^2$ $\qquad \prod_2 = \mu/\rho V D$

6. Need for Non–Dimensional Numbers: In fluid dynamics, we generally want to know the flow and its characteristics such as stream line, turbulence etc. when we try to understand a flow of fluid dynamics we need the parameters which are involved in the flow. These parameters may be larger in number. It creates difficulties in presentation. Therefore we want to concise the presentation. It is dimensional analysis, which helps us in doing so. Let us suppose that two flows one is prototype and other is model are geometrically similar with characteristic length l_1, l_2 and the free stream velocities U_1, U_2. The corresponding characteristic time becomes,

$t_1 = l_1/U_2$ and $t_2 = l_2/U_2$. Then governing equation [16] of a characteristic fluid particle in

Each case will reduce to,

$$\rho_1(dq_1 / dt_1) = \rho_1 g_1 - \nabla_1 p_1 + \mu_1 \nabla_1^2 q_1 \quad (5)$$

$$\rho_2(dq_2 / dt_2) = \rho_2 g_2 - \nabla_2 p_2 + \mu_2 \nabla_2^2 q_2 \quad (6)$$

As the two flows are dynamically similar therefore corresponding ratios of the like forms will be the same. $\frac{\rho_2(\frac{dq_2}{dt_2})}{\rho_1(\frac{dq_1}{dt_1})} = \frac{\rho_2 g_2}{\rho_1 g_1} = \frac{\nabla_2 p_2}{\nabla_1 p_1} = \frac{\mu_2 \nabla_2^2 q_2}{\mu_1 \nabla_1^2 q_1}$ (7)

$$\frac{\rho_2(\frac{dq_2}{dt_2})}{\rho_1(\frac{dq_1}{dt_1})} = \frac{\rho_2 g_2}{\rho_1 g_1} \quad (8)$$

As the rate of change of velocity at each point depends on the initial and boundary conditions. Then the ratio will be,

$$\frac{U^2{}_2}{l_2 g_2} = \frac{U^2{}_1}{l_1 g_1} \quad (9)$$

Thus, $U^2/l\,g$ is a non-dimensional number known as Froude number (F_r).

$$F_r = \frac{Internal\ Force}{Gravitational\ Force} = U^2/l\,g$$

This number is useful in calculations of hydraulic jumps as well as in determining the resistance of ships.

7. Derivation of some non-dimensional numbers from the governing equation of fluid dynamics:

Now we take the other two ratios from governing equation of corresponding ratios (7) as follow.

$$\frac{\rho_2(dq_2 / dt)}{\rho_1(dq_1 / dt)} = \frac{\nabla_2 p_2}{\nabla_1 p_1}$$

Or, $$\frac{\nabla_2 p_2}{\rho_1 ({}^{dq_2}/_{dt})} = \frac{\nabla_1 \rho_1}{\rho_1 ({}^{dq_1}/_{dt})}$$

$$\frac{{}^{p_2}/_{l_2}}{\rho_2 U_2/(l_2/U_2)} = \frac{{}^{p_1}/_{l_1}}{\rho_1 U_1 (l_1/U_1)}$$

$$\frac{p_2}{\rho_2 U_2^2} = \frac{p_1}{\rho_1 U_1^2} = C_P \qquad (10)$$

Here, C_p is non-dimensional number, which is known as Pressure coefficient [8]. So in such type of flow where inertia force and pressure forces predominate, there the pressure coefficient must be the same.

Again we take other two ratios from equation (7) and get as follow,

$$\frac{\rho_2 (dq_{2/dt_2})}{\rho_1 (dq_1/dt_1)} = \frac{\mu_2 \nabla^2{}_2 q_2}{\mu_1 \nabla^2{}_1 q_1} \quad \text{or,} \quad \frac{\rho_1 ({}^{dq_1}/_{dt_1})}{\mu_1 \nabla_1^2 q_1} = \frac{\rho_2 ({}^{dq_2}/_{dt_2})}{\mu_2 \nabla_2^2 q_2}$$

Or, $$\frac{\rho_2 {}^{U_2}/({}^{l_2}/_{U_2})}{\mu_2 U_2/l_2^2} = \frac{\rho_1 {}^{U_1}/(l_1/U_1)}{\mu_1 U_1/l_1^2}$$

Or, $$\frac{\rho_1 U_1 l_1}{\mu_1} = \frac{\rho_2 U_2 l_2}{\mu_2} = R_e \qquad (11)$$

This non-dimensional parameter $\frac{\rho U l}{\mu}$ is called Reynold number [2]. This number is relative measure of inertial force to that of viscous force. If we add a buoyant force $\rho\ g\ \beta\ d\ T$ in the equation of motion, having temperature difference $d\ T$, the ratio of forces will be,

$$\frac{\rho_2 dq_2/dt_2}{\rho_2 g_2 \beta_2 dT_2} = \frac{\rho_1 dq_1/dt_1}{\rho_1 g_1 \beta_1 dT_1}$$

Or, $$\frac{g_1 \beta_1 dT_1 l_1}{U_1^2} = \frac{g_2 \beta_2 dT_1 l_2}{U_2^2} = G_r$$

Here β is thermal expansion. This non-dimensional parameter G_r is known as Grashof number [15]. It is the measure of relative strength of temperature dependent buoyant forces to that of inertial forces.

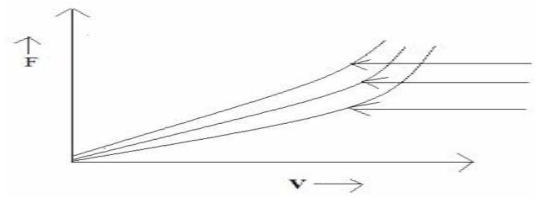
8. How will we use the non-dimensional number?

Let the Different parameters for flow past a sphere are as follow,

V – Freestream speed

D – Diameter of sphere

ρ - Density of fluid $\qquad$ μ –Viscosity of the fluid

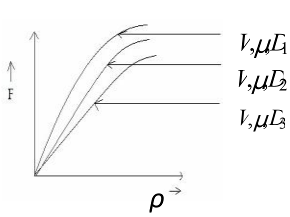


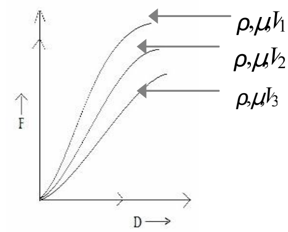
If we will not take the concept of dimensional analysis and use of non-dimensional number, then the process will be lengthy and complex. For example we want to study the drag force in above condition then we will draw graph, drag force (F) vs one parameter keeping other parameters constant.

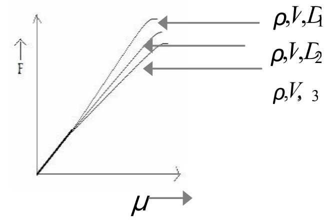
ρ, μ, D_1

$\rho, \mu, D_2 \qquad \rho, \mu, D_3$

If each curve demands at least ten points for good experiment. Then it is clear that a large number of experiments are needed, which is a very lengthy as well as complex task. Therefore non-dimensional numbers are used to make the process easy as well as simple to get results.



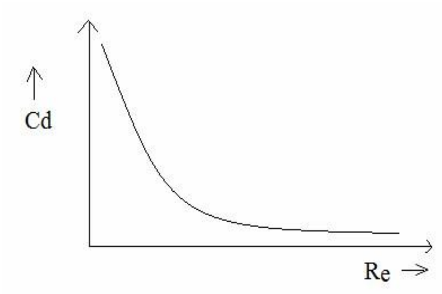




Dimensional less numbers are very needful. The number of parameters can be reduced by determining the proper dimensional less number for the flow. As it is seen that during calculation of drag for the flow about a sphere parameters are chosen V, ρ, μ and D. But with the help of non-dimensional numbers, we can get a relationship as follow,

$C_d = \frac{F}{1/2 \rho V^2 A}$ $\qquad$ [C_d – Drag coefficient]

$R_e = \frac{\rho V D}{\mu}$ $\qquad$ [R_e – Reynold number].



It decreases the complexities of equation with various parameters.

Another advantage of non-dimensional numbers is that the results are independent of the units of measurement. So we can get same result by applying any of units system. The biggest advantage of these non-dimensional numbers are similarity in fluid dynamics. As it is known that the concept of similarity is very important. On the basis of similarity a model is made to test the prototype. Still an infinite numbers of model are present in our nature. We have to recognise them and then we have to make a prototype for wellbeing of human kind. It will open a number of new works in science, engineering as well as in mathematics.

CONCLUSIONS

As we found that similarity is very important concept in science as well as in mathematics. On the basis of similarity concept, we have observed a large number of new works which are beneficial for human beings. A number of new models have been made by using the concept of similarity from nature as the prototype, in laboratory. We have also identified, the importance of non-dimensional numbers in fluid dynamics. With the help of non-dimensional numbers in fluid dynamics, the complexities of the problems can be decreased by the reducing the parameters of related system. We can also investigate new non-dimensional numbers by going through given non-dimensional numbers in this paper. Through this research paper we have found that still new works can be obtained with the help of similarity Researcher can investigate prototype from the infinite nature and put them in the form of model, on the basis of similarity concept in fluid dynamics. Surely this research paper will strengthen the concept of similarity and non-dimensional number for further research in fluid dynamics.

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This research paper has been prepared to accept the nature and natural designs, which helps in the improvement of human kind welfare as well as scientific mathematical development.

REFERENCES

[1] Kline, Stephen J., “Similitude and Approximation Theory”, Springer-Verlag, New York, 1986. ISBN 0-387 – 16518 – 5.

[2] Sommerfeld, Arnold (1908). (A contribution to Hydrodynamic Explanation of

[3] Turbulent Fluid Motions.) International Congress of Mathematicians. 3: 116 –

[4] 124. [1]

[5] Shih 2009, p.7

[6] White 1999, p. 294.

[7] ^ Nancy Hall (ed.). “ Match Number” NASA

[8] White, F. M. (2006). Viscous Fluid Flow (3rd ed.) New York : McGraw – Hill.

[9] ISBN 0-07-240231-8.

[10] White, Frank M. Fluid Mechanics (4th ed.). McGraw Hill.

[11] Anderson, John D (2001) Fundamentals of Aerodynamic 3rd Edition, McGraw-

[12] Hill. ISBN 0-07-237335 – 0

[13] Hoerner, Dr. Sighard F., Fluid – Dynamic Drag, Hoerner Fluid Dynamics, Bricktown New Jersey, 1965.

[14] Clancy, L. J. : Aerodynamics Section 4.15

[15] Clancy, L. J. (1975), Aerodynamics, Sub – section 5.9. Pitman Publishing ISBN 0273011200.

[16] ^ Patankar, Suhas V. (1980). Numerical Heat Transfer and Fluid Flow. New York : McGraw – Hill. p. 102. ISBN 0-89116 – 522 – 3.

[17] “Temperature Profile in Steam Generator Tube Metal” (PDF). Retrieved 23

International Journal of Applied Engineering Research ISSN 0973-4562 Volume 19, Number 1 (2024) pp. 19-24

[18] September 2009.

[19] Hanche – Olsen, Harald (2004). Buckingham's pi-theorem" (PDF) NTNU.

[20] Retrieved April 9, 2007.

[21] Sander, C. J. ; Holman, J. P. (1972) " Franz Grashof and the Grashof Number" . Int. J. Heat Mass Transfer 15:562.

[22] Fletcher, Clive A. J. (1991). Computational Techniques for Fluid Dynamics 2;

[23] Chapter 1; Fluid Dynamics: The Governing Equations. 2. Berlin/ Heidelberg, Germany: Springer Berlin Heidelberg. pp. 1-46. ISBN 978-3-642-58239-4.

[24] 17. ^ Moody, L.F. (1944), "Friction factors for pipe flow" , Transaction of the

[25] ASME, 66(8): 671-684.