Multiresolution Analysis for the Quadratic-phase Fourier Wavelets

Pratap Bhanu Sharma[1], **Pankaj Sharma**[*2]

[1]*Department of Mathematics (School of Science), Natoinal Institute of Technology, Andhra Pradesh- 534101, India.*
E-mail: sharmapbrs@gmail.com
[2]*Department of Mathematics, Pondicherry University, Puducherry-605 014, India.*
Email: pankajsharma@pondiuni.ac.in

Abstract :
In this paper, we have introduced a multiresolution analysis linked to the quadratic-phase Fourier wavelet transform. Additionally, we explore various properties of orthonormal wavelets associated with the quadratic-phase Fourier wavelet transform.

Keywords : Fourier transform; Quadratic-phase Fourier transform; Multiresolution analysis, Quadratic-phase orthonormal wavelets.

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1 INTRODUCTION

The Quadratic Phase Fourier Transform (QPFT) has gained significant popularity for its applications in image and signal processing. A recent contribution by Castro et al. [3] has established the QPFT as a generalization of several integral transforms, including the Fourier transform, fractional Fourier transform [1, 20] and the linear canonical transform [9, 10] etc. Furthermore, Prasad and Sharma [13] have developed the theory of QPFT, delving into the investigation of the continuous wavelet transform within the QPFT domain [3] and deriving several noteworthy results.

If a function $\phi \in L^1(\mathbb{R})$ or $L^2(\mathbb{R})$, the QPFT is given as [13]

$$(\mathcal{Q}^{a,b,c}_{r,s}\phi)(\omega) = \hat{\phi}(\omega) = \int_{\mathbb{R}} \mathcal{K}^{a,b,c}_{r,s}(\omega, x)\phi(x)dx, \qquad (1.1)$$

where

$$\mathcal{K}^{a,b,c}_{r,s}(\omega, x) = \sqrt{\frac{b}{2\pi i}}\, e^{i\Omega^{a,b,c}_{r,s}(\omega,x)}, \qquad (1.2)$$

is the kernel of QPFT, and $\Omega^{a,b,c}_{r,s}(\omega, x) = ax^2 + bx\omega + c\omega^2 + rx + s\omega$ is known as the quadratic-phase function, $a, b \neq 0, c, r, s \in \mathbb{R}$.

The inverse quadratic-phase Fourier transform (QPFT) of $\hat{\phi} \in L^1(\mathbb{R})$ or $L^2(\mathbb{R})$ is given by

$$\phi(x) = \int_{\mathbb{R}} \mathcal{K}^{-c,-b,-a}_{-s,-r}(x, \omega)(\mathcal{Q}^{a,b,c}_{d,e}\phi)(\omega)d\omega, \qquad (1.3)$$

where

$$\mathcal{K}^{-c,-b,-a}_{-s,-r}(x, \omega) = \sqrt{\frac{bi}{2\pi}}\, e^{-i\Omega^{c,b,a}_{s,r}(\omega,x)}.$$

In this context, it is observed that when the real parameters a, b, c, r, s are assigned as $b = \pm 1$ and $a = c = r = s = 0$, then QPFT turns out to the Fourier transform (FT) and inverse FT respectively.

If $a = c = \cot\theta$, $b = -\csc\theta$ and $r = s = 0$, it can be viewed as fractional Fourier transform [15, 21]. Moreover, the linear canonical transform [2, 10] is attained by setting $r = s = 0$.

2 PRELIMINARIES

In this section, we present certain definitions and fundamental outcomes that will be instrumental in advancing the theory of quadratic-phase Fourier wavelets in the subsequent sections.

The wavelet comprises a set of functions generated through the translation and dilation of a singular function $\psi \in L^2(\mathbb{R})$, known as the mother wavelet [6, **?**]. This family of functions is defined by:

$$\psi_{\beta,\alpha}(x) = \frac{1}{\sqrt{\alpha}}\psi\Big(\frac{x-\beta}{\alpha}\Big), \quad \beta \in \mathbb{R},\ \alpha > 0, \qquad (2.1)$$

where α is known as the scaling parameter and β is a translation parameter.

On discretization of the scaling parameter α and translation parameter β, we choosing $\alpha = \alpha_0^m$ and $\beta = n\beta_0\alpha_0^m$, where α_0 and β_0 are fixed positive constants. Therefore, the discretized family of wavelets are given by

$$\psi_{n,m}(x) = \alpha_0^{\frac{-m}{2}}\psi(x\alpha_0^{-m} - n\beta_0), \qquad (2.2)$$

where both m and n are in $\mathbb{Z}$. In particular, if we choose $\alpha_0 = 2, \beta_0 = 1$, then we have

$$\psi_{n,m}(x) = 2^{\frac{-m}{2}}\psi(x2^{-m} - n), \qquad \forall m, n \in \mathbb{Z}. \qquad (2.3)$$

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[*] Corresponding Author

Definition 2.1. If a function $\psi \in L^2(\mathbb{R})$, then the orthonormal wavelet for the sequence of functions $\{\psi_{n,m}(x) : m, n \in Z\}$ is called orthonormal basis for $L^2(\mathbb{R})$ and is given by [4, 8]

$$\psi_{n,m}(x) = 2^{\frac{m}{2}}\psi(2^m x - n). \tag{2.4}$$

Therefore, the set of functions $\psi_{m,n}$ is generated from an individual function ψ through a process involving dilation by 2^{-m} and dyadic translation by $n2^{-m}$.

There are two simple equations that completely characterize all orthonormal wavelets. They are given by

$$\sum_{m=-\infty}^{\infty} \left|\hat{\psi}(2^m\omega)\right|^2 = 1, \; for\ a.e.\ \omega \in R \tag{2.5}$$

and for every odd integer k,

$$\sum_{m=0}^{\infty} \hat{\psi}(2^m\omega)\overline{\hat{\psi}(2^m(\omega + 2\pi k))} = 0, \; for\ a.e.\ \omega \in R \tag{2.6}$$

Prasad and Sharma[13] initially introduced the quadratic phase Fourier wavelet (QPFW). Furthermore, numerous results in the QPFW domain have been achieved, as evidenced in[14, 15, 16, 17, 18, 19]. The QPFW seamlessly integrates the benefits of quadratic-phase Fourier and wavelet, forming a novel wavelet that combines the distinctive properties of each.

Prasad and Sharma [13] defined the quadratic-phase Fourier wavelet (QPFW) of a function $\phi(x)$ for each real parameters a, r is given by

$$\psi^{\Omega_r^a}_{\beta,\alpha}(x) = \frac{1}{\sqrt{\alpha}}\psi\Big(\frac{x-\beta}{\alpha}\Big)e^{-ia(x^2-\beta^2)-ir(x-\beta)}, \tag{2.7}$$
$$\beta \in \mathbb{R},\ \alpha > 0.$$

For a mathematical perspective, the parameters α and β in equation (2.8) can be transformed into a discrete one by taking only the integer vales of α and β. To discrete the parameters, we have chosen $\alpha = \alpha_0^{-p}$, $\beta = q\alpha_0^{-p}\beta_0$, where $p, q \in \mathbb{Z}$ and $\alpha_0 > 1$, $\beta_0 > 0$ are fixed. Then (2.8) becomes

$$\psi^{\Omega_r^a}_{q,p}(x) = \tag{2.8}$$
$$\alpha_0^{\frac{p}{2}}\psi\Big(\alpha_0^p x - q\beta_0\Big)e^{-ia(x^2-(q\alpha_0^{-p}\beta_0)^2)-ir(x-q\alpha_0^{-p}\beta_0)}.$$

In particular, for computational efficiency we have $\alpha_0 = 2, \beta_0 = 1$. Hence, Eq. (2.8) may be written as:

$$\psi^{\Omega_r^a}_{k,p}(x) = 2^{\frac{p}{2}}\psi\Big(2^p x - q\Big)e^{-ia\left(x^2-(\frac{q}{2^p})^2\right)-ir\left(x-\frac{q}{2^p}\right)}. \tag{2.9}$$

Taking QPFT of (2.9) we get

$$\mathcal{Q}^{a,b,c}_{r,r}\Big(\psi^{\Omega_r^a}_{q,p}\Big)(\omega) = \int_{\mathbb{R}} \mathcal{K}^{a,b,c}_{r,s}(\omega, x)\psi^{\Omega_r^a}_{q,p}(x)\,dx$$

$$\begin{aligned}
\hat{\psi}^{\Omega_r^a}_{q,p}(\omega) &= \sqrt{\frac{b}{2\pi i}}\int_{\mathbb{R}} e^{i(ax^2+bx\omega+c\omega^2+rx+s\omega)}2^{\frac{p}{2}}\psi\Big(2^p x - k\Big) \\
&\times\ e^{-ia\left(x^2-(\frac{q}{2^p})^2\right)-ir\left(x-\frac{q}{2^p}\right)}dx \\
&= 2^{\frac{p}{2}}\sqrt{\frac{b}{2\pi i}}\int_{\mathbb{R}} e^{i\left(a(\frac{q}{2^p})^2+bx\omega+c\omega^2+r(\frac{q}{2^p})+s\omega\right)}\psi\Big(2^p x - q\Big)dx.
\end{aligned} \tag{2.10}$$

Substituting $2^p x - q = y$, we obtain

$$\begin{aligned}
\hat{\psi}^{\Omega_r^a}_{q,p}(\omega) &= 2^{\frac{-p}{2}}\sqrt{\frac{b}{2\pi i}}\int_{\mathbb{R}} e^{i\left(a(\frac{q}{2^p})^2+b(\frac{q}{2^p})\omega+c\omega^2+r(\frac{q}{2^p})+s\omega\right)}e^{i(\frac{b\omega}{2^p})y}\psi(y)dy. \\
&= 2^{\frac{-p}{2}}\mathcal{K}^{a,b,c}_{r,s}\big(\omega, \frac{q}{2^p}\big)\int_{\mathbb{R}} e^{i(\frac{b\omega}{2^p})y}\psi(y)dy. \\
&= 2^{\frac{p}{2}}\mathcal{K}^{a,b,c}_{r,s}\big(\omega, \frac{q}{2^p}\big)\tilde{\Psi}(\frac{b\omega}{2^p}).
\end{aligned} \tag{2.11}$$

Particularly, we have the QPFT of $\psi^{\Omega_r^a}_{q,0}(x)$ and $\psi^{\Omega_r^a}_{0,0}(x)$ are as follows

$$\hat{\psi}^{\Omega_r^a}_{q,0}(\omega) = \sqrt{\frac{b}{2\pi i}}e^{i(aq^2+bq\omega+c\omega^2+rk+s\omega)}\tilde{\Psi}(b\omega). \tag{2.12}$$

Also,

$$\hat{\psi}^{\Omega_r^a}_{0,0}(\omega) = \sqrt{\frac{b}{2\pi i}}e^{i(c\omega^2+s\omega)}\tilde{\Psi}(b\omega), \tag{2.13}$$

where $\hat{\Psi}(b\omega)$ is the Fourier transform.

If $g(n)$ and $h(n)$ are two functions of $L^2(\mathbb{R})$, then their inner product is given by

$$\langle g,h\rangle = \int_{\mathbb{R}} g(x)\overline{h(x)}dx. \tag{2.14}$$

For any two function $g,h \in L^2(\mathbb{R})$, then we have following Parseval's identity for the QPFT

$$\Big\langle (\mathcal{Q}^{a,b,c}_{d,e}g)(\omega), \overline{(\mathcal{Q}^{a,b,c}_{d,e}h)(\omega)}\Big\rangle = \langle g(x), \overline{h(x)}\rangle. \tag{2.15}$$

For $\phi=\psi$ the above (2.15) becomes a Plancherel's identity.

3 MULTIRESOLUTION ANALYSIS FOR THE QUADRATIC-PHASE FOURIER WAVELET

The foundational idea behind multiresolution analysis (MRA) was initially introduced by Mallat [11] and Meyer [12]. This concept plays a pivotal role in advancing the theory of wavelets and establishing their orthonormal bases for $L^2(\mathbb{R})$, formulated as $2^{\frac{m}{2}}\psi(2^m x - n)$.

Shi et al. [22] pioneered the development of the multiresolution analysis theory, achieving orthogonal wavelets within the domain of fractional Fourier wavelet. Subsequently, Guo et al. [7] introduced a more expansive generalization, presenting multiresolution analysis associated with the linear canonical transform. Additionally, Srivastava et al. [23] conducted an in-depth exploration of the fractional wavelet transformation and its various properties.

In a more recent contribution, Sharma [16] put forth a novel MRA specifically tailored for quadratic-phase Fourier wavelets. Furthermore, Sharma successfully derived quadratic-phase orthonormal wavelets within the proposed MRA framework.

MRA for quadratic-phase Fourier wavelets is a sequence of closed subspaces $\{U_p^{\Omega_d^a} : p \in \mathbb{Z}\}$ of $L^2(\mathbb{R})$ that fulfill the properties given by [16]:

(i) $U_p^{\Omega_r^a} \subset U_{p+1}^{\Omega_r^a}$, for all $p\in\mathbb{Z}$

(ii) $\overline{\cup_{p\in\mathbb{Z}} U_p^{\Omega_r^a}} = L^2(\mathbb{R})$ and $\cap_{p\in\mathbb{Z}} U_p^{\Omega_r^a} = \{0\}$

(iii) $f(x)\in U_p^{\Omega_r^a}$ if and only if
$f(2x)e^{-ia(4x^2-\beta^2)-ir(2x-\beta)} \in U_{p+1}^{\Omega_r^a}$ for all $p\in\mathbb{Z}$

(iv) There exists a function $\phi_{q,0}^{\Omega_r^a}\in U_0^{\Omega_r^a}$ such that
$\{\phi_{q,0}^{\Omega_r^a} = \phi(x-q)e^{-ia(x^2-q^2)-ir(x-q)} : q\in\mathbb{Z}\}$
forms an orthonomal basis of $U_0^{\Omega_r^a}$.

That is

$$\|f\|^2 = \int_{\mathbb{R}} |f(x)|^2dx = \sum_{n=-\infty}^{\infty} |\langle f, \phi_{q,0}^{\Omega_r^a}\rangle|^2 \text{ for all } f\in U_0^{\Omega_r^a}.$$

When, the function $\phi_{q,0}^{\Omega_r^a}$ holds the property (iv), then it is called a scaling function for MRA. In (vi), if the sequence of functions $\{\phi_{q,0}^{\Omega_r^a} : q\in\mathbb{Z}\}$ is a Riesz basis of $V_0^{\Omega_a^d}$. Then,

$$\psi_{q,p}^{\Omega_r^a}(x) = 2^{\frac{p}{2}}\psi\Big(2^p x - q\Big)e^{-ia(x^2-\frac{q^2}{2^{2p}})-ir(x-\frac{q}{2^p})},\ \forall\, p\in\mathbb{Z}. \tag{3.1}$$

is an orthonormal basis of $U_p^{\Omega_r^a}$.

Theorem 3.1. *Let $\psi_{q,p}^{\Omega_r^a}\in L^2(\mathbb{R})$. Then the set of functions $\{\psi_{q,0}^{\Omega_r^a}(x) = \psi(x-q)e^{-ia(x^2-q^2)-r(x-q)},\ \forall\ q\in\mathbb{Z}\}$ is an orthonormal wavelet iff the following the following conditions are equivalent.*

$$(i) \qquad \sum_{n=-\infty}^{\infty} |\hat{\Psi}(b\omega+2\pi n)|^2 = 1; \tag{3.2}$$

almost everywhere (a.e.) (3.3)

$$(ii) \qquad \sum_{k=-\infty}^{\infty}\Big[\tilde{\Psi}(b\omega+2\pi k)\tilde{\Phi}(b\omega+2\pi k)\Big] = 0, \tag{3.4}$$

almost everywhere. (3.5)

Proof. (i) We have the quadratic-phase wavelet

$$\psi_{q,0}^{\Omega_r^a}(x) = \psi(x-q)e^{-ia(x^2-q^2)-r(x-q)},\ \forall\, q\in\mathbb{Z} \tag{3.6}$$

Taking QPFT of (3.6)

$$\mathcal{Q}^{a,b,c}_{r,s}\Big(\psi_{q,0}^{\Omega_r^a}\Big)(\omega) \;=\; \int_{\mathbb{R}} \mathcal{K}^{a,b,c}_{r,s}(\omega,x)\psi_{q,0}^{\Omega_r^a}(x)\,dx$$

Therefore

$$\begin{aligned}
\hat{\psi}_{q,0}^{\Omega_r^a}(\omega) &= \sqrt{\frac{b}{2\pi i}}\int_{\mathbb{R}} e^{i(ax^2+bx\omega+c\omega^2+rx+s\omega)}\psi(x-q)\\
&\times\ e^{-ia(x^2-q^2)-r(x-q)}dx\\
&= \sqrt{\frac{b}{2\pi i}}\int_{\mathbb{R}} e^{i(bx\omega+c\omega^2+s\omega)}\psi(x-q)e^{i(aq^2+rq)}dx\\
&= e^{i(aq^2+c\omega^2+rq+s\omega)}\sqrt{\frac{b}{2\pi i}}\int_{\mathbb{R}} e^{i(bx\omega)}\psi(x-q)dx
\end{aligned}$$

Putting $x - q = t$ and we obtained

$$\begin{aligned}\hat{\psi}_{q,0}^{\Omega_r^a}(\omega) &= e^{i(aq^2+bq\omega+c\omega^2+rq+s\omega)}\sqrt{\frac{b}{2\pi i}}\int_{\mathbb{R}} e^{i(bt\omega)}\psi(t)dt \\ &= e^{i(c\omega^2+s\omega)}\hat{\psi}_{0,0}^{\Omega_r^a}(\omega)\end{aligned}$$

We have

$$\begin{aligned}&\left\langle \hat{\psi}_{q,0}^{\Omega_d^a}(x), \hat{\psi}_{p,0}^{\Omega_d^a}(x)\right\rangle \\ =\ & \left\langle \hat{\psi}_{0,0}^{\Omega_d^a}(x), \hat{\psi}_{p-q,0}^{\Omega_d^a}(x)\right\rangle \\ =\ & \left\langle \mathcal{Q}_{d,e}^{a,b,c}\big[\psi_{0,0}^{\Omega_d^a}(x)\big], \overline{\mathcal{Q}_{d,e}^{a,b,c}\big[\psi_{p-q,0}^{\Omega_d^a}(x)\big]}\right\rangle \\ =\ & \frac{b}{2\pi}\int_{\mathbb{R}} e^{i(c\omega^2+s\omega)}\tilde{\Psi}(b\omega) \\ \times\ & e^{-i\left(a(p-q)^2+b(p-q)+c\omega^2+r(p-q)+s\omega\right)}\overline{\tilde{\Psi}(b\omega)}dw \\ =\ & \frac{b}{2\pi}\int_{\mathbb{R}} e^{-i\left(a(p-q)^2+b(p-q)\omega+r(p-q)\right)}|\tilde{\Psi}(b\omega)|^2 d\omega \\ =\ & \frac{b}{2\pi}\sum_{n=-\infty}^{\infty}\int_{2\pi n}^{2\pi(n+1)} e^{-i\left(a(p-q)^2+b(p-q)\omega+r(p-q)\right)}|\tilde{\Psi}(b\omega)|^2 d\omega \\ =\ & \frac{b}{2\pi}\int_0^{2\pi} e^{-i\left(a(p-q)^2+b(p-q)\omega+r(p-q)\right)}\sum_{n=-\infty}^{\infty}|\tilde{\Psi}(b\omega+2\pi n)|^2 d\omega.\end{aligned}$$

Hence, it follows from the completeness of $e^{-i\left(a(p-q)^2+b(p-q)\omega+r(p-q)\right)} \in L^2(\mathbb{R})$ i.e., we have

$$\left\langle \hat{\psi}_{q,0}^{\Omega_d^a}, \hat{\psi}_{p,0}^{\Omega_d^a}\right\rangle = \delta_{q,p}$$

if and only if

$$\sum_{n=-\infty}^{\infty}|\tilde{\Psi}(b\omega+2\pi n)|^2 = 1.$$

We will not delve into the proof of (ii) as it closely mirrors the earlier proof (i), and thus, we leave it at this point. ☐

Theorem 3.2. *For any two function $\phi_{n,m}^{\Omega_d^a}(x), \psi_{n,m}^{\Omega_d^a}(x) \in L^2(\mathbb{R})$, the set of functions $\{\psi_{n,0}^{\Omega_d^a}(x) \equiv \psi(x-n)e^{-ia(x^2-n^2)-d(x-n)},\ \forall n \in \mathbb{Z}\}$ and*
$\{\phi_{m,0}^{\Omega_d^a}(x) \equiv \phi(x-m)e^{-ia(x^2-m^2)-d(x-m)},\ \forall m \in \mathbb{Z}\}$ are orthonormal, that is

$$\langle \psi_{n,0}^{\Omega_d^a}, \phi_{m,0}^{\Omega_d^a}\rangle = 0, \forall m, n \in \mathbb{Z}$$

if and only if

$$\sum_{k=-\infty}^{\infty}\left[\tilde{\Psi}(b\omega+2\pi k)\tilde{\Phi}(b\omega+2\pi k)\right] = 0,\ \textit{almost everywhere},$$

where $\phi_{n,m}^{\Omega_d^a}$ defined as previous.

Proof. We have

$$\begin{aligned}\left\langle \hat{\psi}_{n,0}^{\Omega_d^a}(x), \hat{\phi}_{m,0}^{\Omega_d^a}(x)\right\rangle &= \left\langle \hat{\psi}_{0,0}^{\Omega_d^a}(x), \hat{\phi}_{m-n,0}^{\Omega_d^a}(x)\right\rangle \\ &= \left\langle \mathcal{Q}_{d,e}^{a,b,c}\big[\psi_{0,0}^{\Omega_d^a}(x)\big], \mathcal{Q}_{d,e}^{a,b,c}\big[\phi_{m-n,0}^{\Omega_d^a}(x)\big]\right\rangle.\end{aligned}$$

Therefore,

$$\begin{aligned}
& \left\langle \hat{\phi}_{n,0}^{\Omega_d^a}(x), \hat{\psi}_{m,0}^{\Omega_d^a}(x) \right\rangle \\
= & \int_{\mathbb{R}} \mathcal{Q}_{d,e}^{a,b,c}\left[\phi_{0,0}^{\Omega_d^a}(x)\right](\omega), \overline{\mathcal{Q}_{d,e}^{a,b,c}\left[\psi_{m-n,0}^{\Omega_d^a}(x)\right](\omega)} d\omega \\
= & \frac{b}{2\pi} \int_{\mathbb{R}} e^{-ia(m-n)^2-ib(m-n)\omega-id(m-n)}\ \tilde{\Psi}(b\omega)\ \overline{\tilde{\Phi}(b\omega)} d\omega \\
= & \frac{b}{2\pi} \sum_{k=-\infty}^{\infty} \int_{2\pi k}^{2\pi(k+1)} e^{-ia(m-n)^2-ib(m-n)\omega-id(m-n)} \\
\times & \ \tilde{\Psi}(b\omega)\ \overline{\tilde{\Phi}(b\omega)} d\omega \\
= & \frac{b}{2\pi} \int_0^{2\pi} e^{-ia(m-n)^2-ib(m-n)\omega-id(m-n)} \\
\times & \sum_{k=-\infty}^{\infty} \left[\hat{\Psi}(b\omega+2\pi k)\overline{\hat{\Phi}(b\omega+2\pi k)}\right] d\omega \\
= & \ 0, \text{ for all } m,n \in \mathbb{Z}.
\end{aligned}$$

If and only if

$$\sum_{k=-\infty}^{\infty} \left[\tilde{\Psi}(b\omega+2\pi k)\overline{\tilde{\Phi}(b\omega+2\pi k)}\right] = 0, \text{ almost everywhere.}$$

□

CONCLUSION

This study presents a novel multiresolution analysis framework associated with the quadratic-phase Fourier wavelet transform. Furthermore, we have investigated and established several key properties of orthonormal wavelets constructed within this framework, highlighting their theoretical significance and potential applications. These findings not only extend the applicability of wavelet theory but also contribute to the growing body of research on Fourier-domain wavelet transforms, particularly for signals with quadratic phase characteristics.

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