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Fractional Programming Methodology in Hybrid Decision Making Environment using Hexagonal Fuzzy Numbers

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Abstract

An innovative method for solving fuzzy stochastic fractional programming problems having some or all parameters as fuzzy numbers or exponentially distributed fuzzy random variables is proposed in this article. An equivalent fuzzy programing model is developed by applying chance constrained programming methodology to all the probabilistic constraints in a hybrid fuzzy uncertain decision making environment. Hexagonal fuzzy numbers are one of the extensions of fuzzy sets and trapezoidal fuzzy numbers and the concept of alpha-cuts of hexagonal fuzzy numbers is introduced to reduce the fuzzy fractional programming problem to a sub-problem with interval coefficients. After that, a nonlinear programming model is constructed by considering convex combination of each interval parameters. Finally, the modified form of nonlinear programming model is solved for different values of alpha-cuts. To demonstrate the efficiency of the proposed technique an illustrative example, studied previously, is considered and solved. Furthermore, comparison analysis of the proposed methodology is shown to express its advantages over similar existing methodologies.

Keywords: α-Cut, Chance Constrained Programming, Exponential Distribution, Fractional Programming, Hexagonal Fuzzy Number, Fuzzy Programming.

1. Introduction

In many real world decision making problems, viz, production planning, financial and corporate planning, health care and hospital planning etc., the decision makers (DMs) often observed that optimization of ratios of criteria provides more insight into the situation than optimizing each criterion independently. To resolve that situation, fractional programming [1] performed as an optimization tool in which the objective of the model appeared in the form of the ratio of two functions. Mathematical programming model with linear fractional objective was introduced by Charnes and Cooper [2]. Borza et al. [3] proposed a methodology for solving linear fractional programming (LFP) models by considering interval coefficients with the objective. In 2012 Odior [4] solved the LFP problem by algebraic approach which depends on the duality concept and the partial fractions.

However, in most of the practical applications, the DMs frequently face a situation that the parameters values of the mathematical models are not always crisp rather some sort of imprecision or ambiguity lying with them. These uncertainties may be in general probabilistic type or possibilistic type or a combination of both.

Stochastic programming (SP) is a framework for modeling optimization problem that involve probabilistic uncertainty in defining parameters of the model. In 1959, Charnes and Cooper [5] first introduced chance constrained programming (CCP) technique for solving SP problem. Thereafter, different methodological aspects of CCP were discussed by several researchers [6, 7]. In 2005, Chen [8] applied stochastic fractional programming problems to inventory problems. Recently, stochastic fractional programming problem have been applied to sustainable waste management [9], sustainable management of electric power systems [10], etc.

Zimmermann [11] introduced the concept of fuzzy programming (FP) to capture possibilistic uncertainties in decision making problems. There after a plenty of works have done [12, 13] from the view point of its potential applicability in different real life planning problems. Sinha and Baky [14], Mehrjerdi [15] proposed techniques for solving fractional programming problems in a fuzzy decision making environment. Using fuzzy goal programming (FGP) approach quadratic fractional bilevel programming problem was solved by Biswas and Bose [16]. Methodology for solving multiobjective fuzzy linear programming problem using hexagonal fuzzy number (HFN) was developed by Rajarajeswari and Sahaya [17]. The advantage of considering HFNs over triangular or trapezoidal fuzzy numbers is that it captures uncertainties more efficiently as HFNs are the

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generalization of triangular or trapezoidal fuzzy numbers. Another advantage of using HFN is that the parameters, involving different practical problems, can be expressed more proficiently through HFN based on the available data resources. In recent years optimal solution of transportation problem using HFNs is proposed by Rajarajeswari and Sangeetha [18], Thamaraiselvi and Santhi [19]. Solving transportation problem with generalized HFNs by ranking method was developed by Ghadle and Pathade [20]. Again, fuzzy fractional programming problem are applied to various practical applications, like solid transportation problem [21], land use planning in agricultural sector [22], etc.

To deal with co-occurrence of probabilistic and possibilistic uncertainties, hybrid approaches of stochastic programming and fuzzy programming were proposed [23]. The spectrum of research activities in this field of fuzzy stochastic programming are found in the work of Luhandjula [24]. A possibility programming approach for stochastic fuzzy multiobjective LFP was developed by Iskander [25]. However, solution technique for solving LFP problems with parameters as HFNs and extreme value distributed fuzzy random variables (FRVs) is yet to appear in the literature.

In this article a methodology for solving fuzzy stochastic LFP problem consisting of exponentially distributed FRVs associated with right side parameters of the system constraints is developed. The other parameters of the model are considered as HFNs. At first the CCP methodology is applied to the probabilistic constraints to form an equivalent FP. Using α-cut of the HFNs the FP model is reduced to fractional programming problem with interval coefficients. Finally by introducing new variables and by using convex combination of intervals, the fractional programming problem is converted to a nonlinear programming model. Finally developed model is solved to achieve the most satisfactory solution in the hybrid uncertain decision making environment.

2. Preliminary

In this section the terms such as hexagonal fuzzy numbers, $\alpha-$cut of hexagonal fuzzy numbers, fuzzy random variables following exponential distribution which are necessary for the development of the article are presented briefly.

2.1 Hexagonal Fuzzy Number

A HFN $\tilde{A}$ is a $6-$tuple $(a_1, a_2, a_3, a_4, a_5, a_6)$ whose membership function $\mu_{\tilde{A}}(x)$ is defined by

$$\mu_{\tilde{A}}(x) = \begin{cases} \frac{1}{2}\left(\frac{x-a_1}{a_2-a_1}\right) & a_1 \le x \le a_2 \\ \frac{1}{2}+\frac{1}{2}\left(\frac{x-a_2}{a_3-a_2}\right) & a_2 \le x \le a_3 \\ 1 & a_3 \le x \le a_4 \\ 1-\frac{1}{2}\left(\frac{x-a_4}{a_5-a_4}\right) & a_4 \le x \le a_5 \\ \frac{1}{2}\left(\frac{a_6-x}{a_6-a_5}\right) & a_5 \le x \le a_6 \\ 0 & x < a_1 \ or \ x > a_6 \end{cases}$$

Diagrammatically, a HFN $\tilde{A} \cong (a_1, a_2, a_3, a_4, a_5, a_6)$ is expressed as

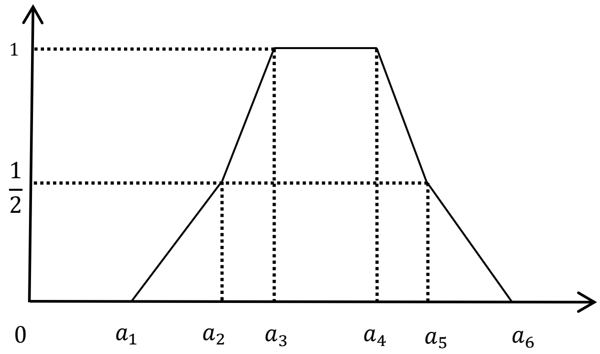

Fig.1: Hexagonal fuzzy number

2.2 $\alpha-$cut of a Hexagonal Fuzzy Number

An α-cut of a fuzzy set $\tilde{A}$ defined on a set X is a crisp set, denoted by $\tilde{A}[\alpha]$ and defined as the set of those elements for which the membership value is greater than or equal to by α, i.e.,

$$\tilde{A}[\alpha] = \{x \in X : \mu_{\tilde{A}}(x) \ge \alpha\ , 0 \le \alpha \le 1\}$$

An α-cut of any fuzzy number is always a closed interval of real numbers.

The α-cut of a HFN $\tilde{A} = (a_1, a_2, a_3, a_4, a_5, a_6)$ is defined as

$$\tilde{A}[\alpha] = \begin{cases} [a_1 + 2\alpha(a_2 - a_1), a_6 - 2\alpha(a_6 - a_5)] & \alpha \in [0, 0.5] \\ [a_2 + (2\alpha - 1)(a_3 - a_2), a_4 + (2 - 2\alpha)(a_5 - a_4)] & \alpha \in (0.5, 1] \end{cases}$$

2.3 Fuzzy Random Variable following Exponential distribution

Let $f(x, \theta)$ be a probability density function of a continuous random variable X, where θ is the parameter of the probability density function. If θ is inexact in nature then θ can be regarded as the fuzzy number $\tilde{\theta}$. Then the continuous random variable with fuzzy parameter $\tilde{\theta}$ is known as continuous FRV $\tilde{X}$.

If $\tilde{X}$ be an exponentially distributed FRV, then its probability density function is written as

$$f(x, \tilde{\lambda}) = s\ exp(-sx)$$

where $s \in \tilde{\lambda}[\alpha]$; $\tilde{\lambda}[\alpha]$ being the α –cut of the fuzzy number $\tilde{\lambda}$, the support of $\tilde{X}$ and $\tilde{\lambda}$ are defined on the set of positive real numbers.

The mean and variance of the FRV $\tilde{X}$ is given by

$m_{\tilde{X}} = E(\tilde{X}) = \frac{1}{\tilde{\lambda}}$ and $\sigma_{\tilde{X}}^2 = Var(\tilde{X}) = \frac{1}{\tilde{\lambda}^2}$, respectively.

3. Fuzzy Stochastic Fractional Programming Model Formulation

An LFP problem in a fuzzy probabilistic decision making environment is presented in matrix form as follows

Find $X(x_1, x_2, \ldots\ldots, x_n)$ so as to

Minimize $\frac{\tilde{C}X+\tilde{u}}{\tilde{D}X+\tilde{v}}$ (1)

Subject to

$\tilde{A}_1 X \lesssim \tilde{B}_1$ (2)

$\tilde{A}_2 X \lesssim \tilde{B}_2$ (3)

$X \geq 0$,

where $\tilde{C} \cong (\tilde{c}_1, \tilde{c}_2, \ldots, \tilde{c}_n)$, $\tilde{D} \cong (\tilde{d}_1, \tilde{d}_2, \ldots, \tilde{d}_n)$ are n dimensional row vectors with entries as HFNs. Also $\tilde{u}$, $\tilde{v}$ are taken as HFNs. Here $X = (x_1, x_2, \ldots\ldots, x_n)$ is a column vector regarded as non-fuzzy decision variables. Again $\tilde{A}_1 \cong (\tilde{a}_{ij})_{m\times n}$ and $\tilde{A}_2 \cong (\tilde{a}_{tj})_{l\times n}$ are the fuzzy matrices with elements as HFNs. The right hand side parameter $\tilde{B}_1 = (\tilde{b}_1,\ \tilde{b}_2, \ldots,\ \tilde{b}_m)$ of the constraints in (2) is m dimensional vector with components as exponentially distributed FRVs and the parameter $\tilde{B}_2 \cong (\tilde{b}_1, \tilde{b}_2, \ldots, \tilde{b}_l)$ for the constraint (3) is l dimensional vector with entries as HFNs.

3.1 Fuzzy Programming model formulation

The right sided parameters of the constraints in (2) are FRVs following exponential distribution. Therefore the constraints in (2) are rewritten in the following form

$Pr(\tilde{A}_1 X \lesssim \tilde{B}_1) \geq \beta$

i.e., in summation convention

$Pr(\sum_{j=1}^{n} \tilde{a}_{ij} x_j \lesssim \tilde{b}_i) \geq \beta_i; \quad i = 1,2,\ldots,m.$

where a m-tuple vector $\beta = (\beta_1, \beta_2, \ldots, \beta_m)$ represents a specific probability level.

As $\tilde{b}_i$ is an exponentially distributed FRV, its probability density function is written as

$f(b_i, \tilde{\lambda}_i) = s\, exp(-s b_i)$ where $s > 0$ and $s \in \tilde{\lambda}_i[\alpha]$.

Using CCP methodology to all the probabilistic constraints $Pr(\sum_{j=1}^{n} \tilde{a}_{ij} x_j \lesssim \tilde{b}_i) \geq \beta_i$, the constraints are modified as follows

$Pr(\sum_{j=1}^{n} \tilde{a}_{ij} x_j \lesssim \tilde{b}_i) \geq \beta_i$; *i.e.,* $Pr(\widetilde{U}_i \lesssim \tilde{b}_i) \geq \beta_i$, where $\widetilde{U}_i \cong \sum_{j=1}^{n} \tilde{a}_{ij} x_j$

i.e., $\left\{\int_u^{\infty} s\, exp(-s b_i) db_i : u \in \widetilde{U}_i[\alpha],\ s \in \tilde{\lambda}_i[\alpha]\right\} \geq \beta_i$; $i = 1,2,\ldots,m$

$i.e., u \leq -\frac{1}{s}\ln(\beta_i); i = 1,2,\ldots,m$ (4)

Since this is true for all $\alpha \in (0,1]$, then the above equation can be written as

$\widetilde{U}_i[\alpha] \leq -\frac{1}{\tilde{\lambda}_i[\alpha]}\ln(\beta_i)\ ; i = 1,2,\ldots,m$ (5)

Applying the first decomposition theorem on (5) it becomes

$\sum_{j=1}^{n} \tilde{a}_{ij} x_j \lesssim -\frac{1}{\tilde{\lambda}_i}\ln(\beta_i)\ ; i = 1,2,\ldots,m$ (6)

Hence the equivalent FP model of the fuzzy stochastic fractional programming model is written as

Find $X(x_1, x_2, \ldots\ldots, x_n)$ so as to

Minimize $\frac{\sum_{j=1}^{n} \tilde{c}_j x_j + \tilde{u}}{\sum_{j=1}^{n} \tilde{d}_j x_j + \tilde{v}}$

Subject to

$\sum_{j=1}^{n} \tilde{a}_{ij} x_j \lesssim -\frac{1}{\tilde{\lambda}_i}\ln(\beta_i)\ ; i = 1,2,\ldots,m$

$\sum_{j=1}^{n} \tilde{a}_{tj} x_j \lesssim \tilde{b}_t\ ; t = 1,2,\ldots,l$

$x_j \geq 0\ ; j = 1,2,\ldots,n$ (7)

3.2 Interval Parameter Fractional Programing Model Formulation

On the basis of α −cut of the HFNs the FP model (7) is converted into fractional programming model with interval coefficients as follows

The different hexagonal fuzzy parameters $\tilde{c}_j$, $\tilde{d}_j, \tilde{u}, \tilde{v}$, $\tilde{a}_{ij}$, $\tilde{a}_{tj}$, $\tilde{b}_t$, $\frac{1}{\tilde{\lambda}_i}$ involved with the model (7) are considered as follows

$\tilde{c}_j \cong (c_{j1}, c_{j2}, c_{j3}, c_{j4}, c_{j5}, c_{j6})$; $\tilde{d}_j \cong (d_{j1}, d_{j2}, d_{j3}, d_{j4}, d_{j5}, d_{j6})$;

$\tilde{u} \cong (u_1, u_2, u_3, u_4, u_5, u_6)$; $\tilde{v} \cong (v_1, v_2, v_3, v_4, v_5, v_6)$;

$\tilde{a}_{ij} \cong (a_{ij1}, a_{ij2}, a_{ij3}, a_{ij4}, a_{ij5}, a_{ij6})$;

$\tilde{a}_{tj} \cong (a_{tj1}, a_{tj2}, a_{tj3}, a_{tj4}, a_{tj5}, a_{tj6})$;

$\tilde{b}_t \cong (b_{t1}, b_{t2}, b_{t3}, b_{t4}, b_{t5}, b_{t6})$; $\frac{1}{\tilde{\lambda}_i} \cong \left(\frac{1}{\lambda_{i1}}, \frac{1}{\lambda_{i2}}, \frac{1}{\lambda_{i3}}, \frac{1}{\lambda_{i4}}, \frac{1}{\lambda_{i5}}, \frac{1}{\lambda_{i6}}\right)$, $(j = 1,2,\ldots,n; i = 1,2,\ldots,m; t = 1,2,\ldots,l)$ (8)

The α −cut of the above expressed HFNs are stated as defined subsection 2.2. as

$$\tilde{c}_j[\alpha] = \begin{cases} [c_{j1} + 2\alpha(c_{j2} - c_{j1}), c_{j6} - 2\alpha(c_{j6} - c_{j5})] & \alpha \in [0, 0.5] \\ [c_{j2} + (2\alpha - 1)(c_{j3} - c_{j2}), c_{j4} + (2 - 2\alpha)(c_{j5} - c_{j4})] & \alpha \in (0.5, 1] \end{cases}$$

$$\tilde{d}_j[\alpha] = \begin{cases} [d_{j1} + 2\alpha(d_{j2} - d_{j1}), d_{j6} - 2\alpha(d_{j6} - d_{j5})] & \alpha \in [0, 0.5] \\ [d_{j2} + (2\alpha - 1)(d_{j3} - d_{j2}), d_{j4} + (2 - 2\alpha)(d_{j5} - d_{j4})] & \alpha \in (0.5, 1] \end{cases} (j = 1,2,\ldots,n)$$

$$\tilde{u}[\alpha] = \begin{cases} [u_1 + 2\alpha(u_2 - u_1), u_6 - 2\alpha(u_6 - u_5)] & \alpha \in [0, 0.5] \\ [u_2 + (2\alpha - 1)(u_3 - u_2), u_4 + (2 - 2\alpha)(u_5 - u_4)] & \alpha \in (0.5, 1] \end{cases}$$

$$\tilde{v}[\alpha] = \begin{cases} [v_1 + 2\alpha(v_2 - v_1), v_6 - 2\alpha(v_6 - v_5)] & \alpha \in [0, 0.5] \\ [v_2 + (2\alpha - 1)(v_3 - v_2), v_4 + (2 - 2\alpha)(v_5 - v_4)] & \alpha \in (0.5, 1] \end{cases}$$

$$\tilde{a}_{ij}[\alpha] = \begin{cases} [a_{ij1} + 2\alpha(a_{ij2} - a_{ij1}), a_{ij6} - 2\alpha(a_{ij6} - a_{ij5})] & \alpha \in [0, 0.5] \\ [a_{ij2} + (2\alpha - 1)(a_{ij3} - a_{ij2}), a_{ij4} + (2 - 2\alpha)(a_{ij5} - a_{ij4})] & \alpha \in (0.5, 1] \end{cases}$$

$$\tilde{a}_{tj}[\alpha] = \begin{cases} [a_{tj1} + 2\alpha(a_{tj2} - a_{tj1}), a_{tj6} - 2\alpha(a_{tj6} - a_{tj5})] & \alpha \in [0, 0.5] \\ [a_{tj2} + (2\alpha - 1)(a_{tj3} - a_{tj2}), a_{tj4} + (2 - 2\alpha)(a_{tj5} - a_{tj4})] & \alpha \in (0.5, 1] \end{cases}$$

(4)

$(i = 1,2,\ldots,m; j = 1,2,\ldots,n; t = 1,2,\ldots,l)$

$\tilde{b}_t[\alpha] =$ (5)

$$\begin{cases} [b_{t1} + 2\alpha(b_{t2} - b_{t1}), b_{t6} - 2\alpha(b_{t6} - b_{t5})] & \alpha \in [0, 0.5] \\ [b_{t2} + (2\alpha - 1)(b_{t3} - b_{t2}), b_{t4} + (2 - 2\alpha)(b_{t5} - b_{t4})] & \alpha \in (0.5, 1] \end{cases} (t = 1,2,\ldots,l)$$

$$\frac{1}{\tilde{\lambda}_i}[\alpha] = \begin{cases} \left[\frac{1}{\lambda_{i1}} + 2\alpha\left(\frac{1}{\lambda_{i2}} - \frac{1}{\lambda_{i1}}\right), \frac{1}{\lambda_{i6}} - 2\alpha\left(\frac{1}{\lambda_{i6}} - \frac{1}{\lambda_{i5}}\right)\right] & \alpha \in [0, 0.5] \\ \left[\frac{1}{\lambda_{i2}} + (2\alpha - 1)\left(\frac{1}{\lambda_{i3}} - \frac{1}{\lambda_{i2}}\right), \frac{1}{\lambda_{i4}} + (2 - 2\alpha)\left(\frac{1}{\lambda_{i5}} - \frac{1}{\lambda_{i4}}\right)\right] & \alpha \in (0.5, 1] \end{cases}$$

$(i = 1,2, \dots, m)$ (9)

Using the above defined $\alpha -$cut of HFNs, the fractional programming model with interval coefficients is presented as

Find $X(x_1, x_2, \dots \dots, x_n)$ so as to

Minimize
$$\frac{\sum_{j=1}^{n}[c_{j1}+2\alpha(c_{j2}-c_{j1}),c_{j6}-2\alpha(c_{j6}-c_{j5})]x_j+[u_1+2\alpha(u_2-u_1),u_6-2\alpha(u_6-u_5)]}{\sum_{j=1}^{n}[d_{j1}+2\alpha(d_{j2}-d_{j1}),d_{j6}-2\alpha(d_{j6}-d_{j5})]x_j+[v_1+2\alpha(v_2-v_1),v_6-2\alpha(v_6-v_5)]}$$

Subject to

$\sum_{j=1}^{n}[a_{ij1} + 2\alpha(a_{ij2} - a_{ij1}), a_{ij6} - 2\alpha(a_{ij6} - a_{ij5})]x_j \leq (-\ln(\beta_i))\left[\frac{1}{\lambda_{i1}} + 2\alpha\left(\frac{1}{\lambda_{i2}} - \frac{1}{\lambda_{i1}}\right), \frac{1}{\lambda_{i6}} - 2\alpha\left(\frac{1}{\lambda_{i6}} - \frac{1}{\lambda_{i5}}\right)\right]$; $i = 1,2, \dots, m$

$\sum_{j=1}^{n}[a_{tj1} + 2\alpha(a_{tj2} - a_{tj1}), a_{tj6} - 2\alpha(a_{tj6} - a_{tj5})]x_j \leq [b_{t1} + 2\alpha(b_{t2} - b_{t1}), b_{t6} - 2\alpha(b_{t6} - b_{t5})]$; $t = 1,2, \dots, l$

$x_j \geq 0$; $j = 1,2, \dots, n$; $0 \leq \alpha \leq 0.5$ (10)

and,

Find $X(x_1, x_2, \dots \dots, x_n)$ so as to

Minimize
$$\frac{\sum_{j=1}^{n}[c_{j2}+(2\alpha-1)(c_{j3}-c_{j2}),c_{j4}+(2-2\alpha)(c_{j5}-c_{j4})]x_j+[u_2+(2\alpha-1)(u_3-u_2),u_4+(2-2\alpha)(u_5-u_4)]}{\sum_{j=1}^{n}[d_{j2}+(2\alpha-1)(d_{j3}-d_{j2}),d_{j4}+(2-2\alpha)(d_{j5}-d_{j4})]x_j+[v_2+(2\alpha-1)(v_3-v_2),v_4+(2-2\alpha)(v_5-v_4)]}$$

Subject to

$\sum_{j=1}^{n}[a_{ij2} + (2\alpha - 1)(a_{ij3} - a_{ij2}), a_{ij4} + (2 - 2\alpha)(a_{ij5} - a_{ij4})]x_j \leq (-\ln(\beta_i))\left[\frac{1}{\lambda_{i2}} + (2\alpha - 1)\left(\frac{1}{\lambda_{i3}} - \frac{1}{\lambda_{i2}}\right), \frac{1}{\lambda_{i4}} + (2 - 2\alpha)\left(\frac{1}{\lambda_{i5}} - \frac{1}{\lambda_{i4}}\right)\right]$; $i = 1,2, \dots, m$

$\sum_{j=1}^{n}[a_{tj2} + (2\alpha - 1)(a_{tj3} - a_{tj2}), a_{tj4} + (2 - 2\alpha)(a_{tj5} - a_{tj4})]x_j \leq [b_{t2} + (2\alpha - 1)(b_{t3} - b_{t2}), b_{t4} + (2 - 2\alpha)(b_{t5} - b_{t4})]$; $t = 1,2, \dots, l$

$x_j \geq 0$; $j = 1,2, \dots, n$; $0.5 < \alpha \leq 1$ (11)

Now to remove the fractional nature of the objective, the following variables are introduced by defining

$z = \left(\sum_{j=1}^{n}[d_{j1} + 2\alpha(d_{j2} - d_{j1}), d_{j6} - 2\alpha(d_{j6} - d_{j5})]x_j + [v_1 + 2\alpha(v_2 - v_1), v_6 - 2\alpha(v_6 - v_5)]\right)^{-1}$,

Or,

$z = \left(\sum_{j=1}^{n}[d_{j2} + (2\alpha - 1)(d_{j3} - d_{j2}), d_{j4} + (2 - 2\alpha)(d_{j5} - d_{j4})]x_j + [v_2 + (2\alpha - 1)(v_3 - v_2), v_4 + (2 - 2\alpha)(v_5 - v_4)]\right)^{-1}$

and $y_j = x_j z$; $j = 1,2, \dots \dots, n$,

On the basis of the new variables the model (10) and (11) reduce to the following form,

Minimize $\sum_{j=1}^{n}[c_{j1} + 2\alpha(c_{j2} - c_{j1}), c_{j6} - 2\alpha(c_{j6} - c_{j5})]y_j + [u_1 + 2\alpha(u_2 - u_1), u_6 - 2\alpha(u_6 - u_5)]z$

Subject to

$\sum_{j=1}^{n}[d_{j1} + 2\alpha(d_{j2} - d_{j1}), d_{j6} - 2\alpha(d_{j6} - d_{j5})]y_j + [v_1 + 2\alpha(v_2 - v_1), v_6 - 2\alpha(v_6 - v_5)]z = 1$

$\sum_{j=1}^{n}[a_{ij1} + 2\alpha(a_{ij2} - a_{ij1}), a_{ij6} - 2\alpha(a_{ij6} - a_{ij5})]y_j + \ln(\beta_i)\left[\frac{1}{\lambda_{i1}} + 2\alpha\left(\frac{1}{\lambda_{i2}} - \frac{1}{\lambda_{i1}}\right), \frac{1}{\lambda_{i6}} - 2\alpha\left(\frac{1}{\lambda_{i6}} - \frac{1}{\lambda_{i5}}\right)\right] z \leq 0$; $i = 1,2, \dots, m$

$\sum_{j=1}^{n}[a_{tj1} + 2\alpha(a_{tj2} - a_{tj1}), a_{tj6} - 2\alpha(a_{tj6} - a_{tj5})]y_j - [b_{t1} + 2\alpha(b_{t2} - b_{t1}), b_{t6} - 2\alpha(b_{t6} - b_{t5})]z \leq 0$; $t = 1,2, \dots, l$

$x_j \geq 0$; $j = 1,2, \dots, n$; $0 \leq \alpha \leq 0.5$ (12)

and,

Minimize $\sum_{j=1}^{n}[c_{j2} + (2\alpha - 1)(c_{j3} - c_{j2}), c_{j4} + (2 - 2\alpha)(c_{j5} - c_{j4})]y_j + [u_2 + (2\alpha - 1)(u_3 - u_2), u_4 + (2 - 2\alpha)(u_5 - u_4)]z$

Subject to

$\sum_{j=1}^{n}[d_{j2} + (2\alpha - 1)(d_{j3} - d_{j2}), d_{j4} + (2 - 2\alpha)(d_{j5} - d_{j4})]y_j + [v_2 + (2\alpha - 1)(v_3 - v_2), v_4 + (2 - 2\alpha)(v_5 - v_4)]z = 1$

$\sum_{j=1}^{n}[a_{ij2} + (2\alpha - 1)(a_{ij3} - a_{ij2}), a_{ij4} + (2 - 2\alpha)(a_{ij5} - a_{ij4})]y_j + \ln(\beta_i)\left[\frac{1}{\lambda_{i2}} + (2\alpha - 1)\left(\frac{1}{\lambda_{i3}} - \frac{1}{\lambda_{i2}}\right), \frac{1}{\lambda_{i4}} + (2 - 2\alpha)\left(\frac{1}{\lambda_{i5}} - \frac{1}{\lambda_{i4}}\right)\right] z \leq 0$; $i = 1,2, \dots, m$

$\sum_{j=1}^{n}[a_{tj2} + (2\alpha - 1)(a_{tj3} - a_{tj2}), a_{tj4} + (2 - 2\alpha)(a_{tj5} - a_{tj4})]y_j - [b_{t2} + (2\alpha - 1)(b_{t3} - b_{t2}), b_{t4} + (2 - 2\alpha)(b_{t5} - b_{t4})]z \leq 0$; $t = 1,2, \dots, l$

$x_j \geq 0$; $j = 1,2, \dots, n$; $0.5 < \alpha \leq 1$ (13)

Further the concept of convex combination be applied to each interval of the model (12) and (13), which yields the following non-linear models as

Minimize $\sum_{j=1}^{n}\left[\rho_j\left(c_{j1}+2\alpha(c_{j2}-c_{j1})\right)+(1-\rho_j)\left(c_{j6}-2\alpha(c_{j6}-c_{j5})\right)\right]y_j+\left[\rho_{n+1}(u_1+2\alpha(u_2-u_1))+(1-\rho_{n+1})(u_6-2\alpha(u_6-u_5))\right]z$

subject to

$\sum_{j=1}^{n}\left[\delta_j\left(d_{j1}+2\alpha(d_{j2}-d_{j1})\right)+(1-\delta_j)\left(d_{j6}-2\alpha(d_{j6}-d_{j5})\right)\right]y_j+\left[\delta_{n+1}(v_1+2\alpha(v_2-v_1))+(1-\delta_{n+1})(v_6-2\alpha(v_6-v_5))\right]z=1$

$\sum_{j=1}^{n}\left[\tau_{ij}\left(a_{ij1}+2\alpha(a_{ij2}-a_{ij1})\right)+(1-\tau_{ij})\left(a_{ij6}-2\alpha(a_{ij6}-a_{ij5})\right)\right]y_j+\ln(\beta_i)\left[\tau_{i(n+1)}\left(\frac{1}{\lambda_{i1}}+2\alpha\left(\frac{1}{\lambda_{i2}}-\frac{1}{\lambda_{i1}}\right)\right)+(1-\tau_{i(n+1)})\left(\frac{1}{\lambda_{i6}}-2\alpha\left(\frac{1}{\lambda_{i6}}-\frac{1}{\lambda_{i5}}\right)\right)\right]z\le 0\ ;i=1,2,\dots,m$

$\sum_{j=1}^{n}\left[\theta_{tj}\left(a_{tj1}+2\alpha(a_{tj2}-a_{tj1})\right)+(1-\theta_{tj})\left(a_{tj6}-2\alpha(a_{tj6}-a_{tj5})\right)\right]y_j-\left[\theta_{t(n+1)}(b_{t1}+2\alpha(b_{t2}-b_{t1}))+(1-\theta_{t(n+1)})(b_{t6}-2\alpha(b_{t6}-b_{t5}))\right]z\le 0\ ;t=1,2,\dots,l$

$0\le\alpha\le 0.5;\ y_j\ge 0$ for $j=1,2,\dots,n,$

$z\ge 0, 0\le\rho_j\le 1, 0\le\delta_j\le 1\ ;j=1,2,\dots,n+1$
$0\le\tau_{ij}\le 1; 0\le\theta_{tj}\le 1; i=1,2,\dots,m; t=1,2,..,l; j=1,2,\dots,n+1$ (14)

and,

Minimize $\sum_{j=1}^{n}\left[\rho_j^*\left(c_{j2}+(2\alpha-1)(c_{j3}-c_{j2})\right)+(1-\rho_j^*)\left(c_{j4}+(2-2\alpha)(c_{j5}-c_{j4})\right)\right]y_j+\left[\rho_{n+1}^*(u_2+(2\alpha-1)(u_3-u_2))+(1-\rho_{n+1}^*)(u_4+(2-2\alpha)(u_5-u_4))\right]z$

subject to

$\sum_{j=1}^{n}\left[\delta_j^*(d_{j2}+(2\alpha-1)(d_{j3}-d_{j2}))+(1-\delta_j^*)\left(d_{j4}+(2-2\alpha)(d_{j5}-d_{j4})\right)\right]y_j+\left[\delta_{n+1}^*(v_2+(2\alpha-1)(v_3-v_2))+(1-\delta_{n+1}^*)(v_4+(2-2\alpha)(v_5-v_4))\right]z=1$

$\sum_{j=1}^{n}\left[\tau_{ij}^*\left(a_{ij2}+(2\alpha-1)(a_{ij3}-a_{ij2})\right)+(1-\tau_{ij}^*)\left(a_{ij4}+(2-2\alpha)(a_{ij5}-a_{ij4})\right)\right]y_j+\ln(\beta_i)\left[\tau_{i(n+1)}^*\left(\frac{1}{\lambda_{i2}}+(2\alpha-1)\left(\frac{1}{\lambda_{i3}}-\frac{1}{\lambda_{i2}}\right)\right)+(1-\tau_{i(n+1)}^*)\left(\frac{1}{\lambda_{i4}}+(2-2\alpha)\left(\frac{1}{\lambda_{i5}}-\frac{1}{\lambda_{i4}}\right)\right)\right]z\le 0\ ;i=1,2,\dots,m$

$\sum_{j=1}^{n}\left[\theta_{tj}^*\left(a_{tj2}+(2\alpha-1)(a_{tj3}-a_{tj2})\right)+(1-\theta_{tj}^*)\left(a_{tj4}+(2-2\alpha)(a_{tj5}-a_{tj4})\right)\right]y_j-\left[\theta_{t(n+1)}^*(b_{t2}+(2\alpha-1)(b_{t3}-b_{t2}))+(1-\theta_{t(n+1)}^*)(b_{t4}+(2-2\alpha)(b_{t5}-b_{t4}))\right]z\le 0\ ;t=1,2,\dots,l,$

$0.5<\alpha\le 1;\ y_j\ge 0$ for $j=1,2,\dots,n,$

$z\ge 0, 0\le\rho_j^*\le 1, 0\le\delta_j^*\le 1\ ;j=1,2,\dots,n+1$
$0\le\tau_{ij}^*\le 1; 0\le\theta_{tj}^*\le 1; i=1,2,\dots,m; t=1,2,..,l; j=1,2,\dots,n+1$ (15)

Now model (14) and (15) is solved to find the satisfactory solution in the hybrid uncertain decision making environment.

3.3 Solution Algorithm

The developed methodology for solving fuzzy stochastic fractional programming problem in hybrid environment is presented in the form of an algorithm as follows:

Step 1: Apply CCP technique to all the fuzzy probabilistic constraints, to form a fuzzy fractional programming model with HFNs as parameters.
Step 2: Using the α −cut of HFNs, a fractional programing model with interval coefficients is developed.
Step 3: Some variables are introduced to remove the fractional nature of the objective.
Step 4: Taking convex combination of each interval to form an equivalent nonlinear programming model.
Step 5: Developed nonlinear programming model is solved to achieve most satisfactory solution in hybrid uncertain environment.
Step 6: Stop.

4. Numerical Illustration

To illustrate the proposed approach for solving fuzzy stochastic fractional programming model with HFNs, a modified version of the fractional programming problem studied earlier by Borza et al. [3] is considered.

Minimize $\frac{\widetilde{-2}x_1+\tilde{3}x_2+\widetilde{(-1)}}{\tilde{1}x_1+\tilde{1}x_2+\tilde{4}}$

Subject to $Pr(\widetilde{-1}x_1+\tilde{1}x_2\lesssim\tilde{b}_1)\ge 0.70;$
$Pr(\tilde{2}x_1+\tilde{3}x_2\lesssim\tilde{b}_2)\ge 0.45;$
$\tilde{1}\mathrm{x}_1+(\widetilde{-1})\mathrm{x}_2\lesssim\tilde{5};\ x_1,x_2\ge 0.$ (16)

Here $\tilde{b}_1,\tilde{b}_2$ are independent FRVs following exponential distribution with mean represented by the following HFNs

$m_{\tilde{b}_1}\cong\tilde{2}\cong(1.95,1.98,2,2.02,2.04,2.05);$
$m_{\tilde{b}_2}\cong\widetilde{14}\cong(13.95,13.97,14,14.02,14.03,14.05).$

Again all the parameters $\widetilde{-2},\widetilde{-1},\tilde{1},\tilde{3},\tilde{4}$ of the objectives are taken as HFNs of the form

$\widetilde{-2} \cong (-3, -2.75, -2.25, -2, -1.5, -1)$,
$\widetilde{-1} \cong (-2, -1.75, -1.25, -1, -0.5, 0)$,
$\tilde{1} \cong (0.5, 0.75, 1, 1.15, 1.25, 1.5)$,
$\tilde{3} \cong (2, 2.5, 3, 3.25, 3.75, 4)$,
$\tilde{4} \cong (3, 3.5, 4, 4.25, 4.75, 5)$.

Also, remaining parameters of all the constraints are considered as HFNs with the following form

$\widetilde{-1} \cong (-1.05, -1.02, -1, -0.98, -0.96, -0.95)$,
$\tilde{1} \cong (0.92, 0.96, 1, 1.01, 1.04, 1.06)$, $\tilde{2} \cong (1.95, 1.98, 2, 2.02, 2.04, 2.05)$,
$\tilde{3} \cong (2.95, 2.97, 3, 3.01, 3.03, 3.05)$, $\tilde{5} \cong (4.95, 4.98, 5, 5.25, 5.75, 5.94)$.

Using CCP technique and α-cut of HFNs, the problem (16) is reduced to a fractional programming problem with interval coefficients as described in section 3.2.

By introducing the variables z, y_1, y_2 and on the basis of convex combination of the α −cut the HFNs associated with the model, the model with interval parameter, is reduced to the form as

Minimize $[\rho_1(-3+0.5\alpha)+(1-\rho_1)(-1-\alpha)]y_1+[\rho_2(2+\alpha)+(1-\rho_2)(4-0.5\alpha)]y_{2+}[\rho_3(-2+0.5\alpha)+(1-\rho_3)(-\alpha)]z$

subject to

$[\delta_1(0.5+0.5\alpha)+(1-\delta_1)(1.5-0.5\alpha)]y_1+[\delta_2(0.5+0.5\alpha)+(1-\delta_2)(1.5-0.5\alpha)]y_2+[\delta_3(3+\alpha)+(1-\delta_3)(5-0.5\alpha)]z=1$

$[\tau_{11}(-1.05+0.06\alpha)+(1-\tau_{11})(-0.95-0.02\alpha)]y_1+[\tau_{12}(0.92+0.08\alpha)+(1-\tau_{12})(1.06-0.04\alpha)]y_2 \leq -z[\tau_{13}(1.95+0.06\alpha)+(1-\tau_{13})(2.05-0.02\alpha)]\ ln(0.70)$,

$[\tau_{21}(1.95+0.06\alpha)+(1-\tau_{21})(2.05-0.02\alpha)]y_1+[\tau_{22}(2.95+0.04\alpha)+(1-\tau_{22})(3.05-0.04\alpha)]y_2 \leq -z[\tau_{23}(13.95+0.04\alpha)+(1-\tau_{23})(14.05-0.04\alpha)]\ ln(.45)$

$[\theta_{11}(0.92+0.08\alpha)+(1-\theta_{11})(1.06-0.04\alpha)]y_1+[\theta_{12}(-1.05+0.06\alpha)+(1-\theta_{12})(-0.95-0.02\alpha)]y_2 \leq [\theta_{13}(4.95+0.06\alpha)+(1-\theta_{13})(5.94-0.38\alpha)]z$

$0 \leq \alpha \leq 0.5; 0 \leq \rho_j \leq 1; 0 \leq \delta_j \leq 1; 0 \leq \tau_{ij} \leq 1; 0 \leq \theta_{tj} \leq 1$
$y_1, y_2, z \geq 0$; $(j=1,2,3; i=1,2; t=1)$ (17)

and,

Minimize $[\rho_1^*(-3.25+\alpha)+(1-\rho_1^*)(-1-\alpha)]y_1+[\rho_2^*(2+\alpha)+(1-\rho_2^*)(4.25-\alpha)]y_{2+}[\rho_3^*(-2.25+\alpha)+(1-\rho_3^*)(-\alpha)]z$

subject to

$[\delta_1^*(0.5+0.5\alpha)+(1-\delta_1^*)(1.35-0.2\alpha)]y_1+[\delta_2^*(0.5+0.5\alpha)+(1-\delta_2^*)(1.35-0.2\alpha)]y_2+[\delta_3^*(3+\alpha)+(1-\delta_3^*)(5.25-\alpha)]z=1$

$[\tau_{11}^*(-1.04+0.04\alpha)+(1-\tau_{11}^*)(-1.02+0.04\alpha)]y_1+[\tau_{12}^*(0.92+0.08\alpha)+(1-\tau_{12}^*)(0.95+0.06\alpha)]y_2 \leq -z[\tau_{13}^*(1.96+0.04\alpha)+(1-\tau_{13}^*)(1.98+0.04\alpha)]\ ln(0.70)$,

$[\tau_{21}^*(1.96+0.04\alpha)+(1-\tau_{21}^*)(1.98+0.04\alpha)]y_1+[\tau_{22}^*(2.94+0.06\alpha)+(1-\tau_{22}^*)(2.97+0.04\alpha)]y_2 \leq -z[\tau_{23}^*(13.94+0.06\alpha)+(1-\tau_{23}^*)(14+0.02\alpha)]\ ln(.45)$

$[\theta_{11}^*(0.92+0.08\alpha)+(1-\theta_{11}^*)(0.95+0.06\alpha)]y_1+[\theta_{12}^*(-1.04+0.04\alpha)+(1-\theta_{12}^*)(-1.02+0.04\alpha)]y_2 \leq [\theta_{13}^*(4.96+0.04\alpha)+(1-\theta_{13}^*)(5+0.5\alpha)]z$

$0.5 < \alpha \leq 1;\ 0 \leq \rho_j^* \leq 1; 0 \leq \delta_j^* \leq 1; 0 \leq \tau_{ij}^* \leq 1; 0 \leq \theta_{tj}^* \leq 1$
$y_1, y_2, z \geq 0$; $(j=1,2,3; i=1,2; t=1)$ (18)

The software LINGO (11.0) is used to solve the problem.

The optimal solutions of the problem obtained for different values of α are shown in the following table. In this table the optimal solution for lower values of α are only considered, as the α −cut for lower values of α contain the α −cut for higher values of α.

Table 1: Solution for different values of α

Value of α	Solution Point	Objective value
0	$y_1 = 0.98, y_2 = 0, z = 0.17$	−3.280
0.02	$y_1 = 0.97, y_2 = 0, z = 0.17$	−3.225
0.05	$y_1 = 0.948, y_2 = 0, z = 0.164$	−3.145
0.06	$y_1 = 0.942, y_2 = 0, z = 0.16$	−3.119
0.07	$y_1 = 0.936, y_2 = 0, z = 0.163$	−3.094

The solution obtained by using the methodology developed by Borza et al. [3] by considering transformation of variables approach is $y_1 = 0.9091$, $y_2 = 0$, $z = 0.1818$ with the corresponding objective value -3.0909.

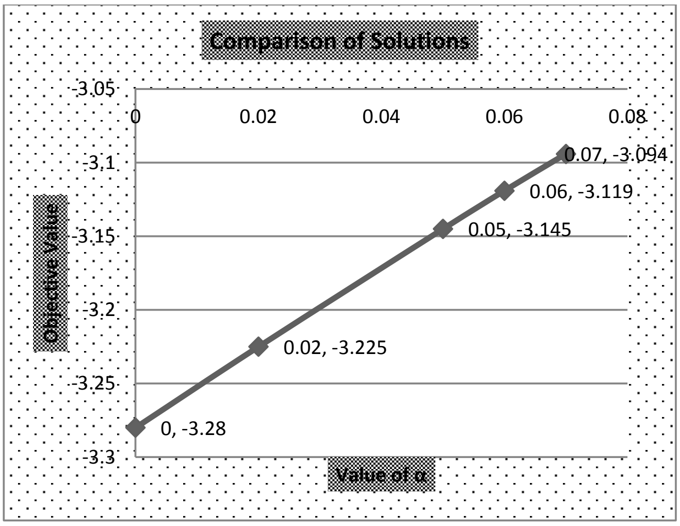

Fig. 2: Comparison of solutions

From this diagram it is clear that the decision obtained by developed methodology discussed in this article is most satisfactory in the fuzzy chance constrained decision making arena over the existing methodology. Also the proposed methodology is more flexible than the existing methodology as the value of α can be modified according to the requirement of the DMs. Thus, the proposed methodology is more acceptable to the DMs than the existing methodology for solving LFP problems in hybrid uncertain environment.

5. Conclusions

In this article an innovative methodology for solving linear fractional programming problem with HFNs or exponentially distributed FRVs as parameters is proposed in hybrid uncertain decision making environment. A crisp counterpart nonlinear programming model is developed using $\alpha -$cut of fuzzy numbers and by introducing new variables. The proposed model can be extended in a hybrid uncertain decision making environment in which all the parameters of the model would be represented by FRVs following various continuous probability distributions. The developed methodology can be applied to solve hybrid uncertain bilevel or multilevel, multiobjective programming problems. Furthermore, the proposed methodology can also be applied to various real life applications related to fractional programming problems, viz., production planning, financial and corporate planning, health care, hospital management, etc. Finally, it is hoped the proposed methodology may open up new dimensions into the way of solving fractional programming problems under the joint occurrence of fuzziness and randomness simultaneously.

References

[1] Schaible S & Ibaraki T (1983), "Fractional Programming, Invited Review", *European Journal of Operational Research* 12, 325-338.

[2] Charnes A & Cooper WW (1962), "Programming with linear fractional functions", *Naval Research Logistics* 9, 181–186.

[3] Borza M, Rambely AS & Saraj M (2012), "Solving linear fractional programming problems with interval coefficients in the objective function, A new approach", *Applied Mathematical Sciences* 6, 3443-3452.

[4] Odior AO (2012), "An approach for solving linear fractional programming problems", *International Journal of Engineering and Technology* 1, 298-304.

[5] Charnes A & Cooper WW (1959), "Chance Constrained Programming", *Management Sciences* 6, 73 –79.

[6] Kataoka S (1963), "A stochastic programming model", *Econometrica* 31, 181–196.

[7] Pre/kopa A (1973), "Contribution to the Theory of Stochastic Programming", *Management Sciences* 4, 202 – 221.

[8] Chen YF (2005), "Fractional programming approach to two stochastic inventory problems", *European Journal of Operational Research* 160, 63-71.

[9] Zhu H & Huang GH (2011), "A stochastic linear fractional programming approach for sustainable waste management", *Waste Management* 31, 2612-2619.

[10] Zhu H & Huang GH (2013), "Dynamic stochastic fractional programming for sustainable management of electric power systems", *International Journal of Electrical Power & Energy Systems* 53, 553-563.

[11] Zimmermann HJ (1976), "Description and optimization of fuzzy systems", *International Journal of General Systems* 2, 209-215.

[12] Dubois D & Prade H, *Fuzzy Sets and Systems: Theory and Applications*, Academic Press, New York, (1980).

[13] Pal BB & Moitra BN (2003), "A Goal Programming Procedure for Solving Problems with Multiple Fuzzy Goals Using Dynamic Programming", *European Journal of Operational Research* 144, 480 – 491.

[14] Abo-Sinna MA & Baky IA (2010), "Fuzzy goal programming procedure to bilevel multiobjective linear fractional programming problems", *International Journal of Mathematics and Mathematical Sciences* 2010, 1 – 15.

[15] Mehrjerdi YZ (2011), "Solving fractional programming problem through fuzzy goal setting and approximation", *Applied Soft Computing* 11, 1735 – 1742.

[16] Biswas A & Bose K, "Fuzzy goal programming approach for quadratic fractional bilevel programming", *Proceedings of the 2011 International Conference on Scientific Computing*, (2011), pp. 143-149.

[17] Rajarajeswari P & Sahaya AS (2013), "Ranking of Hexagonal Fuzzy Numbers for Solving Multiobjective Fuzzy Linear Programming Problem", *International Journal of Computer Applications* 84, 14-19.

[18] Rajarajeswari P & Sangeetha M (2015), "An Effect for Solving Fuzzy Transportation Problem Using Hexagonal Fuzzy Numbers", *International Journal of Research in Information Technology* 3, 295-307.

[19] Thamaraiselvi A & Santhi R (2015), "Optimal solution of Fuzzy Transportation Problem using Hexagonal Fuzzy Numbers", *International Journal of Scientific & Engineering Research* 6, 40 – 45.

[20] Ghadle KP & Pathade PA (2017), "Solving Transportation Problem with Generalized Hexagonal and Generalized Octagonal Fuzzy Numbers by Ranking Method", *Global Journal of Pure and Applied Mathematics* 13, 6367-6376.

[21] Radhakrishnan B & Anukokila P (2014), "Fractional Goal Programming for Fuzzy Solid Transportation Problem with Interval Cost", *Fuzzy Information and Engineering* 6, 359-377.

[22] Mishra B, Nishad AK & Singh SR (2014), "Fuzzy Multi-fractional Programming for Land Use Planning in Agricultural Production System", *Fuzzy Information Engineering* 6, 245-262.

[23] Biswal MP & Shinha SB (1997), "Fuzzy Programming Approach to Multi-Objective Stochastic Linear Programming Problems", *Fuzzy Sets and Systems* 88, 173 – 181.

[24] Luhandjula MK (2011), "On fuzzy random – valued optimization", *American Journal of Operations Research* 1, 259 – 267.

[25] Iskander MG (2004), "A possibility programming approach for stochastic fuzzy multiobjective linear fractional programs", *Computer & Mathematics with applications* 48, 1603-1609.