Subdivision of Heronian Mean Labeling of Graphs

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Abstract

In this paper, we contribute some new results for Heronian Mean labeling of graphs. We prove that subdivision of Heronian Mean Graphs are Heronian Mean Graphs. We use some standard graphs to derive the results for subdivision of graphs.

Keywords: Graph, Heronian Mean Graph, Comb, Ladder, Triangular Snake.

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1. INTRODUCTION

By a graph we mean a finite undirected graph without loops or parallel edges. For all detailed survey of graph labeling, we refer to J.A. Gallian [1]. For all other standard terminology and notations we follow Harary[2]. The concept of Mean labeling has been introduced by S. Somasundaram and R. Ponraj [3] in 2004. S.Somasundaram and S.S.Sandhya introduced Harmonic mean labeling [4] in 2012. Motivated by the above works we introduced a new type of labeling called Heronian Mean Labeling in [5].

In this paper we investigate the Subdivision of Heronian Mean Labeling of graphs. We will provide brief summary of definitions and other information which are necessary for our present investigation.
A Path $P_n$ is a walk in which all the vertices are distinct. The graph obtained by joining a single pendant edge to each vertex of a Path is called a Comb. The Ladder $L_n$ is the product graph $P_2 \times P_n$. A Triangular Snake $T_n$ is obtained from a path $u_1, u_2, \ldots, u_n$ by joining $u_i$ and $u_{i+1}$ to a new vertex $v_i$ for $1 \leq i \leq n - 1$. That is every edge of a path is replaced by a triangle $C_3$.

**Definition 1.1:**
A graph $G=(V,E)$ with $p$ vertices and $q$ edges is said to be a Heronian Mean graph if it is possible to label the vertices $x \in V$ with distinct labels $f(x)$ from $1, 2, \ldots, q+1$ in such a way that when each edge $e = uv$ is labeled with,

$$f(e = uv) = \left\lfloor \frac{f(u) + \sqrt{f(u)f(v)} + f(v)}{3} \right\rfloor$$

then the edge labels are distinct. In this case $f$ is called a Heronian Mean labeling of $G$.

**Definition 1.2:**
If $e=uv$ is an edge of $G$ and $w$ is not a vertex of $G$ then $e$ is said to be subdivided when it is replaced by the edges $uw$ and $wv$. The graph obtained by subdividing each edge of a graph $G$ is called the subdivision of $G$ and is denoted by $S(G)$.

**Theorem 1.3:** Any Comb $P_n \circ K_1$ is a Heronian mean graph.

**Theorem 1.4:** Any Ladder $L_n$ is a Heronian mean graph.

**Theorem 1.5:** Any Triangular Snake $T_n$ is a Heronian mean graph.

2. MAIN RESULTS

**Theorem 2.1**
Subdivision of any Comb $P_n \circ K_1$ is a Heronian mean graph.

**Proof:**
Let $P_n \circ K_1$ be a graph obtained from a path $u_1u_2 \ldots u_n$ by joining the vertex $u_i$ to pendant vertices $v_i$.

Let $G = S(P_n \circ K_1)$ be a graph obtained by subdividing all the edges of $P_n \circ K_1$.

Here we consider the following cases.

**Case (i):**
Let $G$ be a graph obtained by subdividing each edge $u_iu_{i+1}$ of $P_n \circ K_1$.

Let $w_i, 1 \leq i \leq n - 1$ be the vertices which subdivide $u_i$ and $u_{i+1}$.

Define a function $f: V(G) \rightarrow \{1, 2, 3, \ldots, q+1\}$ by

$$f(u_i) = 3i - 1, 1 \leq i \leq n.$$  

$$f(v_i) = 3i - 2, 1 \leq i \leq n.$$  

$$f(w_i) = 3i, 1 \leq i \leq n - 1.$$  

Edges are labeled with $f(u_iv_i) = 3i - 2, 1 \leq i \leq n$,
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\[ f(u_iw_i) = 3i - 1, \ 1 \leq i \leq n \]
\[ f(w_iu_{i+1}) = 3i, \ 1 \leq i \leq n. \]

Clearly \( f \) is a Heronian Mean labeling.

The labeling pattern is displayed below.

**Figure: 1**

**Case (ii):**

Let \( G \) be a graph obtained by subdividing the edge \( u_iv_1 \) of \( P_n \odot K_1 \).

Let \( w_i, 1 \leq i \leq n-1 \) be the vertices which subdivide \( u_i \) and \( v_1 \).

Let \( t_i, 1 \leq i \leq n \) be the vertices which subdivide \( u_i \) and \( v_1 \).

Define a function \( f: V(G) \rightarrow \{1,2,3, \ldots , q+1\} \) by
\[ f(u_i) = 3i - 2, \ 1 \leq i \leq n. \]
\[ f(v_i) = 3i, \ 1 \leq i \leq n. \]
\[ f(t_i) = 3i - 1, \ 1 \leq i \leq n. \]

Edges are labeled with
\[ f(u_iu_{i+1}) = 3i, 1 \leq i \leq n - 1, \]
\[ f(u_it_i) = 3i - 2, 1 \leq i \leq n \]
\[ f(t_iv_i) = 3i - 1, 1 \leq i \leq n. \]

Clearly \( f \) is a Heronian Mean labeling.

The labeling pattern is displayed below.

**Figure: 2**

**Case (iii):**

Let \( G \) be a graph obtained by subdividing all the edges of \( P_n \odot K_1 \).

Let \( w_i, 1 \leq i \leq n - 1 \) be the vertices which subdivide \( u_i \) and \( u_{i+1} \).

Let \( t_i, 1 \leq i \leq n \) be the vertices which subdivide \( u_i \) and \( v_1 \).

Define a function \( f: V(G) \rightarrow \{1,2,3, \ldots , q+1\} \) by
\[ f(u_i) = 4i - 3, \ 1 \leq i \leq n. \]
\[ f(v_i) = 4i - 1, 1 \leq i \leq n. \]
\[ f(w_i) = 4i, 1 \leq i \leq n - 1. \]
\[ f(t_i) = 4i - 2, 1 \leq i \leq n. \]

Edges are labeled with,
\[ f(u_iw_i) = 4i - 1, 1 \leq i \leq n - 1 \]
\[ f(w_iu_{i+1}) = 4i, 1 \leq i \leq n. \]
\[ f(u_it_i) = 4i - 3, 1 \leq i \leq n \]
\[ f(t_iv_i) = 4i - 2, 1 \leq i \leq n. \]

Clearly \( f \) is a Heronian Mean labeling.

The labeling pattern is displayed below.

![Figure:3](image)

From all the above three cases, we conclude that \( G = S(P_n \circ K_1) \) is a Heronian mean graph.

**Theorem: 2.2**
Subdivision of any Ladder \( L_n \) is a Heronian mean graph.

**Proof:**
Let \( L_n \) be a ladder connecting two paths \( u_1u_2 \ldots u_n \) and \( v_1v_2 \ldots v_n \).
Let \( G = S(L_n) \) be a graph obtained by subdividing all the edges of \( L_n \).
Here we consider the following cases.

**Case (i):**
Let \( G \) be a graph obtained by subdividing each edge \( u_iu_{i+1} \) and \( v_iv_{i+1} \) of \( L_n \).
Let \( x_i, y_i, 1 \leq i \leq n - 1 \) be the vertices which subdivide the edges \( u_iu_{i+1} \) and \( v_iv_{i+1} \).
Define a function \( f: V(G) \rightarrow \{1, 2, 3, \ldots, q + 1\} \) by
\[ f(u_i) = 5i - 4, 1 \leq i \leq n. \]
\[ f(v_i) = 5i - 3, 1 \leq i \leq n. \]
\[ f(x_i) = 5i - 2, 1 \leq i \leq n - 1. \]
\[ f(y_i) = 5i - 1, 1 \leq i \leq n - 1. \]

Edges are labeled with,
\[ f(u_iv_i) = 5i - 4, 1 \leq i \leq n, \]
\[ f(u_ix_i) = 5i - 3, 1 \leq i \leq n - 1, \]
\[ f(x_iu_{i+1}) = 5i - 1, 1 \leq i \leq n - 1, \]
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\[ f(v_iy_i) = 5i - 2, 1 \leq i \leq n - 1, \]
\[ f(y_iv_{i+1}) = 5i, 1 \leq i \leq n - 1. \]

Clearly \( f \) is a Heronian Mean labeling.

The labeling pattern is displayed below.

**Figure: 4**

**Case (ii):**

Let \( G \) be a graph obtained by subdividing the edge \( u_iv_i \) of \( L_n \).

Let \( z_i, 1 \leq i \leq n \) be the vertices which subdivide \( u_i \) and \( v_i \).

Define a function \( f: V(G) \to \{1, 2, 3, \ldots, q + 1\} \) by

\[ f(u_i) = 4i - 3, 1 \leq i \leq n. \]
\[ f(v_i) = 4i - 1, 1 \leq i \leq n. \]
\[ f(z_i) = 4i - 2, 1 \leq i \leq n. \]

Edges are labeled with,

\[ f(u_iu_{i+1}) = 4i - 1, 1 \leq i \leq n - 1, \]
\[ f(v_iv_{i+1}) = 4i, 1 \leq i \leq n - 1, \]
\[ f(u_iz_i) = 4i - 3, 1 \leq i \leq n \]
\[ f(z_iv_i) = 4i - 2, 1 \leq i \leq n. \]

Clearly \( f \) is a Heronian Mean labeling.

The labeling pattern is displayed below.

**Figure: 5**

**Case (iii):**

Let \( G \) be a graph obtained by subdividing all the edges of \( L_n \).

Let \( x_i, y_i, 1 \leq i \leq n - 1 \) be the vertices which subdivide the edges \( u_iu_{i+1} \) and \( v_iv_{i+1} \).

Let \( z_i, 1 \leq i \leq n \) be the vertices which subdivide \( u_i \) and \( v_i \).
Define a function $f : V(G) \rightarrow \{1, 2, 3, \ldots, q + 1\}$ by 
\[ f(u_i) = 6i - 5, \quad 1 \leq i \leq n. \]
\[ f(v_i) = 6i - 3, \quad 1 \leq i \leq n. \]
\[ f(z_i) = 6i - 4, \quad 1 \leq i \leq n. \]
\[ f(x_i) = 6i - 2, \quad 1 \leq i \leq n - 1. \]
\[ f(y_i) = 6i - 1, \quad 1 \leq i \leq n - 1. \]

Edges are labeled with,
\[ f(u_i z_i) = 6i - 5, \quad 1 \leq i \leq n \]
\[ f(z_i v_i) = 6i - 4, \quad 1 \leq i \leq n. \]
\[ f(u_i x_i) = 6i - 3, \quad 1 \leq i \leq n - 1, \]
\[ f(x_i u_{i+1}) = 6i - 1, \quad 1 \leq i \leq n - 1, \]
\[ f(v_i y_i) = 6i - 2, \quad 1 \leq i \leq n - 1, \]
\[ f(y_i v_{i+1}) = 6i, \quad 1 \leq i \leq n - 1. \]

Clearly $f$ is a Heronian Mean labeling.
The labeling pattern is displayed below.

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure6}
\caption{Figure:6}
\end{figure}

From all the above three cases, we conclude that $G = S(L_n)$ is a Heronian mean graph.

**Theorem: 2.3**

Subdivision of any Triangular Snake $T_n$ is a Heronian mean graph.

**Proof:**

Let $T_n$ be a Triangular Snake obtained from a path $u_1 u_2 \ldots u_n$ by joining $u_i$ and $u_{i+1}$ to a new vertex $v_i$, $1 \leq i \leq n - 1$.

Let $G = S(T_n)$ be a graph obtained by subdividing all the edges of $T_n$.

Here we consider the following cases.

**Case (i):**

Let $G$ be a graph obtained by subdividing each edge $u_i u_{i+1}$ of $T_n$.

Let $w_i$, $1 \leq i \leq n - 1$ be the vertices which subdivide $u_i$ and $u_{i+1}$.

Define a function $f : V(G) \rightarrow \{1, 2, 3, \ldots, q + 1\}$ by 
\[ f(u_i) = 4i - 3, \quad 1 \leq i \leq n. \]
\[ f(v_i) = 4i - 2, \quad 1 \leq i \leq n - 1. \]
\[ f(w_i) = 4i, \quad 1 \leq i \leq n - 1. \]
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Edges are labeled with,
\[ f(u_i v_i) = 4i - 3, 1 \leq i \leq n - 1, \]
\[ f(u_{i+1} v_i) = 4i - 1, 1 \leq i \leq n - 1, \]
\[ f(u_i w_i) = 4i - 2, 1 \leq i \leq n - 1, \]
\[ f(w_i u_{i+1}) = 4i, 1 \leq i \leq n - 1. \]
Clearly \( f \) is a Heronian Mean labeling.

The labeling pattern is displayed below.

Case (ii):
Let \( G \) be a graph obtained by subdividing the edges \( u_i v_i \) and \( u_{i+1} v_i \) of \( L_n \).
Let \( x_i \) and \( y_i \), \( 1 \leq i \leq n - 1 \) be the vertices which subdivide the edges \( u_i v_i \) and \( u_{i+1} v_i \).
Define a function \( f: V(G) \to \{1, 2, 3, \ldots, q + 1\} \) by
\[ f(u_i) = 5i - 4, 1 \leq i \leq n. \]
\[ f(v_i) = 5i - 2, 1 \leq i \leq n - 1. \]
\[ f(x_i) = 5i - 3, 1 \leq i \leq n - 1. \]
\[ f(y_i) = 5i - 1, 1 \leq i \leq n - 1. \]
Edges are labeled with,
\[ f(u_i u_{i+1}) = 5i - 2, 1 \leq i \leq n - 1, \]
\[ f(u_i x_i) = 5i - 4, 1 \leq i \leq n - 1. \]
\[ f(x_i v_i) = 5i - 3, 1 \leq i \leq n - 1. \]
\[ f(u_{i+1} y_i) = 5i - 1, 1 \leq i \leq n - 1. \]
\[ f(y_i v_i) = 5i - 1, 1 \leq i \leq n - 1. \]
Clearly \( f \) is a Heronian Mean labeling.

The labeling pattern is displayed below.
Case (iii):
Let $G$ be a graph obtained by subdividing all the edges of $T_n$.

Let $x_i$ and $y_i$, $1 \leq i \leq n - 1$ be the vertices which subdivide the edges $u_iv_i$ and $u_{i+1}v_i$.

Let $w_i$, $1 \leq i \leq n - 1$ be the vertices which subdivide $u_i$ and $u_{i+1}$.

Define a function $f: V(G) \rightarrow \{1, 2, 3, \ldots, q + 1\}$ by 

\[
\begin{align*}
  f(u_i) &= 6i - 5, 1 \leq i \leq n. \\
  f(v_i) &= 6i - 3, 1 \leq i \leq n - 1. \\
  f(w_i) &= 6i, 1 \leq i \leq n - 1. \\
  f(x_i) &= 6i - 4, 1 \leq i \leq n - 1. \\
  f(y_i) &= 6i - 2, 1 \leq i \leq n - 1.
\end{align*}
\]

Edges are labeled with,

\[
\begin{align*}
  f(u_iw_i) &= 6i - 3, 1 \leq i \leq n - 1 \\
  f(w_iu_{i+1}) &= 6i, 1 \leq i \leq n - 1. \\
  f(u_ix_i) &= 6i - 5, 1 \leq i \leq n - 1, \\
  f(x_iv_i) &= 6i - 4, 1 \leq i \leq n - 1, \\
  f(u_{i+1}y_i) &= 6i - 1, 1 \leq i \leq n - 1, \\
  f(y_iv_i) &= 6i - 2, 1 \leq i \leq n - 1.
\end{align*}
\]

Clearly $f$ is a Heronian Mean labeling.

The labeling pattern is displayed below.

![Figure 9](image_url)

From all the above three cases, we conclude that $G = S(T_n)$ is a Heronian mean graph.

3. CONCLUSION:

The Study of labeled graph is important due to its diversified applications. It is very interesting to investigate subdivision of Heronian mean graphs which admit Heronian Mean Labeling. The derived results are demonstrated by means of sufficient illustrations which provide better understanding. It is possible to investigate similar results for several other graphs.
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REFERENCES
