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# Solutions to a Discrete Right-Focal Fractional Boundary Value Problem

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**Abstract**

In this paper, we introduce a well-posed discrete right-focal fractional boundary value problem in the case where the order $\nu$ of the fractional difference satisfies $1 < \nu \leq 2$. We deduce Green's function for this problem and prove certain properties about Green's function. We show in the case $\nu = 2$ that our results agree with the previously known results for second-order discrete boundary value problems but that new results are obtained if $1 < \nu < 2$. In particular, we show that in great contrast to the case when $\nu = 2$, Green's function is not monotone in the case when $1 < \nu < 2$. Finally, we deduce some conditions under which positive solutions to the boundary value problem exist as well as some conditions under which the boundary value problem will have a unique solution.

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# 1 Introduction

In this paper we consider existence results for a certain two-point boundary value problem of right-focal type for a fractional difference equation. A recent paper by Atici and Eloe [6] produced a well-posed fractional boundary value problem (FBVP) of the type we consider in the present work. However, their paper considered only the case of Dirichlet boundary conditions. Given the interest in right-focal BVPs in the classical

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literature, the present paper can be considered an important extension of and parallel to [6].

In the continuous case, the fractional calculus and associated differential equations have been studied since the late 1600s. However, only in recent years has there been an explosion of research in this area – see, for example, [2, 8, 9, 17, 21] and the references therein. Moreover, it has been shown that fractional differential equations have nontrivial applications in numerous diverse fields including electrical engineering, chemistry, mathematical biology, control theory, and the calculus of variations – see, for example [2, 18–20]. Especially interesting, as mentioned in [20], is that the fractional calculus may provide more mathematically accurate epidemic models – an area of current and important mathematical research.

However, as is mentioned in [6], there has been little progress made in developing the theory of fractional difference equations or, moreover, the general theory of the fractional calculus on an arbitrary time scale. In particular, the works cited in the previous paragraph each explore fractional differential equations. Recently, though, there have appeared a number of papers on the discrete fractional calculus, which has helped to build up some of the basic theory of this area. For example, a recent work by Atici and Şengül [7] shows that fractional difference equations may provide for useful biological models. Moreover, each of the papers [5, 11] explore certain properties of fractional IVPs. In addition to these works, one can consult [3–6, 12–15] and the references therein to see the additional progress that has been made in the discrete fractional calculus. This paper, then, can be considered a contribution to this new, emerging area of mathematics.

In this paper, we will be interested in the nonlinear finite discrete FBVP given by

$$\begin{cases} -\Delta^{\nu} y(t) = f(t+\nu-1, y(t+\nu-1)) \\ y(\nu-2) = 0 = \Delta y(\nu+b), \end{cases} \tag{1.1}$$

where $t \in [0, b+1]_{\mathbb{N}_0}$, $\nu \in (1, 2]$, $f : [\nu-1, \nu+b]_{\mathbb{N}_{\nu-1}} \times \mathbb{R} \to \mathbb{R}$, and $b \in \mathbb{N}_0$. Thus, this paper will offer results that complement and extend the exposition given in [6].

In particular, the outline of this paper is as follows. After stating some preliminary results, which may be easily found in the existing literature on the discrete fractional calculus, we first deduce the existence of a unique solution to the FBVP

$$\begin{cases} -\Delta^{\nu} y(t) = h(t+\nu-1) \\ y(\nu-2) = 0 = \Delta y(\nu+b), \end{cases} \tag{1.2}$$

where $\nu \in (1, 2]$, $t \in [0, b+1]_{\mathbb{N}_0}$, and $h : [\nu-1, \nu+b]_{\mathbb{N}_{\nu-1}} \to \mathbb{R}$, by means of an appropriate Green's function. After this, we shall prove that Green's function for (1.2) satisfies the sort of useful properties that are available in the case when $\nu = 2$. Finally, we shall use these properties to show that under certain conditions that mirror those given in the case when $\nu = 2$, the more general problem (1.1) is guaranteed to have either at least one positive solution (provided that we additionally assume that $f(t, y)$ is nonnegative) or a unique solution.

## 2 Preliminaries

We first wish to collect some basic lemmas that will be important to us in the sequel. These and other related results and their proofs can be found, for example, in [3–6]. We begin with some basic properties regarding the discrete fractional derivative. These results will play a decisive role in our proofs later in this paper.

**Definition 2.1.** The $\nu$th fractional sum of a function $f$ defined on $\mathbb{N}_a := \{a, a+1, \dots\}$, for $\nu > 0$, is defined to be

$$\Delta^{-\nu} f(t) = \Delta^{-\nu} f(t; a) := \frac{1}{\Gamma(\nu)} \sum_{s=a}^{t-\nu} (t-s-1)^{\underline{\nu-1}} f(s),$$

where $t \in \{a+\nu, a+\nu+1, \dots\} =: \mathbb{N}_{a+\nu}$. We also define the $\nu$th fractional difference, where $\nu > 0$ and $0 \le N-1 < \nu \le N$ with $N \in \mathbb{N}$, to be

$$\Delta^{\nu} f(t) := \Delta^{N} \Delta^{-(N-\nu)} f(t),$$

where $t \in \mathbb{N}_{a+N-\nu}$.

We also recall the definition of the $\nu$th power falling.

**Definition 2.2.** We define

$$t^{\underline{\nu}} := \frac{\Gamma(t+1)}{\Gamma(t+1-\nu)},$$

for any $t$ and $\nu$ for which the right-hand side is defined. We also appeal to the convention that if $t+1-\nu$ is a pole of the Gamma function and $t+1$ is not a pole, then $t^{\underline{\nu}} = 0$.

Next we require some operational properties of the fractional sum operator.

**Lemma 2.3.** *Let $t$ and $\nu$ be any numbers for which $t^{\underline{\nu}}$ and $t^{\underline{\nu-1}}$ are defined. Then*

$$\Delta t^{\underline{\nu}} = \nu t^{\underline{\nu-1}}.$$

**Lemma 2.4.** *Assume $\mu$, $\nu > 0$ and $f : \mathbb{N}_a \to \mathbb{R}$ be a real-valued function. Moreover, let $\mu$, $\nu > 0$. Then we find that*

$$\Delta^{-\nu}\left[\Delta^{-\mu} f(t)\right] = \Delta^{-(\mu+\nu)} f(t) = \Delta^{-\mu}\left[\Delta^{-\nu} f(t)\right],$$

*where $t \in \mathbb{N}_{\mu+\nu+a}$.*

**Lemma 2.5.** *Let $0 \le N-1 < \nu \le N$. Then*

$$\Delta^{-\nu}\Delta^{\nu} y(t) = y(t) + C_1 t^{\underline{\nu-1}} + C_2 t^{\underline{\nu-2}} + \cdots + C_N t^{\underline{\nu-N}},$$

*for some $C_i \in \mathbb{R}$, with $1 \le i \le N$.*

We shall find Lemma 2.5 of special use in the next section of this paper.

# 3 Derivation of Green's Function

In order to help us analyze the nonlinear problem (1.1), we now wish to derive a Green's function for (1.2). Of particular note, we shall observe at the end of this section that in case $\nu = 2$, Green's function we obtain in Theorem 3.1 below matches Green's function obtained in the case when $\nu = 2$. Before stating this useful theorem, let us introduce the following notation, which will be important in the sequel:

$$T_1 := \left\{(t,s) \in [\nu-1, \nu+b+1]_{\mathbb{N}_{\nu-1}} \times [0, b+1]_{\mathbb{N}_0} \ : \ 0 \le s < t-\nu+1 \le b+2\right\},$$

$$T_2 := \left\{(t,s) \in [\nu-1, \nu+b+1]_{\mathbb{N}_{\nu-1}} \times [0, b+1]_{\mathbb{N}_0} \ : \ 0 \le t-\nu+1 \le s \le b+2\right\}.$$

We now state a result that is important in the sequel.

**Theorem 3.1.** *The unique solution of the FBVP* (1.2) *is given by*

$$y(t) := \sum_{s=0}^{b+1} G(t,s)h(s+\nu-1),$$

*where* $G(t,s)$ *is Green's function for the problem*

$$-\Delta^{\nu} y(t) = 0, \, y(\nu-2) = 0 = \Delta y(\nu+b), \tag{*}$$

*where* $1 < \nu \le 2$*, which is given by*

$$G(t,s) := \frac{1}{\Gamma(\nu)} \begin{cases} \dfrac{\Gamma(b+3)t^{\underline{\nu-1}}}{\Gamma(\nu+b+1)}(\nu+b-s-1)^{\underline{\nu-2}} - (t-s-1)^{\underline{\nu-1}}, & (t,s) \in T_1 \\ \dfrac{\Gamma(b+3)t^{\underline{\nu-1}}}{\Gamma(\nu+b+1)}(\nu+b-s-1)^{\underline{\nu-2}}, & (t,s) \in T_2. \end{cases}$$

*Proof.* Observe that by inverting the fractional difference operator coupled with an application of Lemma 2.5, we find that a general solution of the fractional difference equation in (1.2) is

$$y(t) = -\Delta^{-\nu} h(t+\nu-1) + C_1 t^{\underline{\nu-1}} + C_2 t^{\underline{\nu-2}},$$

whence, by Definition 2.1, we get that

$$y(t) = -\frac{1}{\Gamma(\nu)} \sum_{s=0}^{t-\nu} (t-s-1)^{\underline{\nu-1}} h(s+\nu-1) + C_1 t^{\underline{\nu-1}} + C_2 t^{\underline{\nu-2}}.$$

We now would like to determine the values of $C_1$ and $C_2$ so that the boundary conditions in (*) hold. To this end, applying the boundary condition $y(\nu-2) = 0$, we find that

$$0 = -\Delta^{-\nu} h(t+\nu-1)\big|_{t=\nu-2} + C_1(\nu-2)^{\underline{\nu-1}} + C_2(\nu-2)^{\underline{\nu-2}}. \tag{3.1}$$

Now, it is easy to show by Definition 2.2 that $(\nu-2)^{\underline{\nu-1}}=0$. Similarly, Definition 2.2 implies at once that $(\nu-2)^{\underline{\nu-2}}=\Gamma(\nu-1)$. Finally,

$$-\Delta^{-\nu}h(t+\nu-1)\big|_{t=\nu-2}=-\frac{1}{\Gamma(\nu)}\sum_{s=0}^{-2}(t-s-1)^{\underline{\nu-1}}h(s+\nu-1)=0$$

by the standard convention on sums. So, in summary, we find that (3.1) implies that $C_2=0$.

Similarly, we can apply the right boundary condition – namely, $\Delta y(\nu+b)=0$. Doing so, we find that

$$0=\Delta y(\nu+b)=\left\{\Delta\left[-\Delta^{-\nu}h(t+\nu-1)\right]\right\}_{t=\nu+b}+\Delta\left[C_1t^{\underline{\nu-1}}\right]_{t=\nu+b}. \tag{3.2}$$

Note that

$$\Delta\left[t^{\underline{\nu-1}}\right]_{t=\nu+b}=(\nu-1)\cdot\frac{\Gamma(\nu+b+1)}{\Gamma(b+3)} \tag{3.3}$$

and, as it is known that $\Delta\Delta^{-\nu}=\Delta^{1-\nu}$, that

$$\begin{aligned}\left\{\Delta\left[\Delta^{-\nu}h(t+\nu-1)\right]\right\}_{t=\nu+b}&=\left[\Delta^{1-\nu}h(t+\nu-1)\right]_{t=\nu+b}\\&=\left[\frac{1}{\Gamma(\nu-1)}\sum_{s=0}^{t-\nu+1}(t-s-1)^{\underline{\nu-2}}h(s+\nu-1)\right]_{t=\nu+b}\\&=\frac{1}{\Gamma(\nu-1)}\sum_{s=0}^{b+1}(\nu+b-s-1)^{\underline{\nu-2}}h(s+\nu-1).\end{aligned} \tag{3.4}$$

Putting (3.2), (3.3), and (3.4) together, it is a simple matter to show that

$$C_1=\frac{\left[\Delta\Delta^{-\nu}h(t+\nu-1)\right]_{t=\nu+b}}{\frac{(\nu-1)\Gamma(\nu+b+1)}{\Gamma(b+3)}}=\frac{\Gamma(b+3)}{\Gamma(\nu)\Gamma(\nu+b+1)}\sum_{s=0}^{b+1}(\nu+b-s-1)^{\underline{\nu-2}}h(s-\nu+1). \tag{3.5}$$

But with (3.5) in hand, we can determine $y(t)$ exactly. In particular, we find that

$$\begin{aligned}y(t)&=-\Delta^{-\nu}h(t+\nu-1)+C_1t^{\underline{\nu-1}}\\&=-\frac{1}{\Gamma(\nu)}\sum_{s=0}^{t-\nu}(t-s-1)^{\underline{\nu-1}}h(s+\nu-1)\\&\quad+\frac{\Gamma(b+3)t^{\underline{\nu-1}}}{\Gamma(\nu)\Gamma(\nu+b+1)}\sum_{s=0}^{b+1}(\nu+b-s-1)^{\underline{\nu-2}}h(s-\nu+1)\\&=\frac{1}{\Gamma(\nu)}\left\{\sum_{s=0}^{t-\nu}\left[\frac{\Gamma(b+3)t^{\underline{\nu-1}}}{\Gamma(\nu+b+1)}(\nu+b-s-1)^{\underline{\nu-2}}-(t-s-1)^{\underline{\nu-1}}\right]h(s+\nu+1)\right\}\\&\quad+\frac{1}{\Gamma(\nu)}\sum_{s=t-\nu+1}^{b+1}\frac{\Gamma(b+3)t^{\underline{\nu-1}}}{\Gamma(\nu+b+1)}(\nu+b-s-1)^{\underline{\nu-2}}h(s-\nu+1),\end{aligned}$$

from which it is immediately clear that we may write

$$y(t) = \sum_{s=0}^{b+1} G(t,s)h(s+\nu-1),$$

where

$$G(t,s) := \frac{1}{\Gamma(\nu)} \begin{cases} \dfrac{\Gamma(b+3)t^{\underline{\nu-1}}}{\Gamma(\nu+b+1)}(\nu+b-s-1)^{\underline{\nu-2}} - (t-s-1)^{\underline{\nu-1}}, & (t,s) \in T_1 \\ \dfrac{\Gamma(b+3)t^{\underline{\nu-1}}}{\Gamma(\nu+b+1)}(\nu+b-s-1)^{\underline{\nu-2}}, & (t,s) \in T_2 \end{cases}$$

is Green's function for (*). This completes the proof. □

*Remark* 3.2. Observe that $G(\nu-2,s)=0$, for each $s \in [0,b+1]_{\mathbb{N}_0}$.

*Remark* 3.3. Let us note for the reader that in case we put $\nu = 2$ in Theorem 3.1, it follows that we get the "usual" Green function, just as we might hope would happen. Indeed, in case $\nu = 2$ we find that (in case $a = 0$)

$$G(t,s) = \begin{cases} s+1, & 0 \le s < t-1 \le b+2 \\ t, & 0 \le t-1 \le s \le b+2, \end{cases}$$

with $G(t,s)$ defined on $[1,b+3]_{\mathbb{N}_0} \times [0,b+1]_{\mathbb{N}_0}$, which accords with the usual results.

*Remark* 3.4. As is implied by the definition of the sets $T_1$ and $T_2$ as well the form of $G(t,s)$ as given in Theorem 3.1, we have that Green's function, $G(t,s)$, is defined on the set $[\nu-2,\nu+b+1]_{\mathbb{N}_{\nu-2}} \times [0,b+1]_{\mathbb{N}_0}$. Incidentally, it is easy to show that $G(t,b+2)=0$, for each admissible $t$. So, $G$ could be extended to $[\nu-2,\nu+b+1]_{\mathbb{N}_{\nu-2}} \times [0,b+2]_{\mathbb{N}_0}$ without difficulty, but we do not require this in the sequel.

# 4 Properties of Green's Function

In this section of the paper, we wish to prove that our Green's function $G(t,s)$ satisfies, with appropriate and simple modifications, the usual classical properties. Certain of these properties will be crucial when we prove our existence theorems in the final section of this paper. We begin by stating a lemma.

**Lemma 4.1 (See [6]).** *Let $\nu$ be any positive real number and let $a$ and $b$ be two real numbers satisfying $\nu < a \le b$. Then the following hold.*

(i) $\dfrac{1}{x^{\underline{\nu}}}$ *is a decreasing function for* $x \in (\nu,+\infty)_{\mathbb{N}}$.

(ii) $\dfrac{(a-x)^{\underline{\nu}}}{(b-x)^{\underline{\nu}}}$ *is a decreasing function for* $x \in [0,a-\nu)_{\mathbb{N}_0}$.

We now state and prove the first of a trio of propositions regarding $G(t,s)$.

**Proposition 4.2.** *The function $G(t,s)$ defined in Theorem 3.1 satisfies $G(t,s) \geq 0$ for all $t \in [\nu-1, \nu+b+1]_{\mathbb{N}_{\nu-1}}$ and $s \in [0, b+1]_{\mathbb{N}_0}$, where $[\nu-1, \nu+b+1]_{\mathbb{N}_{\nu-1}} := \{\nu-1, \nu, \ldots, \nu+b+1\}$.*

*Proof.* To prove this proposition, we shall show directly that $G(t,s) > 0$ for each $(t,s) \in [\nu-1, \nu+b+1]_{\mathbb{N}_{\nu-1}} \times [0, b+1]_{\mathbb{N}_0}$. For simplicity, we shall look at $\Gamma(\nu)G(t,s)$, for $\Gamma(\nu) > 0$ so that if $\Gamma(\nu)G(t,s) > 0$, then at once it follows that $G(t,s) > 0$, too.

First notice that for $(t,s) \in T_2$, we have that

$$\begin{aligned}\Gamma(\nu)G(t,s) &= \frac{\Gamma(b+3)t^{\underline{\nu-1}}}{\Gamma(\nu+b+1)}(\nu+b-s-1)^{\underline{\nu-2}} \\ &= \frac{\Gamma(b+3)\Gamma(t+1)\Gamma(\nu+b-s)}{\Gamma(\nu+b+1)\Gamma(t-\nu+2)\Gamma(b-s+2)} \\ &> 0,\end{aligned}$$

clearly.

On the other hand, for $(t,s) \in T_1$, we find that

$$\begin{aligned}\Gamma(\nu)G(t,s) &= \frac{\Gamma(b+3)t^{\underline{\nu-1}}(\nu+b-s-1)^{\underline{\nu-2}}}{\Gamma(\nu+b+1)} - (t-s-1)^{\underline{\nu-1}} \\ &= \frac{\Gamma(b+3)\Gamma(t+1)\Gamma(\nu+b-s)}{\Gamma(\nu+b+1)\Gamma(t+2-\nu)\Gamma(b-s+2)} - \frac{\Gamma(t-s)}{\Gamma(t-s-\nu+1)}.\end{aligned}$$

We claim that

$$\frac{\Gamma(b+3)\Gamma(t+1)\Gamma(\nu+b-s)}{\Gamma(\nu+b+1)\Gamma(t+2-\nu)\Gamma(b-s+2)} - \frac{\Gamma(t-s)}{\Gamma(t-s-\nu+1)} > 0.$$

To see that this is true, note that it suffices to show that

$$\frac{\Gamma(b+3)\Gamma(t+1)\Gamma(\nu+b-s)\Gamma(t-s-\nu+1)}{\Gamma(\nu+b+1)\Gamma(t+2-\nu)\Gamma(b-s+2)\Gamma(t-s)} > 1$$

whenever $(t,s) \in T_1$. To prove this latter claim, we shall show that for each admissible $s$ and $t$, we have both that

$$\frac{\Gamma(b+3)\Gamma(\nu+b-s)}{\Gamma(\nu+b+1)\Gamma(b-s+2)} \geq 1 \tag{4.1}$$

and that

$$\frac{\Gamma(t+1)\Gamma(t-s-\nu+1)}{\Gamma(t+2-\nu)\Gamma(t-s)} > 1, \tag{4.2}$$

from which the desired claim will follow at once, clearly.

To see that (4.1) holds, let $s_0$ be an arbitrary but fixed element of $[0, b+1]_{\mathbb{N}_0}$. Then we find that

$$\begin{aligned}\frac{\Gamma(b+3)\Gamma(\nu+b-s)}{\Gamma(\nu+b+1)\Gamma(b-s+2)} &= \frac{\Gamma(b+3)\Gamma\left(\nu+b-s_0\right)}{\Gamma(\nu+b+1)\Gamma\left(b-s_0+2\right)} \\ &= \frac{(b+2)!\Gamma\left(\nu+b-s_0\right)}{\Gamma(\nu+b+1)\left(b-s_0+1\right)!} \\ &= \frac{(b+2)(b+1)\cdots\left(b-s_0+2\right)}{(b+\nu)(b+\nu-1)\cdots\left(b+\nu-s_0\right)}.\end{aligned} \tag{4.3}$$

But notice that $\dfrac{b+2}{b+\nu} \geq 1$, $\dfrac{b+1}{b+\nu-1} \geq 1$, $\ldots$, $\dfrac{b-s_0+2}{b+\nu-s_0} \geq 1$ in expression (4.3) above, with equality occurring if and only if $\nu = 2$. Thus, we conclude that

$$\frac{\Gamma(b+3)\Gamma(\nu+b-s)}{\Gamma(\nu+b+1)\Gamma(b-s+2)} \geq 1,$$

which establishes (4.1).

On the other hand, to see that (4.2) holds, let $s_0$, once again, be arbitrary but fixed such that $s_0 \in [0, b+1]_{\mathbb{N}_0}$. Then we have that for $t$ to be admissible, $t = s_0 + k + \nu$, for some $0 \leq k \leq b - s_0 + 1$ with $k \in \mathbb{N}_0$. But then it follows that

$$\begin{aligned}&\frac{\Gamma(t+1)\Gamma(t-s-\nu+1)}{\Gamma(t+2-\nu)\Gamma(t-s)} \\ &= \frac{\Gamma\left(s_0+k+\nu+1\right)}{\Gamma\left(s_0+k+2\right)} \cdot \frac{\Gamma(k+1)}{\Gamma(k+\nu)} \\ &= \frac{\left(\nu+s_0+k\right)\left(\nu+s_0+k-1\right)\cdots(\nu+k)\,\Gamma(k+\nu)}{\left(s_0+k+1\right)!} \cdot \frac{k!}{\Gamma(k+\nu)} \\ &= \frac{\left(\nu+s_0+k\right)\cdots(\nu+k)\cdot k!}{\left(s_0+k+1\right)!} \\ &= \frac{\left(s_0+k+\nu\right)\left(s_0+k-1+\nu\right)\cdots(k+\nu)}{\left(s_0+k+1\right)\left(s_0+k\right)\cdots(k+1)}.\end{aligned} \tag{4.4}$$

Notice, however, that each of the numerator and denominator in (4.4) has $(s_0+1)$ terms. Moreover, if we consider the terms in pairs, as in $\dfrac{s_0+k+\nu}{s_0+k+1}$, $\dfrac{s_0+k-1+\nu}{s_0+k}$, $\ldots$, $\dfrac{k+\nu}{k+1}$, then we notice that each pair is greater than unity. Indeed, as $1 < \nu \leq 2$, it follows at once, for example, that $\dfrac{s_0+k+\nu}{s_0+k+1} > 1$. As this argument may be applied to each of the $(s_0+1)$ terms in (4.3), it follows that

$$\frac{\Gamma(t+1)\Gamma(t-s-\nu+1)}{\Gamma(t+2-\nu)\Gamma(t-s)} > 1,$$

which establishes (4.2).

Finally, combining (4.1) and (4.2), we see at once that

$$\frac{\Gamma(b+3)\Gamma(t+1)\Gamma(\nu+b-s)\Gamma(t-s-\nu+1)}{\Gamma(\nu+b+1)\Gamma(t+2-\nu)\Gamma(b-s+2)\Gamma(t-s)} > 1,$$

whenever $(t,s) \in T_1$, whence $G(t,s) \geq 0$ whenever $(t,s) \in T_1$. Together with the first part of the proof, we find that $G(t,s) \geq 0$ for all $t \in [\nu-1, \nu+b+1]_{\mathbb{N}_{\nu-1}}$ and $s \in [0, b+1]_{\mathbb{N}_0}$, as claimed. □

Before proving Proposition 4.4 below, we need an easy but important preliminary lemma.

**Lemma 4.3.** *Fix $k \in \mathbb{N}$ and let $\{m_j\}_{j=1}^k$, $\{n_j\}_{j=1}^k \subseteq (0,+\infty)$ such that*

$$\max_{1\leq j\leq k} m_j \leq \min_{1\leq j\leq k} n_j$$

*and that for at least one $j_0$, $1 \leq j_0 \leq k$, we have that $m_{j_0} < n_{j_0}$. Then for fixed $\alpha_0 \in (0,1)$, it follows that*

$$\left(\frac{n_1}{n_1+\alpha_0}\cdot\ldots\cdot\frac{n_k}{n_k+\alpha_0}\right)\left(\frac{m_1+\alpha_0}{m_1}\cdot\ldots\cdot\frac{m_k+\alpha_0}{m_k}\right) > 1.$$

*Proof.* Fix an index $j_0$, where $j_0$ is one of the indices, of which there exists at least one, for which $n_{j_0} > m_{j_0}$. Notice that as $n_{j_0} > m_{j_0}$ and $\alpha_0 > 0$, it follows that $n_{j_0}\alpha_0 > m_{j_0}\alpha_0$, whence $m_{j_0}n_{j_0} + n_{j_0}\alpha_0 > m_{j_0}n_{j_0} + m_{j_0}\alpha_0$, so that

$$\frac{m_{j_0}+\alpha_0}{m_{j_0}} > \frac{n_{j_0}+\alpha_0}{n_{j_0}},$$

whence

$$\frac{n_{j_0}}{n_{j_0}+\alpha_0}\cdot\frac{m_{j_0}+\alpha_0}{m_{j_0}} > 1.$$

But now the claim follows at once by repeating the above steps for each of the remaining $j_0 - 1$ terms and observing that the product of $j$ terms, each of which is at least unity and at least one of which exceeds unity, is greater than unity. □

**Proposition 4.4.** *For $G(t,s)$ defined in Theorem 3.1, it follows that*

$$\max_{t\in[\nu-1,\nu+b+1]_{\mathbb{N}_{\nu-1}}} G(t,s) = G(s+\nu-1,s),$$

*whenever $s \in [0,b+1]_{\mathbb{N}_0}$.*

*Proof.* Before beginning this proof, let us make one preliminary observation. Indeed, note that $[\Delta_t G(t,s)]_{t=\nu+b} = G(\nu+b+1,s) - G(\nu+b,s) = 0$, for each admissible $s$, which is easy to verify by direct computation. Of course, this must be true by virtue of the fact that $G$ must satisfy the right-hand boundary condition in each of FBVPs (1.1) and (1.2). Practically, this means that

$$\max_{t\in[\nu-1,\nu+b+1]_{\mathbb{N}_{\nu-1}}} G(t,s) = \max_{t\in[\nu-1,\nu+b]_{\mathbb{N}_{\nu-1}}} G(t,s),$$

for each admissible $s$. Consequently, this means that in the sequel, we can effectively ignore what happens at $t = \nu+b+1$ on account of the above noted relationship, and we do just that.

Now, let us consider the difference $\Gamma(\nu)\Delta_t G(t,s)$ for $(t,s) \in T_1$. In this case, we find that

$$\begin{aligned}
&\Gamma(\nu)\Delta_t G(t,s)\\
&= \Delta_t\left[\frac{\Gamma(b+3)t^{\underline{\nu-1}}}{\Gamma(\nu+b+1)}(\nu+b-s-1)^{\underline{\nu-2}} - (t-s-1)^{\underline{\nu-1}}\right]\\
&= \frac{\Gamma(b+3)(\nu-1)t^{\underline{\nu-2}}}{\Gamma(\nu+b+1)}(\nu+b-s-1)^{\underline{\nu-2}} - (\nu-1)(t-s-1)^{\underline{\nu-2}}\\
&= \frac{\Gamma(b+3)(\nu-1)t^{\underline{\nu-2}}(\nu+b-s-1)^{\underline{\nu-2}} - (\nu-1)\Gamma(\nu+b+1)(t-s-1)^{\underline{\nu-2}}}{\Gamma(\nu+b+1)}\\
&= \frac{\nu-1}{\Gamma(\nu+b+1)}\left[\Gamma(b+3)t^{\underline{\nu-2}}(\nu+b-s-1)^{\underline{\nu-2}} - \Gamma(\nu+b+1)(t-s-1)^{\underline{\nu-2}}\right].
\end{aligned}$$

Note that it is clear from the above expression that in case $\nu = 2$, we find $\Delta_t G(t,s) = 0$, as expected. Consequently, let us assume in the sequel that $1 < \nu < 2$. Observe that $\dfrac{\nu-1}{\Gamma(\nu+b+1)} > 0$, clearly. So, it follows that $\Gamma(\nu)\Delta_t G(t,s) < 0$ (and thus that $\Delta_t G(t,s) < 0$, seeing as $\Gamma(\nu) > 0$) provided that

$$\Gamma(b+3)t^{\underline{\nu-2}}(\nu+b-s-1)^{\underline{\nu-2}} < \Gamma(\nu+b+1)(t-s-1)^{\underline{\nu-2}}$$

and this is true if and only if

$$\frac{\Gamma(\nu+b+1)\Gamma(t-s)\Gamma(b-s+2)}{\Gamma(b+3)\Gamma(\nu+b-s)\Gamma(t-s-\nu+2)t^{\underline{\nu-2}}} > 1. \tag{4.5}$$

To show that (4.5) holds, let us, as in the proof of Proposition 4.2, suppose that $s_0$ is a fixed but arbitrary element of $[0,b+1]_{\mathbb{N}_0}$. Then it follows, as before, that $t = s_0+k+\nu$, where $k \in \mathbb{N}$ such that $0 \leq k \leq b - s_0$. But we then find that

$$\begin{aligned}
&\frac{\Gamma(\nu+b+1)\Gamma(t-s)\Gamma(b-s+2)}{\Gamma(b+3)\Gamma(\nu+b-s)\Gamma(t-s-\nu+2)t^{\underline{\nu-2}}} \\
&= \frac{\Gamma(\nu+b+1)\Gamma\left(s_0+k+\nu-s_0\right)\Gamma\left(b-s_0+2\right)\Gamma\left(s_0+k+\nu-\nu+3\right)}{\Gamma(b+3)\Gamma\left(\nu+b-s_0\right)\Gamma\left(s_0+k+\nu-s_0-\nu+2\right)\Gamma\left(s_0+k+\nu+1\right)} \\
&= \frac{\Gamma(\nu+b+1)\Gamma(k+\nu)\Gamma\left(b-s_0+2\right)\Gamma\left(s_0+k+3\right)}{\Gamma(b+3)\Gamma\left(\nu+b-s_0\right)\Gamma(k+2)\Gamma\left(s_0+k+\nu+1\right)} \\
&= \frac{\Gamma(\nu+b+1)\Gamma(k+\nu)\left(b-s_0+1\right)!\left(s_0+k+2\right)!}{(b+2)!\Gamma\left(\nu+b-s_0\right)(k+1)!\Gamma\left(s_0+k+\nu+1\right)} \\
&= \frac{\left[(\nu+b)(\nu+b-1)\cdots\left(\nu+b-s_0\right)\right]\left(b-s_0+1\right)!\left(s_0+k+2\right)!}{(b+2)!(k+1)!\left[\left(s_0+k+\nu\right)\left(s_0+k+\nu-1\right)\cdots(k+\nu)\right]} \\
&= \frac{(\nu+b)(\nu+b-1)\cdots\left(\nu+b-s_0\right)}{(b+2)(b+1)\cdots\left(b-s_0+2\right)} \cdot \frac{\left(s_0+k+2\right)\left(s_0+k+1\right)\cdots(k+2)}{\left(s_0+k+\nu\right)\left(s_0+k+\nu-1\right)\cdots(k+\nu)}.
\end{aligned} \tag{4.6}$$

Observe that each of the numerators and denominators of each of the two fractions in (4.6) has exactly $s_0+1$ factors. Moreover, observe that in the case of the first fraction, we can consider this fraction as the product of $s_0+1$ factors as in $\frac{\nu+b}{b+2} \cdot \frac{\nu+b-1}{b+1} \cdot \ldots \cdot \frac{\nu+b-s_0}{b-s_0+2}$. Now, put $\alpha_0 := 2-\nu$ and note that $\alpha_0 \in (0,1)$. Also put $n_j := \nu+b+(1-j)$ for $1 \leq j \leq s_0+1$. Then we find that

$$\frac{(\nu+b)(\nu+b-1)\cdots\left(\nu+b-s_0\right)}{(b+2)(b+1)\cdots\left(b-s_0+2\right)} = \prod_{j=1}^{s_0+1} \frac{n_j}{n_j+\alpha_0},$$

where the finite sequence $\{n_j\}_{j=1}^{s_0+1} \subseteq (0,\infty)$ and the number $\alpha_0$ satisfy the hypotheses of Lemma 4.3. In a completely similar way, if we put $m_j := k+\nu+(j-1)$, then we find that

$$\frac{\left(s_0+k+2\right)\left(s_0+k+1\right)\cdots(k+2)}{\left(s_0+k+\nu\right)\left(s_0+k+\nu-1\right)\cdots(k+\nu)} = \prod_{j=1}^{s_0+1} \frac{m_j+\alpha_0}{m_j},$$

which again is of the form in Lemma 4.3, for $\{m_j\}_{j=1}^{s_0+1} \subseteq (0,\infty)$. Consequently, with $m_j$, $n_j$, and $\alpha_0$ defined as above, we note that

$$\begin{aligned}
&\frac{\Gamma(\nu+b+1)\Gamma(t-s)\Gamma(b-s+2)}{\Gamma(b+3)\Gamma(\nu+b-s)\Gamma(t-s-\nu+2)t^{\underline{\nu-2}}} \\
&= \frac{(\nu+b)(\nu+b-1)\cdots\left(\nu+b-s_0\right)}{(b+2)(b+1)\cdots\left(b-s_0+2\right)} \cdot \frac{\left(s_0+k+2\right)\left(s_0+k+1\right)\cdots(k+2)}{\left(s_0+k+\nu\right)\left(s_0+k+\nu-1\right)\cdots(k+\nu)} \\
&= \left(\prod_{j=1}^{s_0+1} \frac{n_j}{n_j+\alpha_0}\right)\left(\prod_{j=1}^{s_0+1} \frac{m_j+\alpha_0}{m_j}\right).
\end{aligned} \tag{4.7}$$

Now, in order to apply Lemma 4.3 to (4.7) above, we must consider three cases. First, it is possible, depending upon the choice of $s_0$, $k$, and $b$, that there are no repeated factors between the two products in (4.7). In this case, we see that $\max_j m_j < \min_j n_j$, and so, by the argument in the preceding paragraph, we may immediately apply Lemma 4.3 to deduce the bound given in (4.5).

Secondly, it is possible that some factors are repeated between the two products in (4.7). In particular, there may be $p$ such repeated factors, with $1 \leq p \leq s_0$, in each of the numerators and denominators of each of the products in (4.7) that cancel. This cancellation will leave $s_0 + 1 - p$ factors – in particular, in this case it is easy to show that

$$\begin{aligned}
&\frac{\Gamma(\nu+b+1)\Gamma(t-s)\Gamma(b-s+2)}{\Gamma(b+3)\Gamma(\nu+b-s)\Gamma(t-s-\nu+2)t^{\underline{\nu-2}}} \\
&= \left(\prod_{j=1}^{s_0+1} \frac{n_j}{n_j+\alpha_0}\right)\left(\prod_{j=1}^{s_0+1} \frac{m_j+\alpha_0}{m_j}\right) \\
&= \left(\prod_{j=1}^{s_0+1-p} \frac{n_j}{n_j+\alpha_0}\right)\left(\prod_{j=1}^{s_0+1-p} \frac{m_j+\alpha_0}{m_j}\right).
\end{aligned} \tag{4.8}$$

But then Lemma 4.3 may be applied to (4.8) above to yield the bound in (4.5) in this case, too.

Finally, if $k = b - s_0$, then it equally easy to show that product (4.6) is exactly unity – that is,

$$\frac{\Gamma(\nu+b+1)\Gamma(t-s)\Gamma(b-s+2)}{\Gamma(b+3)\Gamma(\nu+b-s)\Gamma(t-s-\nu+2)t^{\underline{\nu-2}}} = \left(\prod_{j=1}^{s_0+1} \frac{n_j}{n_j+\alpha_0}\right)\left(\prod_{j=1}^{s_0+1} \frac{m_j+\alpha_0}{m_j}\right) = 1.$$

However, this corresponds to the case $\Delta_t \left[G\left(t, s_0\right)\right]_{t=\nu+b}$, and we observed at the beginning of this proof that $\Delta_t \left[G\left(t, s_0\right)\right]_{t=\nu+b} = 0$, as it must from the boundary conditions.

So, in summary, in each of the three cases we can safely apply Lemma 4.3 to (4.6) to get that

$$\frac{\Gamma(\nu+b+1)\Gamma(t-s)\Gamma(b-s+2)}{\Gamma(b+3)\Gamma(\nu+b-s)\Gamma(t-s-\nu+2)t^{\underline{\nu-2}}} > 1,$$

so that (4.5) holds. By the earlier observation, then, it follows at once that $\Delta_t G(t,s) < 0$ whenever $0 \leq s < t - \nu + 1 \leq b + 1$, as desired.

We next argue that $\Delta_t G(t,s) > 0$ for $0 \leq t - \nu + 1 \leq s \leq b + 1$. To see that this is true, we simply notice that for $0 \leq t - \nu + 1 \leq s \leq b + 1$,

$$\begin{aligned}\Delta_t G(t,s) &= \Delta_t \left[ \frac{\Gamma(b+3)t^{\underline{\nu-1}}}{\Gamma(\nu+b+1)}(\nu+b-s-1)^{\underline{\nu-2}} \right] \\ &= \frac{\Gamma(b+3)(\nu-1)t^{\underline{\nu-2}}}{\Gamma(\nu+b+1)}(\nu+b-s-1)^{\underline{\nu-2}} \\ &= \frac{\Gamma(b+3)(\nu-1)\Gamma(t+1)\Gamma(\nu+b-s)}{\Gamma(\nu+b+1)\Gamma(t-\nu+3)\Gamma(b-s+2)}.\end{aligned} \tag{4.9}$$

Now, observe that each factor in (4.9) is strictly positive. Therefore, we conclude that $\Delta_t G(t,s) > 0$ in case $0 \leq t - \nu + 1 \leq s \leq b + 1$, whence $G(t,s)$ is increasing on that interval, too.

In summary, then, we have that $G(t,s)$ is increasing for $t - \nu + 1 \leq s \leq b + 1$ and decreasing for $0 \leq s < t - \nu + 1$. And from this we may conclude that

$$\max_{t \in [\nu-1, \nu+b+1]_{\mathbb{N}_{\nu-1}}} G(t,s) = G(s+\nu-1, s),$$

whenever $s \in [0, b+1]_{\mathbb{N}_0}$, as desired. □

*Remark* 4.5. Interestingly, we notice that in case $\nu \in (1,2)$, Proposition 4.4 demonstrates that $G(t,s)$ is not constant for $t > s + \nu - 1$. This contrasts with the classical case, $\nu = 2$, in which Green's function attains its maximum at $t = s$ and then is constant for $t > s$. Furthermore, as $\nu \to 2$ from the left, our Green's function does tend to the known Green function in case $\nu = 2$.

Before proving our final proposition, let us introduce the constants $\gamma_1$ and $\gamma_2$, which will be important not only in the following proposition but also in the final section of this paper:

$$\gamma_1 := \frac{\left(\frac{b+\nu}{4}\right)^{\underline{\nu-1}}}{(b+\nu)^{\underline{\nu-1}}},$$

$$\gamma_2 := \frac{1}{\left(\frac{3(b+\nu)}{4}\right)^{\underline{\nu-1}}} \left[ \left(\frac{3(b+\nu)}{4}\right)^{\underline{\nu-1}} - \frac{b+1}{\Gamma(b+3)} \cdot \frac{\left(\frac{3(b+\nu)}{4} - 1\right)^{\underline{\nu-1}} \Gamma(\nu+b+1)}{(\nu+b-1)^{\underline{\nu-1}}} \right].$$

**Proposition 4.6.** *Assume that* $\left[\frac{b+\nu}{4}, \frac{3(b+\nu)}{4}\right] \cap \mathbb{N}_{\nu-1} \neq \emptyset$. *For* $G(t,s)$ *defined in Theorem 3.1, it follows that there exists a number* $\gamma \in (0,1)$*, where*

$$\gamma := \min\{\gamma_1, \gamma_2\},$$

*with* $\gamma_1$ *and* $\gamma_2$ *as above, such that*

$$\min_{t \in \left[\frac{b+\nu}{4}, \frac{3(b+\nu)}{4}\right]} G(t,s) \geq \gamma \cdot \max_{t \in [\nu-1, \nu+b+1]_{\mathbb{N}_{\nu-1}}} G(t,s) = \gamma G(s+\nu-1, s),$$

*for* $s \in [0, b+1]_{\mathbb{N}_0}$.

*Proof.* Let us begin by noting that

$$\frac{G(t,s)}{G(s+\nu-1,s)} = \begin{cases} \dfrac{t^{\underline{\nu-1}}}{(s+\nu-1)^{\underline{\nu-1}}} - \dfrac{(t-s-1)^{\underline{\nu-1}}\Gamma(\nu+b+1)}{\Gamma(b+3)(s+\nu-1)^{\underline{\nu-1}}(\nu+b-s-1)^{\underline{\nu-2}}}, & (t,s)\in T_1 \\ \dfrac{t^{\underline{\nu-1}}}{(s+\nu-1)^{\underline{\nu-1}}}, & (t,s)\in T_2, \end{cases}$$

which is obtained by direct calculation. Now, for $s \geq t-\nu+1$ and $\frac{b+\nu}{4} \leq t \leq \frac{3(b+\nu)}{4}$, we have that

$$\frac{G(t,s)}{G(s+\nu-1,s)} = \frac{t^{\underline{\nu-1}}}{(s+\nu-1)^{\underline{\nu-1}}} \geq \frac{\left(\frac{b+\nu}{4}\right)^{\underline{\nu-1}}}{((b+1)+\nu-1)^{\underline{\nu-1}}} = \frac{\left(\frac{b+\nu}{4}\right)^{\underline{\nu-1}}}{(b+\nu)^{\underline{\nu-1}}}, \tag{4.10}$$

because $t^{\underline{\alpha}}$ is increasing in $t$ for $\alpha \in (0,1)$.

On the other hand, the proof of Proposition 4.4 shows that $G(t,s)$ is decreasing in case $s < t-\nu+1$. Consequently, for $s < t-\nu+1$ and $t \in \left[\frac{b+\nu}{4}, \frac{3(b+\nu)}{4}\right]$ it follows that

$$\begin{aligned} &\min_{t\in\left[\frac{b+\nu}{4},\frac{3(b+\nu)}{4}\right]} \frac{G(t,s)}{G(s+\nu-1,s)} \\ &= \left[\frac{t^{\underline{\nu-1}}}{(s+\nu-1)^{\underline{\nu-1}}} - \frac{(t-s-1)^{\underline{\nu-1}}\Gamma(\nu+b+1)}{\Gamma(b+3)(s+\nu-1)^{\underline{\nu-1}}(\nu+b-s-1)^{\underline{\nu-2}}}\right]_{t=\frac{3(b+\nu)}{4}} \\ &= \frac{\left(\frac{3(b+\nu)}{4}\right)^{\underline{\nu-1}}}{(s+\nu-1)^{\underline{\nu-1}}} - \frac{\left(\frac{3(b+\nu)}{4}-s-1\right)^{\underline{\nu-1}}\Gamma(\nu+b+1)}{\Gamma(b+3)(s+\nu-1)^{\underline{\nu-1}}(\nu+b-s-1)^{\underline{\nu-2}}}. \end{aligned}$$

Now, put

$$\alpha(s) := \frac{1}{(s+\nu-1)^{\underline{\nu-1}}}\left[\left(\frac{3(b+\nu)}{4}\right)^{\underline{\nu-1}} - \frac{\left(\frac{3(b+\nu)}{4}-s-1\right)^{\underline{\nu-1}}\Gamma(\nu+b+1)}{\Gamma(b+3)(\nu+b-s-1)^{\underline{\nu-2}}}\right].$$

Notice that

$$(\nu+b-s-1)^{\underline{\nu-2}} = \frac{(\nu+b-s-1)^{\underline{\nu-1}}}{b-s+1},$$

which is a simple consequence of Definition 2.2. Furthermore, observe that by Lemma 4.1, part (ii) we find that

$$\frac{\left(\frac{3(b+\nu)}{4}-s-1\right)^{\underline{\nu-1}}}{(\nu+b-s-1)^{\underline{\nu-1}}}$$

is decreasing for $0 \le s \le \dfrac{3(b+\nu)}{4} - \nu + 1$. Consequently, these two observations together with an application of Lemma 4.1, part (i) imply that

$$
\begin{aligned}
&\alpha(s)\\
&= \frac{1}{(s+\nu-1)^{\underline{\nu-1}}}\left[\left(\frac{3(b+\nu)}{4}\right)^{\underline{\nu-1}} - \frac{\left(\frac{3(b+\nu)}{4}-s-1\right)^{\underline{\nu-1}}\Gamma(\nu+b+1)}{\Gamma(b+3)(\nu+b-s-1)^{\underline{\nu-2}}}\right]\\
&= \frac{1}{(s+\nu-1)^{\underline{\nu-1}}}\left[\left(\frac{3(b+\nu)}{4}\right)^{\underline{\nu-1}} - \frac{\left(\frac{3(b+\nu)}{4}-s-1\right)^{\underline{\nu-1}}\Gamma(\nu+b+1)}{\frac{\Gamma(b+3)}{b-s+1}\cdot(\nu+b-s-1)^{\underline{\nu-1}}}\right]\\
&\ge \frac{1}{(s+\nu-1)^{\underline{\nu-1}}}\left[\left(\frac{3(b+\nu)}{4}\right)^{\underline{\nu-1}} - \frac{b+1}{\Gamma(b+3)}\cdot\frac{\left(\frac{3(b+\nu)}{4}-1\right)^{\underline{\nu-1}}\Gamma(\nu+b+1)}{(\nu+b-1)^{\underline{\nu-1}}}\right]\\
&\ge \frac{1}{\left(\frac{3(b+\nu)}{4}\right)^{\underline{\nu-1}}}\left[\left(\frac{3(b+\nu)}{4}\right)^{\underline{\nu-1}} - \frac{b+1}{\Gamma(b+3)}\cdot\frac{\left(\frac{3(b+\nu)}{4}-1\right)^{\underline{\nu-1}}\Gamma(\nu+b+1)}{(\nu+b-1)^{\underline{\nu-1}}}\right],
\end{aligned}
$$

where to get the first inequality we set $s = 0$ in the expression in the square brackets.

As a result of this analysis, we conclude that in case $s < t - \nu + 1$ and $t \in \left[\dfrac{b+\nu}{4}, \dfrac{3(b+\nu)}{4}\right]$,

$$
\begin{aligned}
&\frac{G(t,s)}{G(s+\nu-1,s)}\\
&\ge \frac{1}{\left(\frac{3(b+\nu)}{4}\right)^{\underline{\nu-1}}}\left[\left(\frac{3(b+\nu)}{4}\right)^{\underline{\nu-1}} - \frac{b+1}{\Gamma(b+3)}\cdot\frac{\left(\frac{3(b+\nu)}{4}-1\right)^{\underline{\nu-1}}\Gamma(\nu+b+1)}{(\nu+b-1)^{\underline{\nu-1}}}\right].
\end{aligned}
\tag{4.11}
$$

Finally, then, upon combining (4.10) and (4.11), we deduce that

$$\min_{\frac{b+\nu}{4}\le t\le\frac{3(b+\nu)}{4}} G(t,s) \ge \gamma \max_{t\in[\nu-1,\nu+b+1]_{\mathbb{N}_{\nu-1}}} G(t,s) = \gamma G(s+\nu-1,s),$$

where we put

$$\gamma := \min\{\gamma_1, \gamma_2\},$$

which completes the proof. $\square$

*Remark* 4.7. Note that it is the case that $0 < \gamma < 1$ in Proposition 4.6. Indeed, it is clear that $0 < \dfrac{\left(\frac{b+\nu}{4}\right)^{\underline{\nu-1}}}{(b+\nu)^{\underline{\nu-1}}} < 1$. On the other hand, to see that

$$0 < \frac{1}{\left(\frac{3(b+\nu)}{4}\right)^{\underline{\nu-1}}}\left[\left(\frac{3(b+\nu)}{4}\right)^{\underline{\nu-1}} - \frac{b+1}{\Gamma(b+3)} \cdot \frac{\left(\frac{3(b+\nu)}{4}-1\right)^{\underline{\nu-1}}\Gamma(\nu+b+1)}{(\nu+b-1)^{\underline{\nu-1}}}\right] < 1,$$

we may observe that

$$\begin{aligned} 0 &< \frac{b+1}{\Gamma(b+3)} \cdot \frac{1}{\left(\frac{3(b+\nu)}{4}\right)^{\underline{\nu-1}}} \cdot \frac{\Gamma\left(\frac{3(b+\nu)}{4}\right)\Gamma(\nu+b+1)\Gamma(b+1)}{\Gamma\left(\frac{3(b+\nu)}{4}-\nu+1\right)\Gamma(\nu+b)} \\ &= \frac{1}{b+2} \cdot \frac{\Gamma\left(\frac{3(b+\nu)}{4}\right)\Gamma(\nu+b+1)\Gamma\left(\frac{3(b+\nu)}{4}-\nu+2\right)}{\Gamma\left(\frac{3(b+\nu)}{4}+1\right)\Gamma(\nu+b)\Gamma\left(\frac{3(b+\nu)}{4}-\nu+1\right)} \\ &= \frac{(b+\nu)\left(\frac{3(b+\nu)}{4}-\nu+1\right)}{(b+2)\left(\frac{3(b+\nu)}{4}\right)} \\ &< 1, \end{aligned}$$

which suffices to prove the claim.

*Remark* 4.8. In case we put $\nu = 2$ in Proposition 4.6, we find by direct calculation that $\gamma := \min\left\{\frac{1}{4}, \frac{4}{3b+6}\right\}$.

*Remark* 4.9. It should be noted that while the right-focal problem is simpler than the Dirichlet problem in the case when $\nu = 2$, it is more difficult in the fractional case (i.e., in case $1 < \nu < 2$) as a comparison of the above proofs to the corresponding proofs in [6] shows.

# 5 Existence and Uniqueness Theorems

In this final section of the paper, we wish to deduce certain representative existence and uniqueness theorems. We begin with a preliminary and well known lemma, which can be found, for example, in [1] and is due to Krasnosel'skiĭ.

**Lemma 5.1.** *Let $\mathcal{B}$ be a Banach space and let $\mathcal{K} \subseteq \mathcal{B}$ be a cone. Assume that $\Omega_1$ and $\Omega_2$ are open sets contained in $\mathcal{B}$ such that $0 \in \Omega_1$ and $\overline{\Omega}_1 \subseteq \Omega_2$. Assume, further, that $T : \mathcal{K} \cap \left(\overline{\Omega}_2 \setminus \Omega_1\right) \to \mathcal{K}$ is a completely continuous operator. Then if either*

(i) $\|Ty\| \leq \|y\|$ *for* $y \in \mathcal{K} \cap \partial\Omega_1$ *and* $\|Ty\| \geq \|y\|$ *for* $y \in \mathcal{K} \cap \partial\Omega_2$*, or*

(ii) $\|Ty\| \geq \|y\|$ *for* $y \in \mathcal{K} \cap \partial\Omega_1$ *and* $\|Ty\| \leq \|y\|$ *for* $y \in \mathcal{K} \cap \partial\Omega_2$,

*then* $T$ *has at least one fixed point in* $\mathcal{K} \cap \left(\overline{\Omega}_2 \setminus \Omega_1\right)$.

We now consider the nonlinear equation (1.1). We notice that $y$ solves (1.1) if and only if $y$ is a fixed point of the operator

$$Ty := \sum_{s=0}^{b+1} G(t,s) f(s+\nu-1, y(s+\nu-1)),$$

where $G$ is Green's function derived in this paper and $T : \mathcal{B} \to \mathcal{B}$, where $\mathcal{B}$ is the Banach space $\mathcal{B} := \left\{y : [\nu-2, \nu+b+1]_{\mathbb{N}_{\nu-2}} \to \mathbb{R} : y(\nu-2) = \Delta y(\nu+b) = 0\right\}$ equipped with the usual supremum norm, $\|\cdot\|$.

Let us also make the following declarations, which will be used in the sequel:

$$\eta := \frac{1}{\sum_{s=1}^{b+1} G(s+\nu-1, s)},$$

$$\lambda := \frac{1}{\sum_{s=\lceil \frac{\nu+b}{4} - \nu + 1 \rceil}^{\lfloor \frac{3(\nu+b)}{4} - \nu + 1 \rfloor} G\left(\left\lfloor \frac{b+1}{2} \right\rfloor + \nu, s\right)}.$$

Let us also introduce two conditions on the behavior of $f$ that will be useful in the sequel.

(C1) There exists a number $r > 0$ such that $f(t,y) \leq \eta r$ whenever $0 \leq y \leq r$.

(C2) There exists a number $r > 0$ such that $f(t,y) \geq \lambda r$ whenever $\gamma r \leq y \leq r$.

*Remark* 5.2. The technique that we use to deduce the existence of at least one positive solution is very similar to the techniques found in the classical literature on differential equations – see, for example, [10].

We now can prove the following existence result.

**Theorem 5.3.** *Suppose that there are distinct* $r_1, r_2 > 0$ *such that condition* (*C1*) *holds at* $r = r_1$ *and condition* (*C2*) *holds at* $r = r_2$. *Suppose also that* $f(t,y) \geq 0$ *and continuous. Then the FBVP* (1.1) *has at least one positive solution, say* $y_0$, *such that* $\|y_0\|$ *lies between* $r_1$ *and* $r_2$.

*Proof.* We shall assume without loss of generality that $0 < r_1 < r_2$. Consider the set $\mathcal{K} := \left\{y \in \mathcal{B} : y(t) \geq 0 \text{ and } \min_{t \in \left[\frac{b+\nu}{4}, \frac{3(b+\nu)}{4}\right]} y(t) \geq \gamma\|y\|\right\}$, which is a cone with

$\mathcal{K} \subseteq \mathcal{B}$. Observe that $T : \mathcal{K} \to \mathcal{K}$, for we observe both that

$$\begin{aligned}
\min_{t\in\left[\frac{b+\nu}{4},\frac{3(b+\nu)}{4}\right]}(Ty)(t) &= \min_{t\in\left[\frac{b+\nu}{4},\frac{3(b+\nu)}{4}\right]}\sum_{s=0}^{b+1}G(t,s)f(s+\nu-1,y(s+\nu-1))\\
&\geq \gamma\sum_{s=0}^{b+1}G(s+\nu-1,s)f(s+\nu-1,y(s+\nu-1))\\
&= \gamma\max_{t\in[\nu-1,\nu+b+1]_{\mathbb{N}_{\nu-1}}}\sum_{s=0}^{b+1}G(t,s)f(s+\nu-1,y(s+\nu-1))\\
&= \gamma\|Ty\|,
\end{aligned}$$

and that $(Ty)(t) \geq 0$ whenever $y \in \mathcal{K}$, whence $Ty \in \mathcal{K}$, as claimed. Also, it is easy to see that $T$ is a completely continuous operator.

Now, put $\Omega_1 := \{y \in \mathcal{K} : \|y\| < r_1\}$. Note that for $y \in \partial\Omega_1$, we have that $\|y\| = r_1$ so that condition (C1) holds for all $y \in \partial\Omega_1$. So, for $y \in \mathcal{K} \cap \partial\Omega_1$, we find that

$$\begin{aligned}
\|Ty\| &= \max_{t\in[\nu-1,\nu+b+1]_{\mathbb{N}_{\nu-1}}}\sum_{s=0}^{b+1}G(t,s)f(s+\nu-1,y(s+\nu-1))\\
&\leq \sum_{s=0}^{b+1}G(s+\nu-1,s)f(s+\nu-1,y(s+\nu-1))\\
&\leq \eta r_1\sum_{s=0}^{b+1}G(s+\nu-1,s)\\
&= r_1\\
&= \|y\|,
\end{aligned}$$

whence we find that $\|Ty\| \leq \|y\|$ whenever $y \in \mathcal{K} \cap \partial\Omega_1$. Thus we get that the operator $T$ is a cone compression on $\mathcal{K} \cap \partial\Omega_1$.

On the other hand, put $\Omega_2 := \{y \in \mathcal{K} : \|y\| < r_2\}$. Note that for $y \in \partial\Omega_2$, we have that $\|y\| = r_2$ so that condition (C2) holds for all $y \in \partial\Omega_2$. Also note that

$\left\{\left\lfloor\frac{b+1}{2}\right\rfloor+\nu\right\}\subset\left[\frac{b+\nu}{4},\frac{3(b+\nu)}{4}\right]$. So, for $y\in\mathcal{K}\cap\partial\Omega_2$, we find that

$$\begin{aligned}
Ty\left(\left\lfloor\frac{b+1}{2}\right\rfloor+\nu\right)&=\sum_{s=0}^{b+1}G\left(\left\lfloor\frac{b+1}{2}\right\rfloor+\nu,s\right)f(s+\nu-1,y(s+\nu-1))\\
&\geq\sum_{s=\lceil\frac{\nu+b}{4}-\nu+1\rceil}^{\lfloor\frac{3(\nu+b)}{4}-\nu+1\rfloor}G\left(\left\lfloor\frac{b+1}{2}\right\rfloor+\nu,s\right)f(s+\nu-1,y(s+\nu-1))\\
&\geq\lambda r_2\sum_{s=\lceil\frac{\nu+b}{4}-\nu+1\rceil}^{\lfloor\frac{3(\nu+b)}{4}-\nu+1\rfloor}G\left(\left\lfloor\frac{b+1}{2}\right\rfloor+\nu,s\right)\\
&=r_2,
\end{aligned}$$

whence $\|Ty\|\geq\|y\|$, whenever $y\in\mathcal{K}\cap\partial\Omega_2$. Thus we get that the operator $T$ is a cone expansion on $\mathcal{K}\cap\partial\Omega_2$. So, it follows by Lemma 5.1 that the operator $T$ has a fixed point. But this means that (1.1) has a positive solution, say $y_0$, with $r_1\leq\|y_0\|\leq r_2$, as claimed. $\square$

*Remark* 5.4. Of course, it is possible to extend Theorem 5.3. In particular, one can provide conditions under which multiple positive solutions will exist. As the author has already presented such results in the Dirichlet case (cf., [12]), they will not be repeated with the dual results here.

If we assume that $f$ satisfies a Lipschitz condition, then we can get uniqueness in addition to existence. This is the content of Theorem 5.6 below. We require first a preliminary lemma.

**Lemma 5.5.** *For $G(t,s)$ as defined in Theorem 3.1, we find that*

$$\max_{t\in[\nu-1,\nu+b+1]_{\mathbb{N}_{\nu-1}}}\sum_{s=0}^{b+1}G(t,s)\leq\frac{(b+2)\Gamma(b+\nu+2)}{\Gamma(\nu+1)\Gamma(b+2)}.$$

*Proof.* By invoking Theorem 3.1 together with Proposition 4.4 we find that

$$\begin{aligned}
G(s+\nu-1,s)&=\frac{\Gamma(b+3)(s+\nu-1)^{\underline{\nu-1}}\Gamma(\nu+b-s)}{\Gamma(\nu)\Gamma(\nu+b+1)\Gamma(b-s+2)}\\
&\leq\frac{(b+2)!\Gamma(b+2-s)(s+\nu-1)^{\underline{\nu-1}}}{\Gamma(\nu)(b+1)!\Gamma(b-s+2)}\\
&=\frac{(b+2)}{\Gamma(\nu)}(s+\nu-1)^{\underline{\nu-1}},
\end{aligned}$$

from which it follows that

$$\max_{t\in[\nu-1,\nu+b+1]_{\mathbb{N}_{\nu-1}}}\sum_{s=0}^{b+1}G(t,s)\le\sum_{s=0}^{b+1}\frac{b+2}{\Gamma(\nu)}(s+\nu-1)^{\underline{\nu-1}}$$
$$=\frac{b+2}{\Gamma(\nu)}\left[\frac{1}{\nu}(s+\nu-1)^{\underline{\nu}}\right]_{s=0}^{b+2}$$
$$=\frac{b+2}{\Gamma(\nu)}\cdot\frac{1}{\nu}(b+\nu+1)^{\underline{\nu}}$$
$$=\frac{(b+2)\Gamma(b+\nu+2)}{\Gamma(\nu+1)\Gamma(b+2)},$$

as claimed. □

Now we prove a uniqueness theorem by using the Banach contraction theorem, which can be found, for example, in [22].

**Theorem 5.6.** *Suppose that $f(t,y)$ satisfies a Lipschitz condition in $y$ with Lipschitz constant $\alpha$ – that is, $|f(t,y_2)-f(t,y_1)|\le\alpha|y_2-y_1|$ for all $(t,y_1)$, $(t,y_2)$. Then it follows that if*

$$\frac{(b+2)\Gamma(b+\nu+2)}{\Gamma(\nu+1)\Gamma(b+2)}<\frac{1}{\alpha},$$

*then* (1.1) *has a unique solution.*

*Proof.* Let $y_1,y_2\in\mathcal{B}$, where $\mathcal{B}$ is the Banach space described earlier. Then we find that

$$\|Ty_2-Ty_1\|$$
$$\le\max_{t\in[\nu-1,\nu+b+1]_{\mathbb{N}_{\nu-1}}}\sum_{s=0}^{b+1}\Big[|G(t,s)|$$
$$\cdot\left|f(s+\nu-1,y_2(s+\nu-1))-f(s+\nu-1,y_1(s+\nu-1))\right|\Big]$$
$$\le\alpha\sum_{s=0}^{b+1}G(s+\nu-1,s)\left|y_2(s+\nu-1)-y_1(s+\nu-1)\right|$$
$$\le\alpha\|y_2-y_1\|\sum_{s=0}^{b+1}G(s+\nu-1,s)$$
$$\le\alpha\frac{(b+2)\Gamma(b+\nu+2)}{\Gamma(\nu+1)\Gamma(b+2)}\|y_2-y_1\|.$$

So, as $\alpha\dfrac{(b+2)\Gamma(b+\nu+2)}{\Gamma(\nu+1)\Gamma(b+2)}<1$ by assumption, it follows by the Banach contraction theorem that (1.1) has a unique solution, as claimed. □

**Example 5.7.** Suppose that $\nu := \frac{11}{10}$ and $\alpha := \frac{1}{75}$. If $f(t,y)$ in problem (1.1) is Lipschitz with Lipschitz constant $\alpha$, then Theorem 5.6 implies that (1.1) will have a unique solution provided that

$$\frac{(b+2)\Gamma\left(b+\frac{31}{10}\right)}{\Gamma\left(\frac{21}{10}\right)\Gamma(b+2)} < 75, \tag{5.1}$$

and (5.1) can be solved numerically to get that $b_{\text{max}} \approx 5.960$, where $b_{\text{max}}$ is the largest value of $b$ such that the hypotheses of Theorem 5.6 is satisfied.

*Remark* 5.8. The bound in Theorem 5.6 can be improved if we use a more complicated bound in Lemma 5.5, which may be easily facilitated by the use of a computer. The bound provided by Lemma 5.5 was chosen for computational simplicity.

*Remark* 5.9. Using the bound given by Theorem 5.6 in case $\nu = 2$, yields a unique solution provided that $\frac{(b+3)(b+2)^2}{2} < \frac{1}{\alpha}$, which is not as good as the classical bound (cf., [16]). Once again, however, this bound can be improved by using a more complicated estimate than was used in Lemma 5.5.

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