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Parametric Resonance in Some Dynamic Equations on Time Scales

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Abstract

In this paper some dynamic equations with oscillatory decreasing coefficients on time scales are studied. We show that the small perturbation of the initial time scale may cause the parametric resonance phenomenon.

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1 Introduction

The phenomenon of unbounded oscillations arising in dynamical system under an arbitrarily small periodic (or almost periodic) perturbations of some parameters is known as *parametric resonance*. It is known that parametric resonance occurs when the frequency of perturbation is close to the so-called resonant frequencies defined by the intrinsic properties of the system. We can observe the parametric resonance phenomenon not only in continuous systems but in discrete systems as well. Consider, for example, the following differential equation:

$$\frac{d^2x}{dt^2} + \omega^2\big[1 + \varepsilon f(t)\big]x = 0, \qquad t \in \mathbb{R}, \tag{1.1}$$

where ω is a real parameter, ε is a small parameter and $f(t)$ is a periodic or almost periodic function. A "natural" discrete analog of (1.1) is the following second order difference equation:

$$x(t+2) - 2(\cos\omega)x(t+1) + \big[1 + \varepsilon f(t)\big]x(t) = 0, \qquad t \in \mathbb{N}, \tag{1.2}$$

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where $0 < \omega < \pi$.

In equations (1.1), (1.2) the parametric perturbation is small what is reflected by the presence of small parameter ε. There are physical models in which the external influence is small in some other sense. Namely, the influence decreases as the independent variable (time or space variable) goes to infinity. The examples of such systems are given by equations (1.1), (1.2) if $\varepsilon = \varepsilon(t) \to 0$ as $t \to \infty$. In the latter case these equations are called *adiabatic oscillators*.

In physics and engineering the parametric resonance phenomenon is often used to amplify the small oscillations. To produce large amplitude oscillations we only need to choose an appropriate frequency of parametric perturbation. Suppose we are unable to change the frequency of perturbation to get resonance. Moreover, we assume that we have no possibility to manipulate the natural oscillations of the system. As it is shown below even in this situation we still can amplify the oscillations using the parametric resonance phenomenon.

In this paper we consider the discrete systems with oscillatory decreasing coefficients (see Eq. (1.2), where $\varepsilon = \varepsilon(t) \to 0$ as $t \to \infty$). We show, that parametric resonance in these models occurs if we arrange the elements of physical system at the definite points (if t is a space variable) or if we influence the elements of physical system at the definite moments (if t is a time variable).

The problem we consider here can be formulated in terms of the theory of time scales [2]. Actually, we use the evident fact that the structure of time scale can influence greatly the qualitative behaviour of the solutions of dynamic equations on time scale. The aim of this paper is to construct the example of the dynamic system on time scale that demonstrates the qualitative change in dynamics under a small perturbation of initial time scale. In the example considered below the small perturbation of time scale gives rise to parametric resonance phenomenon.

We are ready to state the problem using the time scales terminology. Let $\mathbb{T}_1$ be a time scale (i.e., a closed nonempty subset of $\mathbb{R}$) that contains only isolated points. We assume that $\mathbb{T}_1$ is bounded below and unbounded above. Hence,

$$\mathbb{T}_1 = \{t_n \in \mathbb{R}, \quad n \in \mathbb{N} \text{ (i.e., } n = 1, 2, 3, \ldots)\}, \qquad t_n < t_{n+1}, \tag{1.3}$$

and $t_n \to \infty$ as $n \to \infty$. Consider the following linear system of dynamic equations on time scale $\mathbb{T}_1$ that, due to the structure of $\mathbb{T}_1$, can be written in the form

$$x^\sigma = A(t)x, \qquad t \in \mathbb{T}_1, \qquad x \in \mathbb{C}^m, \quad \sigma(t_n) = t_{n+1}. \tag{1.4}$$

Here $\mathbb{C}$ is the set of all complex numbers and $\mathbb{C}^m$ is m-dimensional complex arithmetic space. Moreover, $\sigma(t)$ is the forward jump operator

$$\sigma(t) = \inf\{s \in \mathbb{T} : \ s > t, \ t \in \mathbb{T}\},$$

where $\mathbb{T}$ is some time scale and the notation x^σ denotes $x \circ \sigma$. In what follows we assume that the entries of matrix $A(t)$ are defined not only for points of $\mathbb{T}_1$ but for all $t \in \mathbb{R}$.

Suppose now that the time scale $\mathbb{T}_1$ is perturbed to the form of time scale $\mathbb{T}_2$

$$\mathbb{T}_2 = \{\tau_n, \quad n \in \mathbb{N}\}, \qquad \tau_n < \tau_{n+1}, \tag{1.5}$$

where $\tau_n \to \infty$ as $n \to \infty$. We assume that this perturbation is small in the following sense:

$$\lim_{n\to\infty} (t_n - \tau_n) = 0. \tag{1.6}$$

Therefore system (1.4) on time scale $\mathbb{T}_2$ takes the form

$$x^\sigma = A(t)x, \qquad t \in \mathbb{T}_2, \qquad x \in \mathbb{C}^m, \quad \sigma(\tau_n) = \tau_{n+1}. \tag{1.7}$$

In this paper we construct time scales $\mathbb{T}_1$, $\mathbb{T}_2$ with an assumption (1.6) and matrix $A(t)$ having the following properties: all solutions of system (1.4) are bounded, however, system (1.7) has an unbounded solution. Actually, we start with the second order dynamic equation and this will be done in Section 3. In the next section we present some preliminary results concerning the asymptotic behaviour of solutions of systems of difference equations with oscillatory decreasing coefficients.

2 Some Preliminaries

To analyze the behaviour of solutions of difference equations that appear further in this paper we need to use some special technique. The method we apply here was developed for construction the asymptotic formulas for solutions of systems of differential equations with oscillatory decreasing coefficients (see, e.g., [4, 6, 7]). We use here the difference analog of this method that was introduced in paper [3] and in more general form in forthcoming paper [5].

The method we refer to, utilizes two basic ideas: the averaging changes of variable in systems with oscillatory decreasing coefficients and the difference analog of Levinson's fundamental theorem [1]. We will describe now this method in more details.

Consider the following system of difference equations:

$$x(t+1) = \Big(I + \sum_{i=1}^{n} A_i(t)v_i(t) + R(t)\Big)x(t). \tag{2.1}$$

Here $A_1(t), \dots, A_n(t)$, $R(t)$ are $(m \times m)$ square matrices and I is $(m \times m)$ identity matrix. Moreover, $v_1(t), \dots, v_n(t)$ are scalar functions, $x(t) \in \mathbb{C}^m$ and $t \in \mathbb{N}$. Throughout the paper we will use the following notation. We will write $f(t) \in \ell_1$ ($f(t)$: $\mathbb{N} \to \mathbb{C}$) if $\{f(t)\}_{t=1}^\infty \in \ell_1$, i.e.,

$$\sum_{j=1}^{\infty} |f(j)| < \infty.$$

Similarly we will write $R(t) \in \ell_1$, where $R(t)$ is a square matrix, if $f(t) = \|R(t)\| \in \ell_1$ and $\|\cdot\|$ is some matrix norm. Let Σ denote the set of all square matrices whose entries are trigonometric polynomials

$$p(t) = \sum_{j=1}^{N} p_j e^{\mathrm{i}\lambda_j t},$$

where p_j are complex numbers and λ_j are real numbers. The subset of Σ, consisting of matrices with zero mean value, i.e.,

$$\mathrm{M}[A(t)] = \lim_{T\to\infty} \frac{1}{T} \sum_{k=0}^{T-1} A(k) = 0,$$

will be denoted by Σ_0. Finally, symbol Δ will stand for the forward difference operator

$$\Delta f(t) = f(t+1) - f(t).$$

We return now to system (2.1). Let the following conditions be satisfied.

(1°) $v_1(t) \to 0, v_2(t) \to 0, \dots, v_n(t) \to 0$ as $t \to \infty$.

(2°) $\Delta v_1(t), \Delta v_2(t), \dots, \Delta v_n(t) \in \ell_1$.

(3°) $v_i(t)v_j(t) \in \ell_1$ for each pair of indices (i, j).

(4°) Matrices $A_1(t), \dots, A_n(t)$ belong to Σ.

(5°) $R(t) \in \ell_1$.

Let assumptions (1°)–(5°) hold. Then the following theorem is valid.

Theorem 2.1. *System* (2.1) *for sufficiently large* t *by the change of variable*

$$x(t) = \Big[I + \sum_{i=1}^{n} Y_i(t)v_i(t)\Big]y(t), \tag{2.2}$$

where matrices $Y_1(t), \dots, Y_n(t)$ *belong to* Σ_0*, is reduced to the form*

$$y(t+1) = \Big(I + \sum_{i=1}^{n} A_i v_i(t) + R_1(t)\Big)y(t), \quad t \in \mathbb{N}, \tag{2.3}$$

with constant matrices $A_1, \dots, A_n$ *and with* $R_1(t) \in \ell_1$.

Proof. We substitute (2.2) into (2.1) and use (2.3). We obtain

$$\Big[I + \sum_{i=1}^{n} Y_i(t+1)v_i(t+1)\Big]\Big[I + \sum_{i=1}^{n} A_i v_i(t) + R_1(t)\Big]y(t)$$
$$= \Big[I + \sum_{i=1}^{n} A_i(t)v_i(t) + R(t)\Big]\Big[I + \sum_{i=1}^{n} Y_i(t)v_i(t)\Big]y(t). \quad (2.4)$$

Substituting

$$v_i(t+1) = \Delta v_i(t) + v_i(t), \qquad i = 1, \ldots, n, \quad (2.5)$$

we collect terms of the class ℓ_1 on both sides in (2.4). We have

$$\Big[\sum_{i=1}^{n} Y_i(t+1)\Delta v_i(t)\Big]\Big[I + \sum_{i=1}^{n} A_i v_i(t)\Big] + \sum_{i,j=1}^{n} Y_i(t+1)A_j v_i(t)v_j(t)$$
$$+ \Big[I + \sum_{i=1}^{n} Y_i(t+1)v_i(t+1)\Big]R_1(t) = R(t)\Big[I + \sum_{i=1}^{n} Y_i(t)v_i(t)\Big]$$
$$+ \sum_{i,j=1}^{n} A_i(t)Y_j(t)v_i(t)v_j(t). \quad (2.6)$$

Since, by virtue of condition (1°) and the boundedness of the matrices $Y_1(t)$, …, $Y_n(t)$ (which will be defined below), the matrix

$$I + \sum_{i=1}^{n} Y_i(t+1)v_i(t+1)$$

is invertible and the inverse is bounded for $t \geq t_0$, it follows that the matrix $R_1(t)$ can be expressed from (2.6) and obviously belongs to the class ℓ_1. Now we match the terms containing $v_1(t), \ldots, v_n(t)$, on both sides in (2.4), taking into account (2.5). By matching the free terms, we obtain the trivial identity

$$I = I.$$

By matching the coefficients of $v_i(t)$ in (2.4), we obtain the following inhomogeneous difference equations for the matrices $Y_i(t)$ and A_i:

$$Y_i(t+1) - Y_i(t) = A_i(t) - A_i, \qquad i = 1, \ldots, n. \quad (2.7)$$

We find the matrix A_i from the condition that the mean value of the right-hand side in (2.7) is zero; namely,

$$A_i = \mathrm{M}\big[A_i(t)\big]. \quad (2.8)$$

We seek a solution $Y_i(t)$ of equation (2.7) in the form

$$Y_i(t) = \sum_{j=1}^{N} \beta_j^{(i)} e^{\mathrm{i}\lambda_j t}, \quad \lambda_j \neq 2\pi M, \quad M \in \mathbb{Z}, \tag{2.9}$$

where $\beta_j^{(i)}$ are constant matrices to be determined. We substitute the expression (2.9) into (2.7) and use the fact that the entries of the matrices $A_i(t)$ are trigonometric polynomials; then we match the coefficients of like exponentials. We obtain the matrix equations:

$$\beta_j^{(i)} e^{\mathrm{i}\lambda_j} I - I\beta_j^{(i)} = \Upsilon_j^{(i)}, \quad j = 1, \ldots, N.$$

Hence,

$$\beta_j^{(i)} = \frac{1}{e^{\mathrm{i}\lambda_j} - 1} \Upsilon_j^{(i)}.$$

The proof is complete. □

System (2.3) is much simpler than system (2.1) since it does not contain oscillating coefficients. To get the asymptotic representation for the fundamental matrix of system (2.3) we can apply the discrete analog of Levinson's fundamental theorem. In what follows we will construct the asymptotics for solutions of system

$$y(t+1) = \Big(I + A_1 t^{-\alpha_1} + R_1(t)\Big) y(t), \qquad t \in \mathbb{N}, \tag{2.10}$$

where A_1 is a constant matrix, $0 < \alpha_1 \leq 1$ and $R_1(t) \in \ell_1$. Let us now state the asymptotic theorem, that follows from the results of paper [1].

Theorem 2.2. *Suppose that A_1 is a diagonizable matrix. Then the fundamental matrix $\Phi(t)$ of system* (2.10) *has the following asymptotics as $t \to \infty$:*

$$\Phi(t) = \Big[P + o(1)\Big] \prod_{l=t_1}^{t-1} \Lambda(l), \quad t > t_1, \quad t \in \mathbb{N},$$

where P is a constant matrix, whose columns are the eigenvectors of the matrix A_1, corresponding to the eigenvalues $\lambda_1, \ldots, \lambda_m$, and

$$\Lambda(t) = \operatorname{diag}\left(1 + \lambda_1 t^{-\alpha_1}, \ldots, 1 + \lambda_m t^{-\alpha_1}\right).$$

3 Dynamic Equation and Time Scale Perturbation

Consider the following second order dynamic equation:

$$x^{\sigma\sigma} - 2(\cos\omega)x^{\sigma} + \left(1 + \frac{a\sin(\lambda\omega t)}{t^{\rho}}\right) x(t) = 0, \tag{3.1}$$

where t belongs to $\mathbb{T}_1$ or $\mathbb{T}_2$. We choose parameters in (3.1) as follows

$$0<\omega<\pi, \quad \frac{1}{2}<\rho\le 1, \quad a,\lambda\in\mathbb{R}, \quad \lambda\neq 0. \tag{3.2}$$

Evidently, we can rewrite equation (3.1) in the form of the system (1.4). Let time scale $\mathbb{T}_1$ be the set of all natural numbers $\mathbb{N}$, i.e.,

$$\mathbb{T}_1=\{t_n=n, \quad n\in\mathbb{N}\}. \tag{3.3}$$

We define time scale $\mathbb{T}_2$ in the following way:

$$\mathbb{T}_2=\Big\{\tau_n=n+\frac{\alpha\cos(\mu\omega n)}{\lambda\omega n^\beta}, \quad n\in\mathbb{N}\Big\}, \tag{3.4}$$

where $0<\beta\le 1$ and $\alpha,\mu\in\mathbb{R}$, $\alpha\neq 0$. It is clear that assumption (1.6) is fulfilled for every choice of parameters. Moreover, without loss of generality, we may assume that $\tau_{n+1}>\tau_n$ for all $n\in\mathbb{N}$. Otherwise we can simply replace the first points of $\mathbb{T}_2$ by $\tau_n=n$.

3.1 The Asymptotic Analysis of Equation (3.1) on Time Scale $\mathbb{T}_1$

Equation (3.1) on time scale $\mathbb{T}_1$ is actually a discrete adiabatic oscillator of the form

$$x(n+2)-2(\cos\omega)x(n+1)+\left(1+\frac{a\sin(\lambda\omega n)}{n^\rho}\right)x(n)=0. \tag{3.5}$$

We write equation (3.5) as a system

$$\begin{pmatrix}x(n+1)\\ y(n+1)\end{pmatrix}=\left[\begin{pmatrix}0 & 1\\ -1 & 2\cos\omega\end{pmatrix}-q(n)\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\right]\begin{pmatrix}x(n)\\ y(n)\end{pmatrix}, \tag{3.6}$$

where

$$q(n)=\frac{a\sin(\lambda\omega n)}{n^\rho}.$$

Using the change of variable

$$\begin{pmatrix}x(n)\\ y(n)\end{pmatrix}=Cz(n), \qquad C=\begin{pmatrix}1 & 1\\ e^{i\omega} & e^{-i\omega}\end{pmatrix}, \qquad z(n)=\begin{pmatrix}z^{(1)}(n)\\ z^{(2)}(n)\end{pmatrix},$$

we obtain system

$$z(n+1)=\left[\operatorname{diag}\big(e^{i\omega},e^{-i\omega}\big)-\frac{q(n)}{e^{i\omega}-e^{-i\omega}}\begin{pmatrix}1 & 1\\ -1 & -1\end{pmatrix}\right]z(n). \tag{3.7}$$

Using transformation

$$z(n)=\operatorname{diag}\big(e^{i\omega n},e^{-i\omega n}\big)z_1(n),$$

in (3.7), we obtain the system

$$z_1(n+1) = \Big[I + A_1(n)n^{-\rho}\Big] z_1(n), \tag{3.8}$$

where

$$A_1(n) = \frac{a}{4\sin\omega}\begin{pmatrix} e^{-\mathrm{i}\omega}\big(e^{\mathrm{i}\lambda\omega n} - e^{-\mathrm{i}\lambda\omega n}\big) & e^{-\mathrm{i}\omega}\big(e^{\mathrm{i}(\lambda-2)\omega n} - e^{-\mathrm{i}(\lambda+2)\omega n}\big) \\ -e^{\mathrm{i}\omega}\big(e^{\mathrm{i}(\lambda+2)\omega n} - e^{-\mathrm{i}(\lambda-2)\omega n}\big) & -e^{\mathrm{i}\omega}\big(e^{\mathrm{i}\lambda\omega n} - e^{-\mathrm{i}\lambda\omega n}\big) \end{pmatrix}. \tag{3.9}$$

By the change of variable (see Theorem 2.1)

$$z_1(n) = \Big[I + Y_1(n)n^{-\rho}\Big] z_2(n),$$

we bring system (3.8) to the averaged form

$$z_2(n+1) = \Big[I + A_1 n^{-\rho} + O\big(n^{-2\rho}\big)\Big] z_2(n), \tag{3.10}$$

where $A_1 = \mathrm{M}[A_1(n)]$. From now on we assume that

$$(\lambda \pm 2)\omega \neq 0 \mod 2\pi. \tag{3.11}$$

This yields that $A_1 = 0$. Since $\rho > 1/2$ we can apply Theorem 2.2 to construct the asymptotic formulas for solutions of system (3.10). We leave it to the reader to verify that linearly independent solutions of (3.5) have the following asymptotics as $n \to \infty$:

$$x_{1,2}(n) = e^{\pm\mathrm{i}\omega n}\big(1 + o(1)\big). \tag{3.12}$$

Thus, under condition (3.11) all solutions of equation (3.1) are bounded.

3.2 The Asymptotic Analysis of Equation (3.1) on Time Scale $\mathbb{T}_2$

In this case equation (3.1) on time scale $\mathbb{T}_2$ takes the following form:

$$x(\tau_{n+2}) - 2(\cos\omega)x(\tau_{n+1}) + \left(1 + \frac{a\sin(\lambda\omega\tau_n)}{\tau_n^{\rho}}\right)x(\tau_n) = 0, \qquad n \in \mathbb{N}. \tag{3.13}$$

Writing $\tilde{x}(n) = x(\tau_n)$ we get the difference equation

$$\tilde{x}(n+2) - 2(\cos\omega)\tilde{x}(n+1) + (1 + q(n))\tilde{x}(n) = 0, \qquad n \in \mathbb{N}, \tag{3.14}$$

where

$$q(n) = an^{-\rho}\sin\left(\lambda\omega n + \frac{\alpha\cos(\mu\omega n)}{n^{\beta}}\right) + O\Big(n^{-(\rho+\beta+1)}\Big), \qquad n \to \infty. \tag{3.15}$$

The task is now to show that equation (3.14) with function $q(n)$ of the form (3.15) may have unbounded solutions if the parameters α, β, μ are appropriately chosen.

We note that

$$q(n) = an^{-\rho}\left[\sin(\lambda\omega n) + \frac{\alpha\cos(\lambda\omega n)\cos(\mu\omega n)}{n^{\beta}}\right] + O\Big(n^{-(\rho+2\beta)}\Big) + O\Big(n^{-(\rho+\beta+1)}\Big)$$

as $n \to \infty$. We assume further that

$$1 - 2\beta < \rho \leq 1 - \beta, \tag{3.16}$$

and therefore

$$q(n) = an^{-\rho}\left[\sin(\lambda\omega n) + \frac{\alpha\cos(\lambda\omega n)\cos(\mu\omega n)}{n^{\beta}}\right] + \{\ell_1\},$$

where $\{\ell_1\}$ denotes the terms from ℓ_1.

As in the previous case we reduce equation (3.14) to the following system of difference equations:

$$z_1(n+1) = \Big[I + A_1(n)n^{-\rho} + A_2(n)n^{-(\rho+\beta)} + R(n)\Big]z_1(n), \tag{3.17}$$

where $R(n) \in \ell_1$. Here, matrix $A_1(n)$ is defined by formula (3.9) and

$$\begin{aligned} A_2(n) = \frac{\mathrm{i}a\alpha}{8\sin\omega}\Bigg[&\begin{pmatrix} e^{-\mathrm{i}\omega}e^{\mathrm{i}(\lambda+\mu)\omega n} & e^{-\mathrm{i}\omega}e^{\mathrm{i}(\lambda+\mu-2)\omega n} \\ -e^{\mathrm{i}\omega}e^{\mathrm{i}(\lambda+\mu+2)\omega n} & -e^{\mathrm{i}\omega}e^{\mathrm{i}(\lambda+\mu)\omega n} \end{pmatrix} \\ &+ \begin{pmatrix} e^{-\mathrm{i}\omega}e^{\mathrm{i}(\lambda-\mu)\omega n} & e^{-\mathrm{i}\omega}e^{\mathrm{i}(\lambda-\mu-2)\omega n} \\ -e^{\mathrm{i}\omega}e^{\mathrm{i}(\lambda-\mu+2)\omega n} & -e^{\mathrm{i}\omega}e^{\mathrm{i}(\lambda-\mu)\omega n} \end{pmatrix} \\ &+ \begin{pmatrix} e^{-\mathrm{i}\omega}e^{-\mathrm{i}(\lambda-\mu)\omega n} & e^{-\mathrm{i}\omega}e^{-\mathrm{i}(\lambda-\mu+2)\omega n} \\ -e^{\mathrm{i}\omega}e^{-\mathrm{i}(\lambda-\mu-2)\omega n} & -e^{\mathrm{i}\omega}e^{-\mathrm{i}(\lambda-\mu)\omega n} \end{pmatrix} \\ &+ \begin{pmatrix} e^{-\mathrm{i}\omega}e^{-\mathrm{i}(\lambda+\mu)\omega n} & e^{-\mathrm{i}\omega}e^{-\mathrm{i}(\lambda+\mu+2)\omega n} \\ -e^{\mathrm{i}\omega}e^{-\mathrm{i}(\lambda+\mu-2)\omega n} & -e^{\mathrm{i}\omega}e^{-\mathrm{i}(\lambda+\mu)\omega n} \end{pmatrix}\Bigg]. \end{aligned} \tag{3.18}$$

Using Theorem 2.1, we make the averaging change of variable in system (3.17)

$$z_1(n) = \big[I + Y_1(n)n^{-\rho} + Y_2(n)n^{-(\rho+\beta)}\big]z_2(n).$$

Taking into account (3.2), (3.11) and (3.16) we get system

$$z_2(n+1) = \Big[I + A_2 n^{-(\rho+\beta)} + R_1(n)\Big]z_2(n), \tag{3.19}$$

where $A_2 = \mathrm{M}\big[A_2(n)\big]$ and $R_1(n) \in \ell_1$.

The following cases should be considered further:

- **1.** $(\lambda-\mu)\omega = 0 \mod 2\pi$ (or $(\lambda+\mu)\omega = 0 \mod 2\pi$).

It follows from the definition of time scale $\mathbb{T}_2$ (see formula (3.4)) that we should consider only the case $\lambda = \mu$.

- **1.1.** $2\lambda\omega \neq 0 \mod 2\pi$, $2(\lambda \pm 1)\omega \neq 0 \mod 2\pi$.

We have

$$A_2 = \frac{\mathrm{i}a\alpha}{4\sin\omega}\begin{pmatrix} e^{-\mathrm{i}\omega} & 0 \\ 0 & -e^{\mathrm{i}\omega}\end{pmatrix}.$$

We apply Theorem 2.2 to construct the asymptotics for the fundamental matrix $Z_2(n)$ of system (3.19) as $n \to \infty$. We obtain

$$Z_2(n) = \big[I + o(1)\big]\operatorname{diag}\big(\exp\{\xi(n) + \mathrm{i}\varphi(n)\}, \exp\{\xi(n) - \mathrm{i}\varphi(n)\}\big), \tag{3.20}$$

where

$$\xi(n) = \frac{a\alpha}{4}\,\theta(n), \quad \varphi(n) = \frac{a\alpha\cot\omega}{4}\,\theta(n), \quad \theta(n) = \begin{cases} \ln n, & \rho+\beta = 1, \\ \dfrac{n^{1-(\rho+\beta)}}{1-(\rho+\beta)}, & \rho+\beta < 1. \end{cases} \tag{3.21}$$

To obtain the asymptotic representation (3.20), we used the asymptotic formula

$$\sum_{k=n_0}^{n-1} k^{-(\rho+\beta)} = \theta(n) + c(n_0) + o(1), \qquad n \to \infty,$$

where $c(n_0)$ is some constant term. If we return to equation (3.14), we get the following asymptotics for the linearly independent solutions:

$$\tilde{x}_{1,2}(n) = \exp\{\xi(n)\}\exp\{\pm\mathrm{i}(\omega n + \varphi(n))\}\big(1 + o(1)\big), \qquad n \to \infty. \tag{3.22}$$

So, we conclude that in the considered case equation (3.13) has unbounded solutions, if

$$a\alpha > 0. \tag{3.23}$$

- **1.2.** $2\lambda\omega = 0 \mod 2\pi$.

In this case

$$A_2 = \frac{\mathrm{i}a\alpha}{2\sin\omega}\begin{pmatrix} e^{-\mathrm{i}\omega} & 0 \\ 0 & -e^{\mathrm{i}\omega}\end{pmatrix}.$$

We, thus, get the asymptotic formula (3.20) for the fundamental matrix $Z_2(n)$ of system (3.19), where instead of functions $\xi(n)$ and $\varphi(n)$ (these functions are defined in (3.21)) we should write $2\xi(n)$ and $2\varphi(n)$ respectively. Hence, for the fundamental solutions of (3.14) we obtain the asymptotic formulas (3.22), where the analogous changes should be made.

- **1.3.** $2\lambda\omega \neq 0 \mod 2\pi$, $2(\lambda-1)\omega = 0 \mod 2\pi$, $2(\lambda+1)\omega \neq 0 \mod 2\pi$.

We have

$$A_2 = \frac{\mathrm{i}a\alpha}{8\sin\omega}\begin{pmatrix} 2e^{-\mathrm{i}\omega} & e^{-\mathrm{i}\omega} \\ -e^{\mathrm{i}\omega} & -2e^{\mathrm{i}\omega} \end{pmatrix}.$$

The eigenvalues $\frac{\mathrm{i}a\alpha}{8\sin\omega}\,\mu_{1,2}$ of matrix A_2 are defined from the characteristic equation:

$$\mu^2 + 4\mathrm{i}\sin\omega\mu - 3 = 0.$$

The discriminant of this equation has the form

$$D = 12 - 16\sin^2\omega.$$

So the asymptotic representation of the fundamental matrix $Z_2(n)$ will be different in each of the following cases:

- **1.3.1.** $D > 0,\quad \Big((0 < \omega < \frac{\pi}{3})\bigcup(\frac{2\pi}{3} < \omega < \pi)\Big).$
  We get the asymptotic formula

$$Z_2(n) = \big[P + o(1)\big]\,\mathrm{diag}\big(\exp\{\xi(n) + \mathrm{i}\varphi(n)\}, \exp\{\xi(n) - \mathrm{i}\varphi(n)\}\big), \qquad n \to \infty,$$

for the fundamental matrix $Z_2(n)$ of system (3.19). Here, P is a constant matrix, whose columns are the eigenvectors of the matrix A_2; the functions $\xi(n)$ and $\varphi(n)$ are defined by formulas:

$$\xi(n) = \frac{a\alpha}{4}\theta(n), \qquad \varphi(n) = \frac{a\alpha}{8\sin\omega}\sqrt{(3 - 4\sin^2\omega)}\,\theta(n).$$

Finally, function $\theta(n)$ is described by formula (3.21). We will not write out the asymptotic formulas for solutions of (3.14) in this case. Obviously, all nonzero solutions of this equation are unbounded provided the inequality (3.23) holds.

- $D < 0,\quad \left(\frac{\pi}{3} < \omega < \frac{2\pi}{3}\right).$
  In this situation we have the following asymptotics for the fundamental matrix $Z_2(n)$ as $n \to \infty$:

$$Z_2(n) = \big[C + o(1)\big]\,\mathrm{diag}\big(\exp\{\xi(n) + \varphi(n)\}, \exp\{\xi(n) - \varphi(n)\}\big),$$

where

$$\xi(n) = \frac{a\alpha}{4}\,\theta(n), \qquad \varphi(n) = \frac{a\alpha}{8\sin\omega}\sqrt{(4\sin^2\omega - 3)}\,\theta(n).$$

So we remark that under condition (3.23) equation (3.14) has unbounded solutions in all the cases considered above.

- **1.4.** $2\lambda\omega \neq 0 \mod 2\pi$, $2(\lambda - 1)\omega = 0 \mod 2\pi$, $2(\lambda + 1)\omega = 0 \mod 2\pi$.

Clearly, this situation occurs only if $\omega = \dfrac{\pi}{2}$. In this case

$$A_2 = \frac{\mathrm{i}a\alpha}{8\sin\omega}\begin{pmatrix} 2e^{-\mathrm{i}\omega} & 2e^{-\mathrm{i}\omega} \\ -2e^{\mathrm{i}\omega} & -2e^{\mathrm{i}\omega} \end{pmatrix} = \frac{a\alpha}{4}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

The eigenvalues of the matrix A_2 are:

$$\mu_1 = \frac{a\alpha}{2}, \qquad \mu_2 = 0.$$

The fundamental matrix $Z_2(n)$ has the following asymptotics as $n \to \infty$:

$$Z_2(n) = \big[P + o(1)\big]\,\mathrm{diag}\big(\exp\{\xi(n)\}, 1\big),$$

where

$$\xi(n) = \frac{a\alpha}{2}\,\theta(n), \qquad P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{3.24}$$

So this is again the case when equation (3.14) has unbounded solutions, provided inequality (3.23) holds.

- **2.** $(\lambda - \mu - 2)\omega = 0 \mod 2\pi$ (or $(\lambda + \mu + 2)\omega = 0 \mod 2\pi$).

Note, that we should only consider the case $\mu = \lambda - 2$. We may reduce all other situations to this case. As an example, we will now show how to get $\mu = \lambda - 2$ from $\lambda = -\mu - 2$. We have

$$an^{-\rho}\sin\left(\lambda\omega n + \frac{\alpha\cos(\mu\omega n)}{n^\beta}\right) = an^{-\rho}\sin\left((-\mu-2)\omega n + \frac{\alpha\cos(\mu\omega n)}{n^\beta}\right) =$$
$$= -an^{-\rho}\sin\left((\mu+2)\omega n + \frac{(-\alpha)\cos(\mu\omega n)}{n^\beta}\right) = \tilde{a}n^{-\rho}\sin\left(\tilde{\lambda}\omega n + \frac{\tilde{\alpha}\cos(\mu\omega n)}{n^\beta}\right),$$

where $\tilde{a} = -a$, $\tilde{\alpha} = -\alpha$ and $\mu = \tilde{\lambda} - 2$.

- **2.1.** $2\lambda\omega \neq 0 \mod 2\pi$, $2(\lambda-1)\omega \neq 0 \mod 2\pi$, $2(\lambda-2)\omega \neq 0 \mod 2\pi$.
- **2.1.1.** $\omega \neq \dfrac{\pi}{2}$.

We have

$$A_2 = \frac{\mathrm{i}a\alpha}{8\sin\omega}\begin{pmatrix} 0 & e^{-\mathrm{i}\omega} \\ -e^{\mathrm{i}\omega} & 0 \end{pmatrix}.$$

We get the following expression for the eigenvalues of the matrix A_2: $\mu_{1,2} = \pm\dfrac{a\alpha}{8\sin\omega}$. The fundamental matrix $Z_2(n)$ of system (3.19) has the following asymptotic representation as $n \to \infty$:

$$Z_2(n) = \big[P + o(1)\big]\,\mathrm{diag}\big(\exp\{\xi(n)\}, \exp\{-\xi(n)\}\big), \tag{3.25}$$

where

$$\xi(n) = \frac{a\alpha}{8\sin\omega}\,\theta(n). \tag{3.26}$$

Therefore, this is the case when equation (3.14) has unbounded solutions for all nonzero values of parameters a and α.

- **2.1.2.** $\omega = \dfrac{\pi}{2}$.
  We obtain
$$A_2 = \frac{\mathrm{i}a\alpha}{4\sin\omega}\begin{pmatrix} 0 & e^{-\mathrm{i}\omega} \\ -e^{\mathrm{i}\omega} & 0 \end{pmatrix} = \frac{a\alpha}{4}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We, thus, get the asymptotic formula (3.25) for the fundamental matrix $Z_2(n)$, where we should write $2\xi(n)$ instead of $\xi(n)$ and the function $\xi(n)$ is defined in (3.26). The matrix P has the form (3.24).

- **2.2.** $2\lambda\omega = 0 \mod 2\pi$, $2(\lambda-2)\omega \neq 0 \mod 2\pi$.
- **2.2.1.** $\omega \neq \dfrac{\pi}{2}$.
  We get the same matrix A_2 as in the case **2.1.2.**
- **2.2.2.** $\omega = \dfrac{\pi}{2}$.
  We obtain
$$A_2 = \frac{3\mathrm{i}a\alpha}{8\sin\omega}\begin{pmatrix} 0 & e^{-\mathrm{i}\omega} \\ -e^{\mathrm{i}\omega} & 0 \end{pmatrix} = \frac{3a\alpha}{8}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Hence, we get the asymptotic formula (3.25) for the fundamental matrix $Z_2(n)$, where we should write $3\xi(n)$ instead of $\xi(n)$ and the function $\xi(n)$ is defined in (3.26).

- **2.3.** $2\lambda\omega = 0 \mod 2\pi$, $2(\lambda-2)\omega = 0 \mod 2\pi$.
  It is easily seen that this situation occurs only if $\omega = \dfrac{\pi}{2}$. We have
$$A_2 = \frac{\mathrm{i}a\alpha}{2\sin\omega}\begin{pmatrix} 0 & e^{-\mathrm{i}\omega} \\ -e^{\mathrm{i}\omega} & 0 \end{pmatrix} = \frac{a\alpha}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Again we get the asymptotics (3.25) for the fundamental matrix $Z_2(n)$, where we should write $4\xi(n)$ instead of $\xi(n)$ and the latter one is defined in (3.26).

- **2.4.** $2\lambda\omega \neq 0 \mod 2\pi$, $2(\lambda-1)\omega = 0 \mod 2\pi$, $2(\lambda-2)\omega \neq 0 \mod 2\pi$.
- **2.4.1.** $\omega \neq \dfrac{\pi}{2}$.
  We get the same matrix A_2 as in the case **1.3.**
- **2.4.2.** $\omega = \dfrac{\pi}{2}$.
  We get the same matrix A_2 as in the case **1.4.**
- **2.5.** $2\lambda\omega \neq 0 \mod 2\pi$, $2(\lambda-1)\omega \neq 0 \mod 2\pi$, $2(\lambda-2)\omega = 0 \mod 2\pi$.
- **2.5.1.** $\omega \neq \dfrac{\pi}{2}$.
  We get the same matrix A_2 as in the case **2.1.2.**
- **2.5.2.** $\omega = \dfrac{\pi}{2}$.
  We get the same matrix A_2 as in the case **2.2.2.**
- **3.** $(\lambda-\mu+2)\omega = 0 \mod 2\pi$ (or $(\lambda+\mu-2)\omega = 0 \mod 2\pi$).

We remark that we should consider only the case $\lambda = \mu - 2$. It easy to show that this case may be reduced to the situation **2.** This is done as follows:

$$an^{-\rho}\sin\left(\lambda\omega n + \frac{\alpha\cos(\mu\omega n)}{n^\beta}\right) = an^{-\rho}\sin\left((\mu-2)\omega n + \frac{\alpha\cos(\mu\omega n)}{n^\beta}\right) =$$
$$= -an^{-\rho}\sin\left((-\mu+2)\omega n + \frac{(-\alpha)\cos(-\mu\omega n)}{n^\beta}\right) = \tilde{a}n^{-\rho}\sin\left(\tilde{\lambda}\omega n + \frac{\tilde{\alpha}\cos(\tilde{\mu}\omega n)}{n^\beta}\right),$$

where $\tilde{a} = -a$, $\tilde{\alpha} = -\alpha$, $\tilde{\lambda} = -\mu + 2$ and $\tilde{\mu} = -\mu$. Thus, we get the resonance $\tilde{\lambda} = \tilde{\mu} + 2$, i.e., the case **2.**

Finally we note that if none of the cases **1–3** occurs, then the matrix A_2 is a zero matrix and the linearly independent solutions of (3.14) have the asymptotics (3.12) provided the conditions (3.2), (3.11) and (3.16) hold. Hence, in this nonresonant case all solutions of equation (3.13) are bounded.

4 Conclusion

In the equations considered above the parametric resonance phenomenon occurs if the parameters of the dynamic equation are appropriately chosen. Unlike the standard definition of this phenomenon, in our example the cause of the parametric resonance is not the perturbation of the parameters but the perturbation of the time scale on which the dynamic equation is defined.

Finally, we remark that in the examples considered above the small perturbation of the time scale causes the qualitative change in the dynamics of solutions. Therefore, when we study the dynamic equations on time scales the structure of the time scale may play the significant role.

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