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The Stability of the Stochastic Open-Economy Multiplier-Accelerator and the Stationarity of Oscillating Processes

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Abstract

In this paper we reformulate Samuelson's deterministic, closed-economy multiplier-accelerator model as a nonhomogeneous second-order recurrence relation in an open economy. We first derive a stochastic open-economy multiplier-accelerator model and show the existence and stability of the time path of the open-economy accelerator-multiplier. Most importantly, this study establishes a link between the stability of the multiplier-accelerator process and the stationarity of the system and proves that the oscillations of the multiplier-accelerator process are stable if the variables in the second-order recurrence equation are stationary. In order to give some empirical content to our model, we have estimated the structural equations using U.S. quarterly data (1947:I ~ 2019:I) from the perspective of cointegration. We have found that the U.S. open-economy multiplier-accelerator exhibits a dampening oscillating process when the variables in the system are of the same order of integration and are cointegrated among them.

Keywords: Multiplier-accelerator model, nonhomogeneous second-order recurrence relation, time path stability, oscillating process, unit roots, cointegration, stationarity

JEL Classification: C32, C62, E3

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1. INTRODUCTION

The introduction of the multiplier-accelerator model by Samuelson in 1939 brought business cycle theory into prominence. The principle of acceleration and the Keynesian multiplier were incorporated into a discrete-time model that generates endogenous business cycles. This gave an early look at the causal relationship for contractions and expansions of an economy without the need to introduce exogenous stimulating factors. Since then, researchers have made great strides in the expansion of the multiplier-accelerator model by incorporating policy variables into the model whilst keeping the structural assumptions minimal.

Within the tradition of the original multiplier-accelerator model, however, it has been common in the literature to assume a two- or three-sector model in a deterministic setting as opposed to a four-sector model in a stochastic framework with the foreign sector incorporated. With the U.S. economy being increasingly interrelated with the rest of the world, the investigation of the multiplier-accelerator model with foreign trade excluded renders the system of the national income identity incomplete.

Furthermore, the existing literature lacks some empirical content on the link between the stability of the oscillating process of the multiplier-accelerator interaction and the stationarity of variables in the system. We have witnessed sophisticated advances in time series analysis of business cycles over the past several decades, which has made it necessary to recast the multiplier-accelerator model in light of new developments in time series analysis of business cycles.

The purpose of this paper is two-fold: First, this paper aims to derive a stochastic open-economy multiplier-accelerator model and to show the existence and stability of the oscillating time path of the system. To the best of our knowledge, this study is the first attempt to derive a systematic link between the stability of the multiplier-accelerator time path as an oscillating process and the stationarity of the system.

This paper also aims to provide empirical implications for the oscillating process of the multiplier-accelerator within the context of cointegration. We estimate the structural equations to investigate whether the business cycles of the U.S. economy that exhibit a dampening oscillating process are consistent with a cointegrating relation among variables. When the variables in the structural equations are cointegrated, this indicates that there is a long-run equilibrium relationship among the variables. Although in the short run there could be an equilibrium error in the system, which could be a source of short-run oscillations, cointegration among variables is likely to lead to the convergence of a time path with dampening oscillations in the long run.

In order to estimate the model, we have used U.S. quarterly data spanning from 1947:Q1 to 2019:Q1. We have demonstrated that the conditions for the existence, stability, and stationarity of the time path for the U.S. economy are empirically supported: the time path of the multiplier-accelerator process was oscillating and converged over time. Interestingly, the convergence conditions are satisfied when the variables in the system are of the same order of integration and are cointegrated among them. Our open-economy model of the multiplier-accelerator further indicates that the size of the accelerator coefficient becomes smaller in an economy where the variables in the system are cointegrated. This finding implies that the amplitude of an oscillating business cycle is more or less mitigated in an economy characterized with cointegration.

The paper is organized as follows: Chapter 2 reviews the literature on the multiplier-accelerator models. In chapter 3, we address the mathematical foundations of deterministic and stochastic recursive relations. Chapter 4 demonstrates the existence of the multiplier-accelerator time path as an oscillating process and derives sufficient conditions for the stability and stationarity of the relevant time path in an open-economy. Chapter 5 discusses the estimation of the multiplier-accelerator model in the context of unit roots and cointegration and presents the general solution of the model that is consistent with our empirical estimation results. Chapter 6 provides concluding remarks.

2. A REVIEW OF THE LITERATURE

Modern business cycle theory has its origin in Samuelson's (1939) multiplier-accelerator model. By introducing the interaction between the accelerator and the Keynesian multiplier, Samuelson was able to derive endogenous fluctuations in a deterministic discrete-time three-sector economy. He formulated the investment function primarily as an interaction of the capital-output ratio with the first difference of consumption to achieve the desired level of capital stock. In the Samuelson model, economic fluctuations are associated with the deterministic nonhomogeneous second-order linear recurrence relation with constant coefficients yielding complex conjugate roots for its corresponding characteristic polynomial. The stability of the time path for the business cycles is contingent on the sufficient condition that the modulus of the complex conjugate roots be less than unity.

Samuelson's multiplier-accelerator model has been further refined and extended in diverse ways with minimal assumptions employed and the adaptability of the model tailored to policy tools. In the literature concerning the multiplier-accelerator interactions, there are two main differing assumptions made on the investment function. This reformulation was due to discrepancies in the upper and lower bounds of possible investment behavior.

Hicks (1950), in his study of the multiplier-accelerator model, deduced that the accelerator-induced investment must have upper and lower bounds. These upper and lower bounds are primarily due to the inconsistencies found during the expansion and contraction phases of the endogenous oscillations. In the disinvestment phase, capital is used much faster than the depreciation rate, while in the investment phase capital is used at abnormally high rates. Hicks never materialized his proposed floor and ceiling functions. A later study by Puu et al. (2005), aiming to bridge growth theory and business cycle theory, formulated the maximum function on investment and linked the minimum function to national income.

In his reformulation of the multiplier-accelerator model, Hicks also postulated the investment function as a mechanism between the accelerator and the first difference of the lag of national income, as opposed to the capital-output ratio with the first difference of consumption. This yielded inhibited interactions between the marginal propensity to consume with the capital-output ratio. The resulting model is a second order nonhomogeneous deterministic linear recurrence relation with constant coefficients and endogenous cycles, but the sufficient conditions for stability differ from the original Samuelson model. Thus, we have encountered two different versions of the multiplier-accelerator model based on the use of Samuelson's or Hick's assumptions.

With a basic framework solidified in a three-sector economy, some extensions encompassing policy tools have become increasingly important to give a more complete picture of national income dynamics. One of such efforts to include monetary policy was driven by a motivation to dispel the argument against discretionary monetary policy. For example, Lovell and Prescott (1968) introduced money and interest rates into the multiplier-accelerator model of Samuelson with Hicks' (1937) static IS-LM apparatus. Interest rates were proposed to be negatively related to investment, and a portion of the first difference on the lag of national income is denoted as the policy coefficient that corresponds to fluctuations in money. They argue that it is quite possible that the economy would be subject to even greater instability if discretionary monetary policy were replaced by the simple rule of keeping the money supply at a constant rate in both boom and recession phases.

Kendrick and Shoukry (2013) isolated the effects of fiscal policy on the multiplier-accelerator model and found quarterly fiscal adjustments to be optimal in the stabilization of endogenous cycles. This was carried out through implementation of Hick's assumptions and specifications regarding taxes as well as government spending. The model was then converted to state space in which a Monte Carlo experiment was employed. The incorporation of fiscal or monetary policy drastically changes the stability conditions of the multiplier-accelerator model.

Karpetis and Varelas (2012) developed a discrete-time multiplier-accelerator model in which the money market and a balanced government budget constraint are incorporated into Samuelson's model. The modified model has proved to be less stable, and the evolution of income around its equilibrium is more likely to exhibit a sinusoidal way of movement. They have concluded that the inclusion of both policy tools into the Samuelson model leads the oscillations in the time path of national income to be less stable

Bohner et al. (2010) derived a linear second-order dynamic equation which describes the multiplier-accelerator model on time scales. They provided the general form of the dynamic equation, which includes both taxes and foreign trade and examined four special cases of the general multiplier-accelerator model: (1) Samuelson's basic multiplier-accelerator model; (2) Hicks' extension of the basic multiplier-accelerator model; (3) an extended model with taxes; and (4) an extended model with foreign trade. For each of these models, they presented the dynamic equation in both expanded and self-adjoint form and gave examples of particular time scales.

Finally, Chow (1990) has conducted an empirical investigation of Samuelson's basic multiplier-accelerator model in the light of cointegration. In his 1968 paper, Chow investigated the acceleration principle and the nature of business cycles with no government and foreign sectors. He reexamined his previous work with emphasis placed on the stationarity of variables under consideration and cointegration among the variables. He used annual and quarterly data from 1947 to 1989. He has found the existence of a long-run equilibrium relation, i.e., cointegrating relation between consumption and income, which was ruled out in his previous study and made the following remark:

"I happened to be correct in imposing a unit root in the multiplier-accelerator model, but I was incorrect in placing it entirely in the consumption function. I did not know how to impose one unit root in a model of three equations, to be shared by them."

One of the main thrusts of this paper is to fill the gap in the literature by linking the time path stability of the accelerator-multiplier model with the stationarity of the system in a stochastic open economy.

3. THEORETICAL FOUNDATIONS OF DETERMINISTIC AND STOCHASTIC RECURSIVE RELATIONS

This study formulates the multiplier-accelerator model as a nonhomogeneous second-order recurrence relation in the stochastic open-economy framework. Thus, it is essential to develop some mathematical formulations needed for analyzing an oscillating process that arises from stochastic nonhomogeneous second-order linear recurrence relations with constant coefficients.

3.1 Classification of Recurrence Relations

We begin by defining stochastic and deterministic recurrences and then proceed to define sub-classifications of recurrences.

Definition 3.1.1: A stochastic recurrence relation is a relation that recursively defines a sequence for a random variable Y_t such that given the mapping $f: \mathbb{N} \times X^n \times \mathbb{R} \to X$ and the set of initial conditions $(Y_0, \dots, Y_n) \epsilon X$, we have

$$Y_t = f(t, Y_{t-1}, \dots., Y_{t-n}, \varepsilon_t) \text{ for } t > n \tag{3.1}$$

where ε_t is a random disturbance term that is the mapping $\varepsilon_t: \Omega \to \mathbb{R}$, with Ω as the sample space. Furthermore, we assume that $\varepsilon_t \sim N(0, \sigma^2)$ and that $X = \mathbb{R}$.

Definition 3.1.2: A deterministic recurrence relation is a relation that recursively defines a sequence for the variable Y_t such that given the mapping $f: \mathbb{N} \times \mathbb{R}^n \to \mathbb{R}$ and the set of initial conditions $(Y_0, \dots, Y_n) \epsilon \mathbb{R}$, we have:

$$Y_t = f(t, Y_{t-1}, \dots., Y_{t-n}) \text{ for } t > n \tag{3.2}$$

Observe that a deterministic recurrence relation is a degenerate case of a stochastic recurrence relation.

Definition 3.1.3: A stochastic recurrence relation (3.1) is linear if it can be written in the form:

$$P_0(t)Y_t + P_1(t)Y_{t-1} + \cdots + P_{n-1}(t)Y_{t-n-1} + P_n(t)Y_{t-n} = G(t) + \varepsilon_t \tag{3.3}$$

with the force function $G(t)$ as the mapping $G: \mathbb{N} \to \mathbb{R}$. Additionally, we note that the set of sequences $\{P(t)\}_0^n$ represents the mappings $P_n: \mathbb{N} \to \mathbb{R}$ that are linear in t, but are disjoint to ε_t and the set of sequences $\{Y\}_{t-n}^t$.

Definition 3.1.4: A linear stochastic recurrence relation (3.3) has constant coefficients if the set of sequences $\{P(t)\}_0^n$ and the force function $G(t)$ are constants such that:

$$P_0Y_t + P_1Y_{t-1} + \cdots + P_{n-1}Y_{t-n-1} + P_nY_{t-n} = G + \varepsilon_t \quad \text{with } P_0 \neq 0. \tag{3.4}$$

Definitions 3.1.3 through 3.1.4 follow for deterministic reoccurrences as well.

3.2 Solutions of Linear Recurrence Relations with Constant Coefficients

We now aim to classify the family of solutions for recurrence relations. This will then be followed by a methodology showing when such solutions exist for the linear recurrence with constant coefficients of order two.

Definition 3.2.1: The sequence $Y_h: \mathbb{N} \to \mathbb{R}$ is a homogenous solution to (3.3), if $Y_h(t)$ solves (3.3) when $\forall t \in \mathbb{N}(G(t) = 0 = \varepsilon_t)$.

Definition 3.2.2: The sequence $Y_p: \mathbb{N} \to \mathbb{R}$ is a particular solution of (3.3), if $Y_p(t)$ solves (3.3) for $\forall t \in \mathbb{N}(G(t) \wedge \varepsilon_t)$.

Definition 3.2.3: The sequence $Y_g: \mathbb{N} \to \mathbb{R}$ is the general solution to (3.3), if $Y_g(t)$ solves (3.3) for both the homogenous and nonhomogeneous cases such that $Y_g(t) = Y_h(t) + Y_p(t)$.

Now that we have classified the different solutions to a recurrence relation we now proceed to proving their existence for the second-order case exhibiting an oscillating process. We will make use of the characteristic polynomial of the recurrence relation for the homogenous solution. Then we will use the method of undetermined coefficients to determine the particular solution. Thus, let us now consider the standardized stochastic nonhomogeneous second-order linear recurrence relation with constant coefficients:

$$Y_t + \widetilde{P_1}Y_{t-1} + \widetilde{P_2}Y_{t-2} = \tilde{G} + \widetilde{\varepsilon_t} \tag{3.5}$$

where $\widetilde{P_1} = P_1/P_0, \widetilde{P_2} = P_2/P_0, \tilde{G} = \frac{G}{P_0}, \widetilde{\varepsilon_t} = \frac{\varepsilon_t}{P_0}$.

Proposition 1: Suppose λ' and λ'' are homogenous solutions to (3.5), then $(c_1\lambda' + c_2\lambda'')$ is also a homogenous solution to (3.12) for c_1, c_2 arbitrary.

Proof: Suppose homogenous solutions λ' and λ'' for (3.5) such that

$$\lambda'_{t} + \widetilde{P_1}\lambda'_{t-1} + \widetilde{P_2}\lambda'_{t-2} = 0 \tag{3.6a}$$

$$\lambda''_{t} + \widetilde{P_1}\lambda''_{t-1} + \widetilde{P_2}\lambda''_{t-2} = 0 \tag{3.6b}$$

Now let c_1, c_2 be arbitrary. We multiply (3.6a) by c_1 and (3.6b) by c_2. Clearly the resulting equations will still be singular. Now we sum the results to form:

$$c_1(\lambda'_{t} + \widetilde{P_1}\lambda'_{t-1} + \widetilde{P_2}\lambda'_{t-2}) + c_2\left(\lambda''_{t} + \widetilde{P_1}\lambda''_{t-1} + \widetilde{P_2}\lambda''_{t-2}\right) = 0 \tag{3.7}$$

where (3.7) is singular and holds since c_1, c_2 are arbitrary.

Lemma 1: There exists a homogenous solution to (3.5) exhibiting an oscillating process.

Proof: We utilize proposition 1 and infer that the second-order case will be of form:

$$Y_h(t) = c_1(\lambda_1)^t + c_2(\lambda_2)^t \tag{3.8}$$

where λ_1 and λ_2 are roots of the characteristic polynomial for the recurrence relation. Let us assume the homogenous recurrence of (3.5) shifted forward two periods:

$$Y_{t+2} + \widetilde{P_1}Y_{t+1} + \widetilde{P_2}Y_t = 0 \tag{3.9}$$

where we let $Y_{t+n} = \lambda^n$ such that (3.9) can now be restated as

$$char(\lambda) = \lambda^2 + \widetilde{P_1}\lambda + \widetilde{P_2} = 0 \tag{3.10}$$

where $char(\lambda)$ is the characteristic polynomial whose roots solve for the singular case of (3.5). Solving for (3.9) via the quadratic equation produces the roots:

$$-\frac{\widetilde{P_1}}{2} \pm \frac{\sqrt{\widetilde{P_1}^2 - 4\widetilde{P_2}}}{2} \tag{3.11}$$

Thus, the homogenous solution of (3.5) will take different forms contingent on the magnitude of P_1 compared to $\sqrt{4P_0P_2}$. Since we are interested only in a homogenous solution exhibiting an oscillating process, we consider only the case yielding complex conjugate roots. Consider the case of $P_1 < \sqrt{4P_0P_2}$, which results in a pair of complex conjugate roots with value $\lambda_1 = -\frac{\widetilde{P_1}}{2} + \frac{\sqrt{\widetilde{P_1}^2-4\widetilde{P_2}}}{2}i$ and $\lambda_2 = -\frac{\widetilde{P_1}}{2} - \frac{\sqrt{\widetilde{P_1}^2-4\widetilde{P_2}}}{2}i$. It then follows that

$$Y_h(t) = k_1\left(-\frac{\widetilde{P_1}}{2} + \frac{\sqrt{\widetilde{P_1}^2 - 4\widetilde{P_2}}}{2}i\right)^t + k_2\left(-\frac{\widetilde{P_1}}{2} - \frac{\sqrt{\widetilde{P_1}^2 - 4\widetilde{P_2}}}{2}i\right)^t \tag{3.12a}$$

$$= k_1[r(\cos\emptyset + i\sin\emptyset)]^t + k_2[r(\cos\emptyset - i\sin\emptyset)]^t$$

$$= k_1 r^t(\cos\emptyset t + i\sin\emptyset t) + k_2 r^t(\cos\emptyset t - i\sin\emptyset t) \tag{3.12b}$$

Equation (3.12b) results from the use of the polar form of a complex number and De Moivre's theorem. We note that k_1, k_2 are arbitrary constants and $r = |\lambda_1\lambda_2|$ is the modulus of the complex conjugate roots. Notice also that (3.12b) is a complex-valued homogenous solution; we need a real-valued homogenous solution. We then use proposition 1 once more to partition (3.12b) into its real and imaginary components:

$$Y_{h_{Re}}(t) = k_1 r^t \cos\emptyset t + k_2 r^t \cos\emptyset t$$

$$= c_1 r^t \cos\emptyset t \tag{3.13a}$$

$$Y_{h_{Im}}(t) = k_1 r^t i\sin\emptyset t - k_2 r^t i\sin\emptyset t$$

$$= c_2 r^t \sin\emptyset t \tag{3.13b}$$

Now summing (3.13a) and (3.13b) together gives rise to

$$Y_h(t) = Y_{h_{Re}}(t) + Y_{h_{Im}}(t)$$

$$= c_1 r^t \cos\emptyset t + c_2 r^t \sin\emptyset t \tag{3.14}$$

where the homogenous solution (3.14) is as required and $\emptyset = \tan^{-1}\frac{Im(\lambda_1)}{Re(\lambda_2)} = \frac{2\pi}{L}$ is the frequency of the oscillations. The results then follow.

Lemma 4: There exists a particular solution to (3.5) when it exhibits an oscillating process.

Proof: Suppose (3.5), we will make use of the method of undetermined coefficients. Let $Y_t = Y_{t-1} = Y_{t-2} = A$ such that (3.5) can be restated as:

$$\tilde{G} + \tilde{\varepsilon_t} = A + \widetilde{P_1}A + \widetilde{P_2}A \tag{3.15a}$$

$$\rightarrow A = \frac{\tilde{G} + \tilde{\varepsilon_t}}{1 + \widetilde{P_1} + \widetilde{P_2}} = \frac{\tilde{G}}{1 + \widetilde{P_1} + \widetilde{P_2}} + \sum_{j=0}^{\infty} q_j \widetilde{\varepsilon_{t-J}}$$

$$\rightarrow Y_p(t) = \frac{\tilde{G}}{1 + \widetilde{P_1} + \widetilde{P_2}} + \sum_{j=0}^{\infty} q_j \widetilde{\varepsilon_{t-J}} \tag{3.15b}$$

with $\sum_{j=0}^{\infty} q_j = \frac{1}{1 + \widetilde{P_1} + \widetilde{P_2}}$. We now must find the coefficients for the series $\sum_{j=0}^{\infty} q_j \widetilde{\varepsilon_{t-J}}$.

Plugging the latter into $Y_t + \widetilde{P_1}Y_{t-1} + \widetilde{P_2}Y_{t-2} = \widetilde{\varepsilon_t}$ we see that

$$\sum_{j=0}^{t} q_j \widetilde{\varepsilon_{t-j}} + \widetilde{P_1}\sum_{j=0}^{t-1} q_i \widetilde{\varepsilon_{t-1-j}} + \widetilde{P_2}\sum_{j=0}^{t-2} q_j \widetilde{\varepsilon_{t-2-j}} = \widetilde{\varepsilon_t} \tag{3.16}$$

Solving (3.16) when $t = 0$ and $t = 1$ yields initial conditions of $q_0 = 1$ and $q_1 = -\widetilde{P_1}$ respectively. A further inspection of (3.16) reveals that for $t \geq 2$ the coefficients can be found via

$$q_j + \widetilde{P_1}q_{j-1} + \widetilde{P_2}q_{j-2} = 0 \tag{3.17}$$

It follows from lemma 1 that (3.17) has a homogenous solution. Additionally, the characteristic polynomial of (3.17) is identical to that of (3.5). We now state the form of the homogenous solution with complex conjugate roots as needed. Furthermore we solve for uniqueness since initial conditions are known and such identification of this solution will be needed in the discussion of stability and stationarity. Given that $P_1 < \sqrt{4P_0P_2}$ the homogenous solution is of form:

$$q_h(j) = c_1 r^j \cos \emptyset j + c_2 r^j \sin \emptyset j \tag{3.18}$$

Now solving the system when $q_h(0) = 1$ and $q_h(1) = -\widetilde{P_1}$ for (3.18) generates the unique solution:

$$q_c(j) = r^j \cos \emptyset j + [\frac{-\widetilde{P_1} - r\cos\emptyset}{r \sin\emptyset}] r^j \sin \emptyset j \tag{3.19}$$

With the coefficients uniquely found for our relevant case, we now substitute $q_c(j)$ into (3.15b):

$$Y_p(t) = \frac{\tilde{G}}{1+\widetilde{P_1}+\widetilde{P_2}} + \sum_{j=0}^{\infty} q_c(j)\widetilde{\varepsilon_{t-j}} \tag{3.20}$$

Thus the result for a particular solution holds when (3.5) follows an oscillating process.

Theorem 3.2.2: There exists a general solution to the second-order stochastic nonhomogeneous linear recurrence relation with constant coefficients (3.5) that exhibits an oscillating process:

Proof: By lemma 1 and lemma 2 we know homogenous and particular solutions exist for (3.5) when it exhibits an oscillating process. Thus by definition 3.2.3 it follows that $Y_g(t) = Y_h(t) + Y_p(t)$, and the result holds.

We observe that the general solution to (3.5) is $Y_g(t) = Y_h(t) + Y_p(t)$. We may partition $Y_p(t)$ as $Y_p(t) = Y_{p_1}(t) + Y_{p_2}(t)$ where $Y_{p_1}(t)$ corresponds to the particular solution in regards to the force function and $Y_{p_2}(t)$ as the particular solution with respect to the random disturbance term. We then observe that the deterministic counterpart to (3.5) has a general solution of $Y_g(t) = Y_h(t) + Y_{p_1}(t)$.

3.3 Stability of Deterministic Recurrence Relations

After having established the methodology for finding oscillating solutions to second-order nonhomogeneous linear recurrence relations with constant coefficients in the stochastic and deterministic cases, we now turn to the limiting behavior of these recurrences. This section will then be devoted to the stability analysis of the deterministic case. The following section will address the stationarity of the stochastic case and prove that stability implies stationarity under certain conditions. We now focus on the standardized deterministic nonhomogeneous second-order linear recurrence relation with constant coefficients:

$$Y_t + \widetilde{P_1}Y_{t-1} + \widetilde{P_2}Y_{t-2} = \tilde{G} \tag{3.21}$$

Definition 3.3.1: A deterministic recurrence relation (3.2) is stable, if for an arbitrary $\varepsilon \in \mathbb{R}^{++}$, there exists a $t \epsilon \mathbb{N}$ such that whenever $t \geq T$ we also have that $\left|Y_g(t) - Y_{p_1}(t)\right| < \varepsilon$, where $Y_{p_1}(t)$ is the steady state of the sequence.

Theorem 3.3.1: Given a deterministic recurrence relation (3.21) with $Y_g(t)$ oscillating, we argue that the recurrence is stable at $Y_{p_1}(t)$ if and only if $\max(|\lambda_1|, |\lambda_2|) < 1$ for the characteristic polynomial.

Proof: →) Suppose that (3.21) holds with $Y_g(t)$ oscillating and that as t becomes sufficiently large it must be that $\left|Y_g(t) - Y_{p_1}(t)\right| < \varepsilon$ for $\varepsilon \in \mathbb{R}^{++}$. Expanding the latter portion of our stability definition reveals that

$$\left|Y_g(t) - Y_{p_1}(t)\right| < \varepsilon \tag{3.22a}$$

$$\rightarrow \left|Y_h(t) + Y_{p_1}(t) - Y_{p_1}(t)\right| < \varepsilon$$

$$\rightarrow |Y_h(t)| < \varepsilon \tag{3.22b}$$

Eq. (3.22b) amounts to showing that the homogenous solution of the recurrence is smaller than an arbitrary $\varepsilon \in \mathbb{R}^{++}$. To this end we suppose that $\max(|\lambda_1|, |\lambda_2|) \geq 1$ for the roots of the recurrence and see that for our case with oscillations this amounts to $\lim_{t\to\infty} Y_h(t) = \lim_{t\to\infty}(c_1 r^t cos\theta t + c_2 r^t sin\theta t) \neq 0$, and thus it must be that $\max(|\lambda_1|, |\lambda_2|) < 1$ for $\lim_{t\to\infty} Y_h(t) = 0$ due to r^t being degenerate in its limit for this case.

Proof: ←) We prove the contrapositive of the statement. Assuming that (3.21) holds with $Y_g(t)$ oscillating and that definition 3.3.1 does not hold, we want to show $\max(|\lambda_1|, |\lambda_2|) \geq 1$. Since 3.3.1 is not stable, it is the case that for some $\varepsilon \in \mathbb{R}^{++}$, $\left|Y_g(t) - Y_{p_1}(t)\right| > \varepsilon$ holds for any $t \geq T$. As a consequence, the results of the proof follow similarly to the proof for Theorem 3.3.1. Thus, the results follow.

With the conditions for stability now known, we conclude this section with a proof for the set of values for $\widetilde{P_1}$ and $\widetilde{P_1}$ that ensure stability of the oscillations.

Theorem 3.3.2: $char(\lambda) = \lambda^2 + \widetilde{P_1}\lambda + \widetilde{P_2}$ with $\widetilde{P_1}^2 - 4\widetilde{P_2} < 0$ satisfies $\max(|\lambda_1|, |\lambda_2|) < 1$ when $|\widetilde{P_2}| < 1$ holds.

Proof: Suppose $\text{char}(\lambda) = \lambda^2 + \widetilde{P_1}\lambda + \widetilde{P_2}$ with $\widetilde{P_1}^2 - 4\widetilde{P_2} < 0$ and $\max(|\lambda_1|, |\lambda_2|) < 1$. We know that the roots of $\text{char}(\lambda)$ are complex conjugates with values $\lambda_1 = -\frac{\widetilde{P_1}}{2} + \frac{\sqrt{\widetilde{P_1}^2 - 4\widetilde{P_2}}}{2} i$ and $\lambda_2 = -\frac{\widetilde{P_1}}{2} - \frac{\sqrt{\widetilde{P_1}^2 - 4\widetilde{P_2}}}{2} i$. Comparing the magnitude of the modulus $r = |\lambda_1 \lambda_2|$ with respect to one we see that

$$\left| \left(-\frac{\widetilde{P_1}}{2} + \frac{\sqrt{\widetilde{P_1}^2 - 4\widetilde{P_2}}}{2} i\right)\left(-\frac{\widetilde{P_1}}{2} - \frac{\sqrt{\widetilde{P_1}^2 - 4\widetilde{P_2}}}{2} i\right) \right| < 1 \tag{3.23a}$$

$$\rightarrow |\widetilde{P_2}| < 1 \tag{3.23b}$$

And the result follows.

3.4 Stationarity of Stochastic Recurrence Relations

In the previous section we found that the second-order nonhomogeneous linear deterministic recurrence relation with constant coefficients exhibiting an oscillating process is stable when its homogenous solution is degenerate at the limit. This turned out to be equivalent to the characteristic polynomial of the recurrence having a modulus less than one in absolute magnitude. We now would like to express an analog to the latter result in the stochastic setting. Thus, this section will discuss the stationarity of the oscillating process in the stochastic case.

Definition 3.4.1: A stochastic recurrence relation (3.1) is weakly stationary if the following properties hold:

1. The mean of (3.1) is independent of time.

$$\text{E}[Y_t] = \text{E}[Y_{t-s}] \tag{3.24}$$

2. The variance of (3.1) is finite and independent of time.

$$var(Y_t) = var(Y_{t-s}) < \infty \tag{3.25}$$

3. The covariance function of (3.1) is a function of $(t - s)$ but not of t nor s exclusively.

$$cov(Y_t, Y_{t-s}) = cov(Y_{t-v}, Y_{t-v-s}) = \tau(s) \tag{3.26}$$

Theorem 3.4.1: Given the stochastic recurrence relation (3.5) exhibiting an oscillating general solution, then Equation (3.5) is stable, if its deterministic counterpart (3.5) is stable.

Proof: Suppose that (3.5) and (3.21) hold with $Y_g(t)$ oscillating. Additionally, suppose that (3.21) is stable. By theorem 3.2.2 we know that the general solution exists for both. Furthermore by theorem 3.3.1 we know that the modulus of characteristic polynomial is less than one in absolute value such that $\lim_{t\to\infty} Y_g(t) = Y_{p_1}(t) = \frac{\tilde{G}}{1+\widetilde{P_1}+\widetilde{P_2}}$ for (3.21). If we apply the same methodology to (3.5) we see that

$$\lim_{t\to\infty} Y_g(t) = Y_{p_1}(t) + Y_{p_2}(t) \tag{3.27}$$

$$= \frac{\tilde{G}}{1+\widetilde{P_1}+\widetilde{P_2}} + \sum_{j=0}^{\infty} q_c(j)\widetilde{\varepsilon_{t-j}} \tag{3.28}$$

where $\frac{\tilde{G}}{1+\widetilde{P_1}+\widetilde{P_2}}$ and $\sum_{j=0}^{\infty} q_c(j)\widetilde{\varepsilon_{t-j}}$ are finite since $q_c(j)$ is stable such that $\sum_{j=0}^{\infty} q_c(j) = \frac{1}{1+\widetilde{P_1}+\widetilde{P_2}}$ is a geometric series. Our next task is to show that (3.28) satisfies the three properties of a weakly stationary process.

Property 1: $\mathrm{E}[Y_t] = \mathrm{E}[Y_{t-s}]$

Taking the expectations of (3.28) we see that:

$$\mathrm{E}\left[\lim_{t\to\infty} Y_g(t)\right] = \mathrm{E}\left[\frac{\tilde{G}}{1+\widetilde{P_1}+\widetilde{P_2}} + \sum_{j=0}^{\infty} q_c(j)\widetilde{\varepsilon_{t-j}}\right] \tag{3.29}$$

$$= \mathrm{E}\left[\frac{\tilde{G}}{1+\widetilde{P_1}+\widetilde{P_2}}\right] + \mathrm{E}\left[\sum_{j=0}^{\infty} q_c(j)\widetilde{\varepsilon_{t-j}}\right]$$

$$= \frac{\tilde{G}}{1+\widetilde{P_1}+\widetilde{P_2}} \tag{3.30}$$

It is clear that (3.30) is time independent.

Property 2: $var(Y_t) = var(Y_{t-s}) < \infty$

Taking the variance of (3.28) we see that:

$$var\left(\lim_{t\to\infty} Y_g(t)\right) = var\left(\frac{\tilde{G}}{1+\widetilde{P_1}+\widetilde{P_2}} + \sum_{j=0}^{\infty} q_c(j)\widetilde{\varepsilon_{t-j}}\right) \tag{3.31}$$

$$= \mathrm{E}\left[\left(\frac{\tilde{G}}{1+\widetilde{P_1}+\widetilde{P_2}} + \sum_{j=0}^{\infty} q_c(j)\widetilde{\varepsilon_{t-j}} - \mathrm{E}\left[\frac{\tilde{G}}{1+\widetilde{P_1}+\widetilde{P_2}}\right] - \mathrm{E}\left[\sum_{j=0}^{\infty} q_c(j)\widetilde{\varepsilon_{t-j}}\right]\right)^2\right]$$

$$= \mathrm{E}\left[\left(\sum_{j=0}^{\infty} q_c(j)\widetilde{\varepsilon_{t-j}} - \mathrm{E}\left[\sum_{j=0}^{\infty} q_c(j)\widetilde{\varepsilon_{t-j}}\right]\right)^2\right]$$

$$= var\left(\sum_{j=0}^{\infty} q_c(j)\widetilde{\varepsilon_{t-j}}\right)$$

$$= \sigma_\varepsilon^2 \sum_{j=0}^{\infty} q_c(j)^2 \tag{3.32}$$

where (3.32) is time-independent and finite as required.

Property 3: $cov(Y_t, Y_{t-s}) = cov(Y_{t-v}, Y_{t-v-s}) = \tau(s)$

Applying the definition of covariance to (3.28) we see that

$$cov(\lim_{t\to\infty} Y_g(t), \lim_{t\to\infty} Y_g(t-s)$$

$$= \mathrm{E}\left[\left(\frac{\tilde{G}}{1+\widetilde{P_1}+\widetilde{P_2}} + \sum_{j=0}^{\infty} q_c(j)\widetilde{\varepsilon_{t-j}} - \frac{\tilde{G}}{1+\widetilde{P_1}+\widetilde{P_2}} - \mathrm{E}\left[\sum_{j=0}^{\infty} q_c(j)\widetilde{\varepsilon_{t-s-j}}\right]\right)^2\right]$$

$$= cov\left(\sum_{j=0}^{\infty} q_c(j)\widetilde{\varepsilon_{t-j}}, \sum_{j=0}^{\infty} q_c(j)\widetilde{\varepsilon_{t-s-j}}\right)$$

$$= \sigma_\varepsilon^2 \sum_{j=0}^{\infty} q_c(j)\, q_c(s+j) \tag{3.33}$$

It is clear that (3.33) is a function of $(t-s)$. Since all three properties were satisfied, the proposition that stability imply stationarity for (3.5) with $Y_g(t)$ oscillating follows.

We now have the tools necessary for deriving a stochastic open-economy multiplier-accelerator model and analyzing its oscillating time path.

4. CONDITIONS FOR THE STABILITY OF THE MULTIPLIER-ACCELERATOR TIME PATH

In this chapter we derive a stochastic open-economy multiplier-accelerator model using the tools we developed in chapter 3. The aim of the model is to examine national income as a weakly stationary oscillating process. We proceed by imposing structural assumptions on the national income identity, then solving the resulting recurrence relation, noting the stability conditions, and lastly examining the properties of national income as a weakly stationary oscillating process.

4.1 Structural Assumptions

This section is devoted to laying out the structure on an open economy from which a stochastic open-economy multiplier-accelerator model can be derived. Consider an open economy over a denumerable set of time given by the national income accounting identity:

$$Y_t = C_t + I_t + G_t + Ex_t - Im_t \tag{4.1}$$

where Y_t represents national income, C_t consumption expenditures, I_t investment spending, G_t government purchases, Ex_t exports, and Im_t imports. The following assumptions are maintained:

1. Current consumption expenditures are a stochastic process that consists of an autonomous component (α_0) plus a portion of previous national income.

$$C_t = \alpha_0 + \alpha_1 Y_{t-1} + \varepsilon_{c,t} \quad \text{with } \alpha_0 \epsilon \mathbb{R},\ \alpha_1 \epsilon (0,1), \text{ and } \varepsilon_{c,t} \sim N(0, \sigma_c^2). \tag{4.2}$$

2. Current investment spending is a stochastic process that consists of an autonomous component (β_0) plus the first difference in consumption expenditures.

$$I_t = \beta_0 + \beta_1 \Delta C_t + \varepsilon_{I,t} \text{ with } \beta_0 \epsilon \mathbb{R}, \beta_1 \epsilon \mathbb{R}^{++}, \text{ and } \varepsilon_{I,t} \sim N(0, \sigma_I^2). \tag{4.3}$$

$$= \beta_0 + \alpha_1 \beta_1 \Delta Y_{t-1} + \beta_1 \Delta \varepsilon_{c,t} + \varepsilon_{I,t} \tag{4.4}$$

3. Government expenditures are autonomous in all periods.

$$G_t = \bar{G} \quad \text{with } \bar{G} \epsilon \mathbb{R}. \tag{4.5}$$

4. Exports are autonomous in all periods.

$$Ex_t = \overline{Ex} \quad \text{with } \overline{Ex} \epsilon \mathbb{R}. \tag{4.6}$$

5. Current imports are a stochastic process that consists of an autonomous component (γ_0) plus a portion of previous national income.

$$Im_t = \gamma_0 + \gamma_1 Y_{t-1} + \varepsilon_{Im,t} \text{ with } \gamma_0 \epsilon \mathbb{R}, \gamma_1 \epsilon (0,1), \text{ and } \varepsilon_{Im,t} \sim N(0, \sigma_{Im}^2). \tag{4.7}$$

Now that structural assumptions have been imposed on each of the components of national income, we substitute the structural equations into the national income identity to obtain

$$Y_t - (\alpha_1 + \alpha_1 \beta_1 - \gamma_1) Y_{t-1} + \alpha_1 \beta_1 Y_{t-2}$$
$$= \psi + \varepsilon_{c,t} + \varepsilon_{c,t} \beta_1 - \beta_1 \varepsilon_{c,t-1} + \varepsilon_{I,t} - \varepsilon_{Im,t} \tag{4.8}$$

where $\psi = \alpha_0 + \beta_0 - \gamma_0 + \bar{G} + \overline{Ex}$. Notice that (4.8) is a stochastic nonhomogeneous second-order linear recurrence relation with constant coefficients. We can now apply the tools we developed in chapter 3 to derive the time path of (4.8).

4.2 Oscillating General Solution

In this section we derive the oscillating general solution to the recurrence relation (4.8).

Proposition 4.2.1: Let there be some open economy over a denumerable set of time. If assumptions 1-5 are maintained such that we obtain (4.8) with a negative discriminate, then there exists a general solution for national income as an oscillating process.

Proof: Suppose assumptions 1-5 hold such that we yield (4.8) and that $\left(-(\alpha_1 + \alpha_1 \beta_1 - \gamma_1)\right)^2 - 4\alpha_1 \beta_1 < 0$. By lemma 1 we know (4.8) has a homogenous solution of form:

$$Y_h(t) = C_1 (r)^t \cos(\theta t) + C_2 (r)^t \sin(\theta t) \tag{4.9}$$

By lemma 2 we know that a particular solution exists and is of form:

$$Y_p(t) = \frac{\psi}{1-\alpha_1+\gamma_1} + \sum_{j=0}^{\infty} q_c(j)\varepsilon_{Y,t-j} \tag{4.10}$$

Where $\sum_{j=0}^{\infty} q_c(j)\varepsilon_{Y,t-j} = \sum_{j=0}^{\infty} q_c(j)\varepsilon_{c,t-j} + \beta_1\Delta\sum_{j=0}^{\infty} q_c(j)\varepsilon_{c,t-j} + \sum_{j=0}^{\infty} q_c(j)\varepsilon_{I,t-j} - \sum_{j=0}^{\infty} q_c(j)\varepsilon_{Im,t-j}$ for compactness of notation.

Lastly by theorem 3.2.2 we can show that a general solution to (4.8) exists by use of lemma 1 and lemma 2, where the general solution is $Y_g(t) = Y_h(t) + Y_p(t)$. Thus, the results follow.

4.3 Stability Conditions

In the previous section we have derived the oscillating general solution to (4.8). We are now concerned with finding the conditions that ensures (4.8) is weakly stationary in addition to being an oscillating process. By theorem 3.4.1 we have demonstrated that stability implies a weakly stationary process. Thus, we need only examine the stability conditions.

Finding the stability conditions for (4.8) requires the examination of its characteristic polynomial. Theorem 3.3.1 establishes that a deterministic recurrence relation is stable when its modulus is less than one in absolute magnitude. By applying the results of theorem 3.3.2 we see that the latter condition holds when $|\alpha_1\beta_1| < 1$.

With stability conditions now expressed in terms of the parameters of (4.8) we will discuss the implications of the parameter restrictions imposed. Looking at the condition $|\alpha_1\beta_1| < 1$ this implies that the accelerator coefficient is less than one in absolute magnitude. Since the accelerator coefficient is always positive, the absolute magnitude may be ignored. Restating this condition in terms of the individual parameters, the condition holds when the marginal propensity to consume (α_1) dominates the capital output ratio (β_1) such that their product is less than unity. This condition will almost surely hold for developed economies. The rationale is that as time progresses, an economy will utilize its capital more effectively which corresponds to small capital output ratios. Sufficiently small capital output ratios can be scaled to less than unity by the marginal propensity to consume.

4.4 National Income as a Weakly Oscillating Process

From the derived recurrence relation (4.8) we have shown that when the general solution is an oscillating process, the process is weakly stationary through its stability conditions. We have also confirmed that when the marginal propensity to consume dominates the capital output ratio, the relevant stability condition is met. In this section we examine the weakly stationary oscillating general solution of (4.8).

For Equation (4.8) to have a weakly stationary general solution that exhibits an oscillating process we maintain the restrictions of a negative discriminate additionally and an accelerator less than one, both of which will then lead to an explicit representation of the general solution which is of form:

$$Y_g(t) = C_1(r)^t \cos(\emptyset t) + C_2(r)^t \sin(\emptyset \mathrm{t}) + \frac{\psi}{1-\alpha_1+\gamma_1} + \sum_{j=0}^{\infty} q_c(i)\varepsilon_{Y,t-j} \tag{4.11}$$

as found in section 4.2.

We now examine the implications of (4.11). The oscillations of this solution are driven by the trigonometric functions imbedded in its homogenous solution where the frequency was noted to be $\emptyset = \tan^{-1}\frac{Im(\lambda_1)}{Re(\lambda_2)} = \frac{2\pi}{L}$. The oscillations are dampened due to the stationarity of the second-order recurrence equation (4.8). Thus, the process converges to its particular solution which is the sum of multiple disturbance terms and the mean value of the process. The disturbance terms entered the general solution through the structural assumptions imposed on consumption expenditures, investment spending, and imports.

We see that the disturbances entering through consumption expenditures and imports are less volatile than the investment spending disturbances as expected. In general these disturbances will cause volatility in the dampened oscillations and will have the associated coefficients of $q_c(i) = r^i \cos\emptyset i + [\frac{-\widetilde{P_1} - r\cos\emptyset}{r\sin\emptyset}]r^i \sin\emptyset i$. Now taking expectations of the limiting behavior of (4.11) we obtain $\mathrm{E}\left[\lim_{t\to\infty} Y_g(t)\right] = \frac{\psi}{1-\alpha_1+\gamma_1}$. This has the implication that an economy with a higher mean national income value at their limit will on average be better off. Economies will have differing mean national income values based on their domestic multiplier as well as the sum of their autonomous components.

We then see that the spread of national income is $var\left(\lim_{t\to\infty} Y_g(t)\right) = \sigma_{\varepsilon_Y}^2 \sum_{j=0}^{\infty} q_c(j)^2$ and an associated covariance structure of $cov(\lim_{t\to\infty} Y_g(t), \lim_{t\to\infty} Y_g(t-s) = \sigma_{\varepsilon_Y}^2 \sum_{j=0}^{\infty} q_c(j)\, q_c(s+j)$. With the coefficients and properties of (4.11) established we now have sufficient information on national income as a weakly stationary oscillating process, which will be examined in our empirical analysis of the model.

5. EMPIRICAL ANALYSIS

5.1. Methodology

In this chapter we estimate our model to provide empirical support for national income as a weakly stationary oscillating process. We estimate the structural equations from the perspective of cointegration among variables to investigate whether the U.S. economy exhibits a dampening oscillating business cycle. The main purpose of estimating the structural equations in the context of cointegration is to establish a link between the stability of the time path and the stationarity of the oscillating process of the multiplier-accelerator.

When the variables in the structural equations are cointegrated, this indicates that there is a long-run equilibrium relationship among the variables. Although in the short run, there could be an equilibrium error in the system, which could be a source of short-run oscillations, cointegration among variables is likely to lead to a dampening oscillating time path in the long run. Furthermore, in order to obtain more precise estimates in the context of the oscillating path of the accelerator-multiplier model, we must address the stationarity of the variables. In order to estimate the model, we have used U.S. quarterly data spanning from 1947:Q1 to 2019:Q1. We have obtained the data from the Federal Reserve Bank of St. Louis and the Bureau of Economic Analysis.

The model of interest for estimation is the stochastic nonhomogeneous second-order linear recurrence relation with constant coefficients that stems from the structural equations. Structural equations (4.2)-(4.7) are substituted into the national income identity to form (4.8). Note that national income appears on both the left-hand and right-hand sides of the equations. This endogenous feedback effect must be addressed after integration order diagnostics are checked. To remedy the endogenous feedback, we implement a two-stage least squares method by regressing Y_t against the exogenous variables G_t and Ex_t to obtain consistent estimates of national income.

In order to estimate the model, we first conduct unit root tests. If the variables contain a unit root, we next investigate whether the variables in the system are cointegrated. If there is a cointegrating relation among the variables, we could estimate the structural equations in level form. After we perform stationarity and cointegration tests on the variables, we estimate the equations using two different approaches. We first estimate the equations individually assuming that the errors in the equations are not contemporaneously related to each other, and then estimate the equations as a system using seemingly unrelated regression (SUR) to fully take advantage of interactions of the system. As a final step, we use the estimated coefficients to investigate whether the conditions for convergence are satisfied in the U.S. economy. The structural equations are given as follows:

$$
\begin{aligned}
C_t &= \alpha_0 + \alpha_1 Y_{t-1} + \varepsilon_{2t} \\
I_t &= \beta_0 + \beta_1 \Delta Y_{t-1} + \varepsilon_{3t} \\
IM_t &= \gamma_0 + \gamma_1 Y_{t-1} + \varepsilon_{4t}
\end{aligned}
$$

5.2. Tests for Unit Roots and Cointegration

We have performed the augmented Dickey-Fuller test on each variable to examine whether the variables used in this study are stationary or nonstationary. The test results are reported in Tables 1A and 1B. As expected, we have failed to reject the null hypothesis, indicating that all the variables contain a unit root. Next we have conducted the same test on the first-differenced variable, and we have been able to reject the null hypothesis for all the differenced variables. Thus, the first-differenced variables are stationary. This implies that each variable is of the same order of integration. Thus, we are concerned with cointegration among the variables in the system.

Table 1A: Augmented Dickey-Fuller Test

Variable	Obs	Test Statistic	Critical Value(5%)	Critical Value (10%
Y_t	288	-2.313	-2.878	-2.570
C_t	288	-1.536	-2.878	-2.570
I_t	288	-0.785	-2.878	-2.570
G_t	288	-1.180	-2.878	-2.570
Ex_t	288	-2.091	-2.878	-2.570
Im_t	288	-0.631	-2.878	-2.570
ΔY_t	287	-30.127	-2.879	-2.570
ΔC_t	287	-36.031	-2.897	-2.570
ΔI_t	287	-16.733	-2.879	-2.570
ΔG_t	287	-35.552	-2.879	-2.570
ΔEx_t	287	-16.003	-2.879	-2.570
ΔIm_t	287	-15.184	-2.879	-2.570

Table 1B: Engle-Granger Cointegration Test

Variable	Obs	Test Statistic	Critical Value (5%)	Critical Value (10%)
$e_{C,Y}$	288	-5.852	-3.368	-3.067
$e_{I,Y}$	288	-5.145	-3.368	-3.067
$e_{Im,Y}$	288	-3.119	-3.368	-3.067
$e_{Y,C,I,Im}$	288	-6.001	-4.154	-3.853

We have performed a cointeration test on the sample residuals using the Augmented Engle-Granger (AEG) test to check whether the residuals contain a unit root. The test statistic for the cointegration of income with consumption (C) is -5.852, with investment (I) is -5.145, and with imports (IM) is -3.119. These statistics are all greater than the Engle-Yoo critical value at the 5 percent level of significance, which leads to the rejection of the null hypothesis that the sample residuals contain a unit root. (The test statistic for imports is significant at the 6% level.) Thus, we conclude that the variables in the system are cointegrated and have a long-run equilibrium relationship.

5.3. Model Estimation: Individual Equations

Since all the variables in the structural equations are cointegrated, we estimate the equations in level form. In order to ensure the robustness of the estimation results, we estimate the structural equations using various estimation methods. First, we estimate the equations individually using OLS and 2SLS. Next we estimate the equations as a system using OLS and 2SLS.

The result of the individual estimation of the structural equations is presented in Table 2A and 2B.

Table 2A: Structural Equation Regression Results: OLS

	(1)	(2)	(3)
	C_t	I_t	Im_t
Y_{t-1}	0.686***		0.163***
	(0.002)		(0.002)
ΔY_{t-1}		0.917***	
		(0.251)	
Intercept	-0.114***	0.889***	-0.159***
	(0.015)	(0.050)	(0.012)
Obs.	288	287	288
R-squared	0.997	0.045	0.972

Standard errors are in parenthesis

*** *p<0.01*, ** *p<0.05*, * *p<0.1*

Table 2B: Structural Equation Regression Results: 2SLS

	(1)	(2)	(3)
	C_t	I_t	Im_t
Y_{t-1}	0.687***		0.164***
	(0.002)		(0.002)
ΔY_{t-1}		0.937***	
		(0.302)	
Intercept	-0.116***	0.888***	-0.164***
	(0.015)	(0.050)	(0.012)
Obs.	288	287	288
R-squared	0.997	0.045	0.972

Instrumented Var:(Y), Instrument Vars:(G,EX)

Standard errors are in parenthesis

*** *p<0.01*, ** *p<0.05*, * *p<0.1*

The OLS and 2SLS estimations of the consumption function were comparable both in coefficient magnitude as well as the standard error size. The marginal propensity to consume (MPC) was approximately .68 in both cases and statistically significant at the 1 percent level. The investment function was also fairly comparable across the two estimations where the OLS estimates were slightly more precise as expected. In both cases the accelerator coefficient was found less than one and statistically significant at the 1 percent level.

Lastly, the import function was roughly the same across the specifications both in magnitude and standard errors. The marginal propensity to import (MPI) was approximately .16 in both cases and statistically significant at the 1 percent level. Upon further inspection of the estimates it can be seen that estimated recurrence relation follows a stable oscillating process for each case. We now turn to the SUR estimation of the model as a system.

5.4. Model Estimation: Seemingly Unrelated Regression

In order to take advantage of the interrelated error structure, we have now estimated the structural equations as a system using seemingly unrelated regression (SUR) with OLS (SUR-OLS) and 2SLS (SUR-2SLS) employed, respectively. The system estimation gave rise to coefficient estimates fairly similar to the individual structural estimates. We further note that both estimated recurrence relations also follow an oscillating time path but only the SUR-2SLS estimation is stable due to its accelerator coefficient of .957 while the SUR-OLS estimation is unbounded due to its accelerator coefficient being 1.179 which is above unity in magnitude. Thus, three of the four specifications lend empirical support for national income as a weakly stationary oscillating process.

Table 3A: Seemingly Unrelated Regression Results: SUR-OLS

	(1) C_t	(2) I_t	(3) Im_t
Y_{t-1}	0.682***		0.162***
	(0.002)		(0.001)
ΔY_{t-1}		1.179***	
		(0.248)	
Intercept	-0.094***	0.874***	-0.155***
	(0.014)	(0.049)	(0.011)
Obs.	287	287	287
R-squared	0.997	0.041	0.972

Standard errors are in parenthesis
*** *p<0.01*, ** *p<0.05*, * *p<0.1*

Table 3B: Seemingly Unrelated Regression Results: SUR-2SLS

	(1)	(2)	(3)
	C_t	I_t	Im_t
Y_{t-1}	0.673***		0.162***
	(0.003)		(0.001)
ΔY_{t-1}		0.957***	
		(0.238)	
Intercept	-0.050**	0.890***	-0.156***
	(0.025)	(0.049)	(0.010)
Obs.	287	287	287
R-squared	0.991	0.029	0.977

Standard errors are in parenthesis
*** *p<0.01*, ** *p<0.05*, * *p<0.1*

In the next section, we present further empirical support to the convergence conditions by substituting the structural estimates into the particular and homogeneous solutions. This gives empirical content to the theoretical link between the stability of the time path and the stationarity of the oscillating processes of the variables.

5.5. Numerical Values of the Particular and Homogeneous Solutions

Using the estimates from the structural OLS estimates of section 5.3 we will demonstrate the deterministic form of the model and its general solution.

Table 4: Estimated Coefficients

$\alpha_0 = -.114$	$\alpha_1 = .686$	$\alpha_1\beta_1 = .917$	$\psi = 2.531$
$\beta_0 = .889$	$\beta_1 = 1.336$	$\bar{G} = 1.037$	$\emptyset = .228\pi$
$\gamma_0 = -.159$	$\gamma_1 = .163$	$\overline{Ex} = .560$	$r = .956$

We substitute the estimated parameters into the structural equations to obtain the following results:

$$C_t = -.114 + .686Y_{t-1} \tag{7.1}$$

$$I_t = .889 + .917\Delta Y_{t-1} \tag{7.2}$$

$$IM_t = -.159 + .163Y_{t-1} \tag{7.3}$$

$$G_t = 1.037 \tag{7.4}$$

$$EX_t = .560 \tag{7.5}$$

Rearranging (7.1) ~ (7.5) yields:

$$Y_t - (1.44)Y_{t-1} + (.917)Y_{t-2} = 2.531 \tag{7.6}$$

The particular solution of (7.6) is

$$Y_p(t) = 5.306 \tag{7.7}$$

Solving for the homogenous solution of (7.6) produces

$$Y_h(t) = c_1(.956)^t \cos(.228\pi t) + c_2(.956)^t \sin(228\pi t) \tag{7.8}$$

Combining the homogenous solution with the particular solution gives the general solution:

$$Y_g(t) = c_1(.956)^t \cos(.228\pi t) + c_2(.956)^t \sin(228\pi t) + 5.306 \tag{7.9}$$

Here it is worth noting that (7.9) is stable such that $E\left[\lim_{t\to} Y_g(t)\right] = 5.306$ which is approximately the sample mean of national income in the data set which was of value 5.342.

6. CONCLUSION

We have extended Samuelson's multiplier-accelerator model as a nonhomogeneous second-order recurrence relation in a stochastic framework which includes the foreign sector. We have first derived the conditions for the existence and stability of a stochastic time path of the model and then obtained the general solution for the second-order recurrence relation in the complete model.

We have demonstrated that the deterministic nonhomogeneous second-order linear recurrence relation with constant coefficients exhibiting an oscillating process is stable when its homogenous solution is degenerate at the limit. We have further proved that when the variables in the model are weakly stationary, then the oscillating process in the stochastic case is stable. The most important contribution of this study may be found in its attempt to establish a link between the stability of the time path of the multiplier-accelerator and the stationarity of the system.

In order to give a flavor of realism to the working of the accelerator-multiplier with emphasis on the link between the stability and stationarity of the model, we have performed an empirical investigation of our model in the context of cointegration. If the variables in the system are cointegrated, then the system converges to a long-run equilibrium relation.

We have employed SUR to estimate the structural equations as a system using U.S. quarterly data spanning from 1947:Q1 to 2019:Q1. We have found that all the variables in the system contain a unit root, and have the same order of integration. We have further confirmed that the variables in the system are cointegrated, indicating that they are on the same wavelength.

We have found that the conditions for the existence, stability, and stationarity of the oscillating time path for the U.S. economy are empirically supported: the time path generated from the estimates was oscillating and converged over time. Interestingly, we have shown that the convergence conditions are empirically satisfied when the variables in the system are of the same order of integration and are cointegrated among them. We have provided a numerical illustration of the general solution and its expected limiting behavior using the parametric values obtained from the structural estimation of the system. We have confirmed that the multiplier-accelerator model is stable when the system is cointegrated.

Finally we have found that the size of the accelerator-multiplier coefficient becomes smaller in an economy where the variables in the system are cointegrated. This finding indicates that the amplitude of a business cycle is more or less mitigated in an economic system characterized with the stability of the oscillating process of the multiplier-accelerator and the stationarity of the business cycle variables. We conclude that an open-economy multiplier-accelerator in the context of cointegration provides support for the stability of national income as a weakly stationary oscillating process.

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