Forthwith, equation(2) could be solved in the same manner, such that by inserting equation(15) into equation(2), and substituting a collocation points [20] on $[a,b]$, we will have the system of linear equations

$$\sum_{l}^{r}\sum_{n\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}\Xi_{l,n,k}\left(x_i\right)\alpha_{n,k}^{l}=-f\left(x_i\right) \tag{27}$$

Such that

$$\Xi_{i,n,k}\left(x_i\right)=\sum_{l}^{r}\sum_{n\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}\left(\int_0^{x_i}k\left(x_i,t\right)\psi_{l,n,k}dt\right)=f\left(x_i\right)$$

4.2. Nonlinear Volterra Integral Equation of First Kind

The nonlinear Volterra integral equation of first has many different formulas; in this article we will consider the one as in(3). For the nonlinear integral equation(3), we will use the method as in [20], which is based on converting the nonlinear integral equation into a linear integral equation. Firstly, we consider the substitution

$$w(x)=h\left(y(x)\right) \tag{28}$$

and then, by inserting equation (28) into equation(3), we will get a linear Volterra integral equation of the first kind of the form

$$0=f\left(x\right)+\int_0^x k\left(x,y\right)w\left(t\right)dt \tag{29}$$

Which can be solved in the same manner of equation(2), we will have the solution of the function $w(x)$ of equation(29) as;

$$w(x)=\sum_{l}^{r}\sum_{n\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}\alpha_{j,k}^{l}\psi_{l,n,k}\left(x\right) \tag{30}$$

So the solution $y(x)$ for the equation(3) will be obtained from equation(28).

5. ERROR ANALYSIS

In this section we discuss the convergence ate of our method for solving linear Volterra integral equations of second kinds(1). In [9] shows that

$$\left\|y-P_n y\right\|_2^2\leq\left\|y\right\|_\infty\max_l\left\|\psi_l\right\|_1 2^{-(s+1)n}\sum_{l=1}^{r}\sum_{j\geq 0}\sum_{k\in\mathbb{Z}}2^{sj}\left|\left\langle y,\psi_{l,j,k}\right\rangle\right| \tag{31}$$

where $s\geq -1$, $\sum_{l=1}^{r}\sum_{j\geq 0}\sum_{k\in\mathbb{Z}}2^{sj}\left|\left\langle y,\psi_{l,j,k}\right\rangle\right|<\infty$ and $P_n y$ is defined by(15).

Theorem 3: *Suppose that the exact solution* $y(x)$ *in equation(1) is approximated by(19), and the unknown coefficients* $\alpha_{l,n,k}$ *are obtained by solving the matrix equation(23). In addition, we are assuming that the kernel function* $k(x,t)$ *in equation(1) is continuous in* $D=\{(x,t), a\le x,t\le b\}$ *, and the solution* $y(x)$ *satisfies a decay condition with wavelet characterization of Sobolev space* $B_{2,2}^{s}$ *. Then the error* $\|e(x)\|_2=\|y-P_n y\|_2$ *is* $O(2^{-n})$.

Proof: subtracting equation(19) from equation(1), and then applying the L_2 *norm, we get the following equation*

$$\begin{aligned}\|e_n(x)\|_{x\in[a,b]} &\le \max_{i,x_i\in[a,b]}\|e_n(x_i)\| = \max_{i,x_i\in[a,b]}\|y(x_j)-P_n y(x_i)\| = \max_{i,x_i\in[a,b]}\left\|\left(\int_a^{x_i} k(x_i,t)\left(y-P_n y(t)\right)dt\right)\right\| \\ &\le \max_{i,x_i\in[a,b]}\left\|\int_a^{x_i} k(x_i,t)\,dt\right\| \max_{i,x_i\in[a,b]}\left\|\left(\int_a^{x_i} y(t)-P_n y(t)\right)dt\right\| \\ &= c_1 \max_{i,x_i\in[a,b]}\left\|\left(\int_a^{x_i} y(t)-\widetilde{P_n y(t)}\right)dt\right\|\end{aligned} \tag{32}$$

where $c_1=\max_{i,x_i\in[a,b]}\left\|\int_a^{x_i} k(x_i,t)dt\right\|$, *adding and subtracting equation(15) into equation(32) to obtain the following inequality*

$$\begin{aligned}\|e_n(x)\|_{x\in[a,b]} &\le c_1 \max_{i,x_i\in[a,b]}\left\|\int_a^{x_i}\left(y(t)-P_n y(t)+P_n y(t)-P_n y(t)\right)dt\right\| \\ &= c_1 \max_{i,x_i\in[a,b]}\left(\left\|\int_a^{x_i}\left(y-P_n y(t)dt\right)+\int_a^{x_i}\left(P_n y(t)-P_n y(t)dt\right)\right\|\right) \\ &\le c_1 \max_{i,x_i\in[a,b]}\left(\left\|\int_a^{x_i}\left(y-P_n y(t)dt\right)\right\|+\left\|\int_a^{x_i}\left(P_n y(t)-P_n y(t)dt\right)\right\|\right) \\ &\le c_1 \max_{i,x_i\in[a,b]}\left(\int_a^{x_i}\|y-P_n y\|dt+\int_a^{x_i}\left\|\widetilde{P_n y(t)}-P_n y(t)\right\|dt\right)\end{aligned} \tag{33}$$

Now,

$$\begin{aligned}\int_a^{x_i}\left\|P_n y(t)-P_n y(t)\right\| &\le \sum_{l,n,k}\left(\alpha_{l,n,k}-\langle y,\psi_{l,n,k}\rangle\right)\left\|\int_a^b \psi_{l,n,k}(t)dt\right\| \le \sum_{l,n,k} c_{l,n,k}\|\psi_{l,n,k}\|_1 \\ &= c_2 \max_l \|\psi_l\|_1 2^{-n/2}\end{aligned} \tag{34}$$

By using equation (34) and(31), we obtain that, $\|e_n(x)\|$ *is* $O(2^{-n})$.

6. NUMERICAL EXPERIMENTS

Example 6: In equation(1), assume that $f(x)=e^{x}(1-x), k(x,t)=e^{x-t}$ and the exact solution is $y(x)=e^{x}$. The numerical results exists in Table 1 with different values of n.

Example 7: In equation(1), assume that $f(x)=x\sin x, k(x,t)=\cos(x-t)$ and the exact solution is $y(x)=2\sin x$. The numerical results exists in Table 1 with different values of n.

Example 8: In equation(2), assume that $f(x)=1-e^{-x}-x, k(x,t)=1+x-t$ and the exact solution is $y(x)=xe^{-x}$. The numerical results exists in Table 1 with different values of n , such that e_n is absolute error.

Example 9: In equation(3), suppose $k(x,t)=\sin(x-t)+1$, $h(y(x))=\cos(y(x))$ and $f(x)=\frac{-1}{2}(2+x)\sin x$ with the exact solution $y(x)=x$.

Table 1: Numerical results for Examples 6

		UEP			OEP	
x	Exact value	B2, $n=1$	B2 , $n=2$	B4, $n=1$	B2 $n=1$	B2 , $n=2$
0	1	0.992397	0.998376	1	0.999878	0.999978
0.1	1.105170918	1.1061	1.10508	1.10517	1.10515	1.105166
0.2	1.221402758	1.21981	1.2221	1.2214	1.22144	1.221432
0.3	1.349858808	1.34951	1.35064	1.34986	1.34979	1.349797
0.4	1.491824698	1.4952	1.49179	1.49182	1.49203	1.492036
0.5	1.648721271	1.64089	1.64657	1.64872	1.6481	1.648171
0.6	1.822118800	1.82557	1.82195	1.82212	1.82175	1.822175
0.7	2.013752707	2.01025	2.01489	2.01375	2.01502	2.015021
0.8	2.225540928	2.22371	2.22681	2.22554	2.22675	2.226552
0.9	2.459603111	2.46597	2.45954	2.4596	2.45958	2.459578
1	2.718281828	2.70822	2.71534	2.71828	2.71513	2.715122

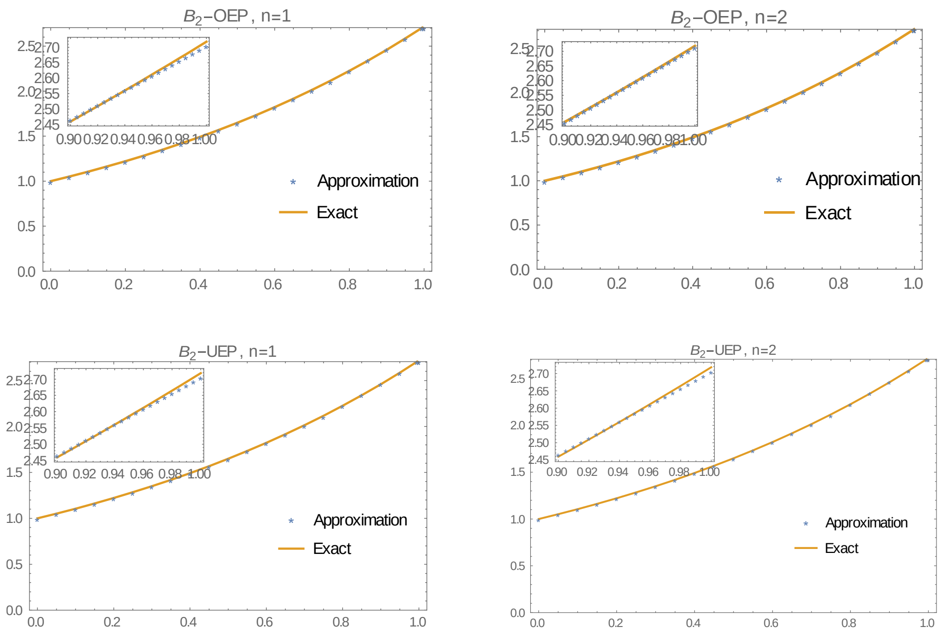

Figure 6: The graph of the exact solution $y(x)$ and the approximation $y_n(x)$ for example 6, with $n=1,2$ by using framelets generated by OEP and UEP.

Table 2: Numerical results for example 7

		UEP			OEP	
X	Exact	B_2, $n=1$	B_2, $n=2$	B_4, $n=1$	B_2, $n=1$	B_2, $n=2$
0.	0	0.000516	0.0000923	$-7.6*10^{-8}$	0.000117063	$6.77199*10^{-7}$
0.1	0.1996668333	0.199368	0.199736	0.199667	0.199723	0.199667
0.2	0.3973386616	0.398221	0.39713	0.397339	0.397117	0.397322
0.3	0.5910404133	0.590718	0.590666	0.59104	0.590621	0.59107
0.4	0.7788366846	0.776861	0.778807	0.778837	0.778767	0.77872
0.5	0.9588510772	0.963003	0.960084	0.958851	0.960302	0.959215
0.6	1.129284947	1.12655	1.12938	1.12928	1.12851	1.12952
0.7	1.288435374	1.29009	1.28768	1.28844	1.28975	1.28762
0.8	1.434712182	1.43514	1.43386	1.43471	1.4335	1.43392
0.9	1.566653819	1.5617	1.56667	1.56665	1.56209	1.56667
1.	1.682941970	1.8827	1.68475	1.68294	1.69067	1.68491

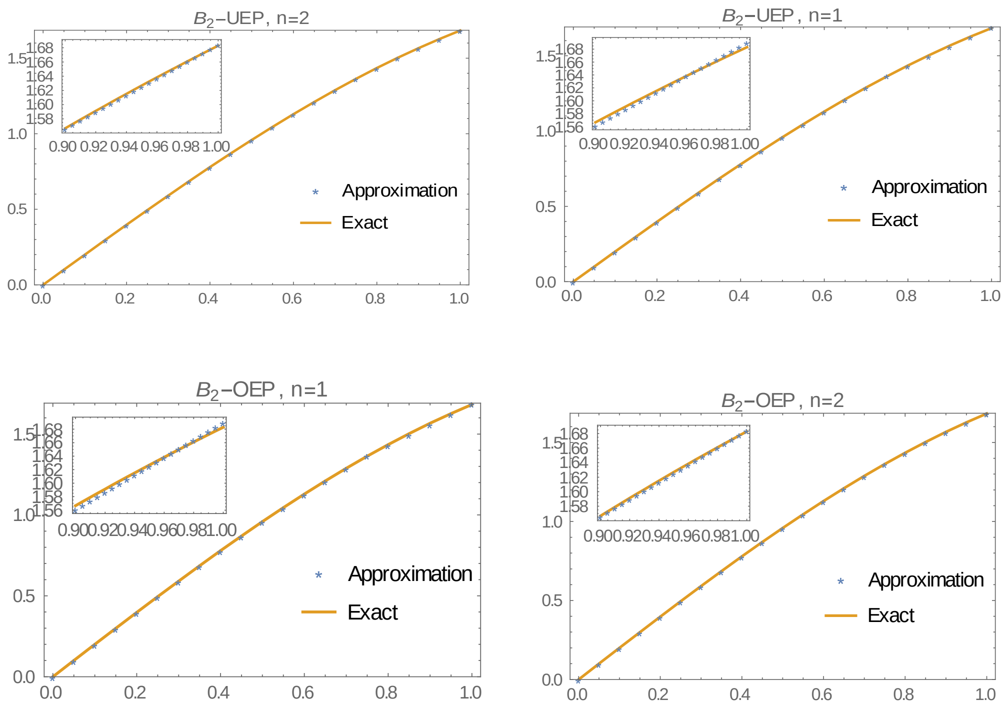

Figure 7: The graph of the exact solution $y(x)$ and the approximation $y_n(x)$ for example 7, with $n=1,2$ by using framelets generated by OEP and UEP.

Table 2: Numerical results for example 8

		UEP			OEP	
X	Exact	B2, $n=1$	B2, $n=2$	B4, $n=1$	B2, $n=1$	B2, $n=2$
0.	0.	0.00955953	0.0024947	1.29861*10-7	0.00251063	0.000161887
0.1	0.0904837	0.0865251	0.0904959	0.0904838	0.0904815	0.09049
0.2	0.163746	0.163491	0.162874	0.163746	0.162892	0.163691
0.3	0.222245	0.223165	0.221552	0.222246	0.221538	0.222284
0.4	0.268128	0.265547	0.268242	0.268128	0.268208	0.267982
0.5	0.303265	0.30793	0.304448	0.303265	0.30468	0.303691
0.6	0.329287	0.327664	0.329283	0.329287	0.328906	0.329377
0.7	0.34761	0.347398	0.347229	0.34761	0.347634	0.347197
0.8	0.359463	0.359842	0.359166	0.359463	0.359139	0.359192
0.9	0.365913	0.364996	0.365965	0.365913	0.365254	0.365943
1.	0.367879	0.37015	0.368484	0.367879	0.371369	0.368545

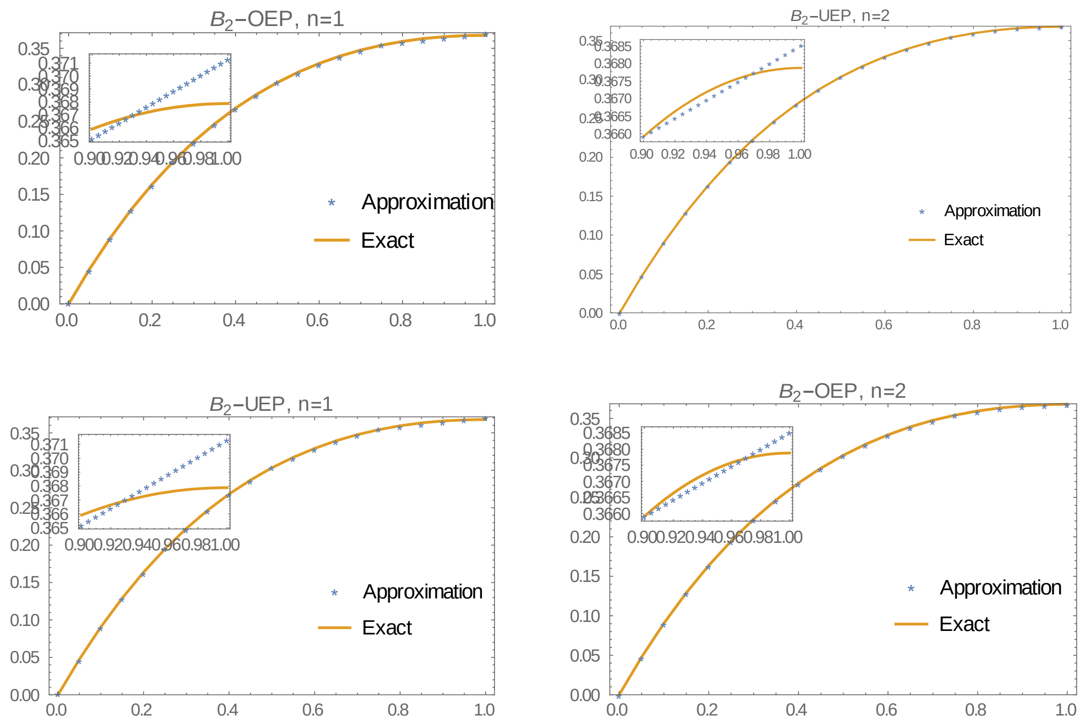

Figure 8: The graph of the exact solution $y(x)$ and the approximation $y_n(x)$ for example 8, with $n=1,2$ by using framelets generated by OEP and UEP.

Table 4: Numerical results for example 9.

		UEP		OEP		
x	Exact	$B_2,\ n=1$	$B_2,\ n=2$	$B_4,\ n=1$	$B_2,\ n=1$	$B_2,\ n=2$
0.	0.	0.0101809	0.005089	0.000065	0.0513185	0.0127835
0.1	0.1	0.12093	0.0995093	0.1	0.0997072	0.0999711
0.2	0.2	0.199293	0.202822	0.2	0.2027	0.200187
0.3	0.3	0.299051	0.301834	0.3	0.301896	0.299899
0.4	0.4	0.405449	0.399845	0.4	0.399964	0.400335
0.5	0.5	0.490209	0.497603	0.5	0.496939	0.499103
0.6	0.6	0.603274	0.599963	0.6	0.600873	0.599803
0.7	0.7	0.700174	0.700685	0.7	0.699827	0.700733
0.8	0.8	0.79953	0.800551	0.8	0.800602	0.800514
0.9	0.9	0.901514	0.899923	0.9	0.901162	0.89995
1.	1.	0.995835	0.998946	1.	0.994259	0.999861

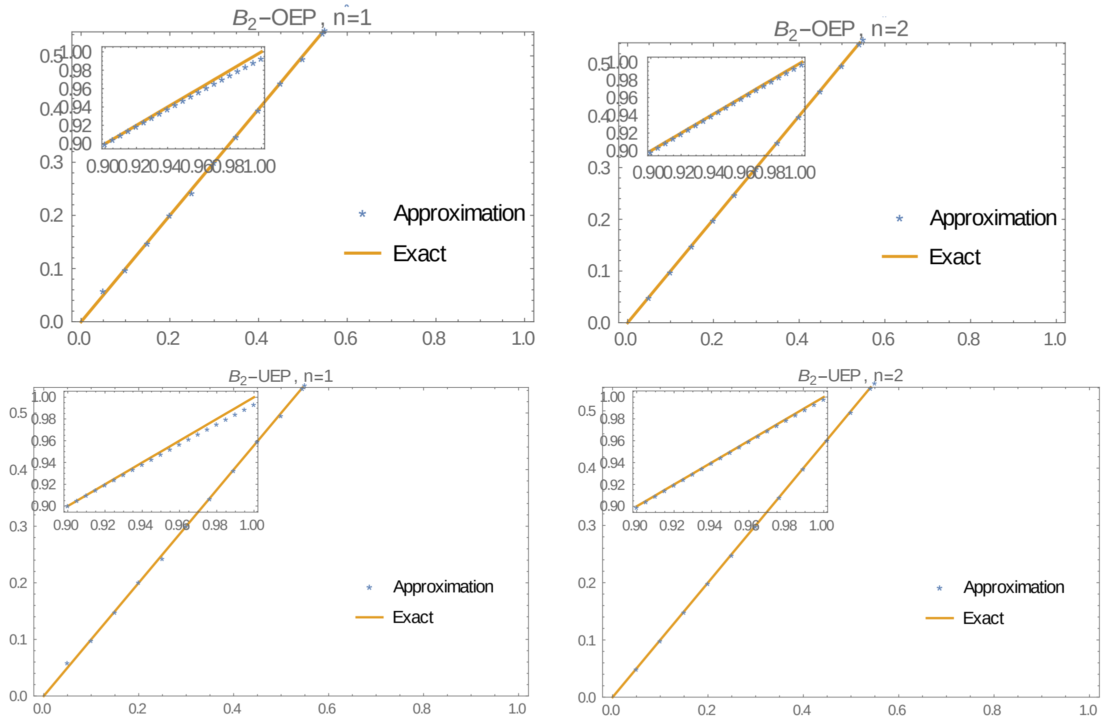

Figure 9: The graph of the exact solution $y(x)$ and the approximation $y_n(x)$ for example 6, $n=1,2$ by using framelets generated by OEP and UEP.

CONCLUSION

In this work, we use our interpolation method to solve Volterra-integral equations. Our method, is based on the framelets system, which is generated by using the unitary extension principle method and Oblique extension principle, applied to the refinable B-spline function of different orders. Clearly, the solution by using framelets generated by OEP gives better results than framelets generated by UEP, and that is, because the OEP has vanishing moments. And then compare our results with those in [7] and [6]. It turns out our method is more efficient with better accuracy. Moreover, our method can be applied to different kind of integral equations, as well as integral-algebraic equations, singular integral equations, partial differential equations and so integro- differential equations. In this article, we construct many framelets systems and we give a direct expression for the framelets system, so it is easy to use for the function approximations.

REFERENCES

[1] T. A. Burton, Volterra Integral and Differntial Equations, New YorkAcademic Press, 1983.

[2] F. Brauer, C. Castillo-Chavez, Mathematical Models in Population Biolgy and Epidemiology, New York: Springer-Verlang, 2001.

[3] F.Brauer, "On a nonlinear Integral Equation for Population Growth Problems," *Journal on Mathematical Analysis,* vol. 6, no. 2, pp. 312-317, 1975.

[4] K. Maleknejad, N. Aghazadeh, "Numerical Solution of Volterra Integral Equations of the Second Kind with Convolution Kernel by Using Taylor-Series Method," *Applied Mathematics and Computation* , vol. 161, no. 3, pp. 915-922, 2005.

[5] J. Steinberg, "Numerical Solution of Volterra Integral Equation," *Numerische Mathematik,* vol. 19, pp. 212-217, 1972.

[6] Yousef Al-Jarrah, En-Bing Lin, "Numerical Solution of Fredholm-Volterra Equations by Using Scaling Function Interpolation Method," *Applied Mathematics,* vol. 4, pp. 204-209, 2013.

[7] Inderdeep Singh, Sheo Kumar, "Haar Wavelet Method for Some Nonlinear Volterra Integral Equations of the First Kind," *Journal of Computational and Applied Mathematics,* vol. 36, pp. 541-552, 2016.

[8] C. Kasumo, "On the Approximate Solutions of Linear Volterra Integral Equations of the First Kind," *Applied Mathematical Sciences,,* vol. 14, no. 20, pp. 141 - 153, 2020.

[9] M. Mohammad, "A Numerical Solution of Fredholm Integral Equations of the Second Kind Based on Tight Framelets Generated by Oblique Extension Principle," *Symmetry,* vol. 11, no. 854, 2019.

[10] R. J. Duffin, A. C. Schaeffer, "A Class of Nonharmonic Fourier Series," *Transaction of the Americal Mathematica Society* , vol. 72, pp. 341-366, 1952.

[11] Daubechies, I., A. Grossman, Y. Meyer, "Painless nonorthogonal Expansions," *Journal of Mathematical Physics,* vol. 27, no. 5, pp. 1271-1283, 1986.

[12] O. Christensen, An Introduction to Frames and Riesz Bases, New York: Springer Science+Busnises Media, 2002.

[13] S. G. Mallat, "Multiresolution Approximations and Wavelet Orthpnormal Bases of L2(R)," *Transaction of the Americal Mathematical Society,* vol. 315, no. 1, pp. 69-87, 1989.

[14] Y. Meyer, Wavelets and Operators, New York: Press Syndicate of the University of Cambridge, 1992.

[15] Bin Dong, Zuowei Shen, MRA-Based Wavelet Frames and Application, IAS Lecture Note Series, 2013.

[16] Ingrid Daubechies, Bin Han, Amos Ron, Shen Zuowei, "Framelets: MRA- Based Constructions of Wavelet Frames," *Applied and Computational Harmounic Analysis,* vol. 14, pp. 1-46, 2003.

[17] C. d. Boor, "A paractical guide to splines," *Applied Mathematical Sciences,* vol. 27, 2001.

[18] Amos Ron, Zuowei Shen, "Affine systems in L_2 (R^d): The Analysis of Analysis Operator," *Journal of Funtional Analysis,* vol. 148, pp. 408-447, 1997.

[19] Bin Han, Qun Mo, Zhenpeng Zhao, Xiaosheng Zhuang, "Directional Compactly Supported Tensor Product Complex Tight Framelets with Applications to Image Denoising and Inpainting," *Journal on Imaging Sciences,* vol. 12, no. 4, pp. 1739-1771, 2019.

[20] Inderdeep Singh, Sheo Kumar, "Haar Wavelet Method for some Nonlinear Volterra integral Equations of the First Kind," *Journal of Computational and Applied Mathematics,* vol. 292, pp. 541-552, 2016.