Gracefulness of a New Class from Copies of $kC_4 \cup P_{2n}$ and $P_2 * nC_3$

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Abstract

Given a graph $G$ with $q$ edges, a labeling of the nodes with distinct integers from the set $\{0, 1, 2, \ldots, q\}$ induces an edge labeling where the label of an edge is the absolute difference of the labels of the two nodes incident to that edge. Such a labeling is graceful if the edge labels are distinct. A graph $G$ is called graceful if there exist a graceful labeling of $G$. In this paper, the gracefulness of a new class, namely $kC_4 \cup P_{2n}$ and $P_2 * nC_3$ are obtained.

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Introduction

When studying graceful labeling, only simple graph is considered. A graceful labeling of an undirected graph $G$ with $q$ edges is a one-to-one function from the set of vertices of $G$ to the set $\{0, 1, 2, \ldots, q\}$ such that the induced edge labels are all distinct. An induced edge label is the absolute value of the difference between the two end vertex labels. Graceful labeling have applications in coding theory, x-ray, crystallography, radar, astronomy, circuit design and communication networks, addressing and data base management.

A complete and current summary of graceful and non-graceful results along with some unproven conjectures can be found in Gallian’s dynamic survey [1] of graceful labeling.

Solairaju and Ambika [2] have proved that the $n$ copies of cycles $C_4$ is graceful.
Solairaju and Ambika [3] have showed that the connected graph $E_k \ast S_{n+1}$ (copies of $k$ number of stars) is a graceful graph. Also they [4] obtained a result that $nC_3$ ($n$ number of $C_3$) and its mirror image are graceful. Similarly they got double $nC_3$ is graceful.

Solairaju and Ambika [5] have showed that a unicycle graph from Copies of Stars on cycles is a graceful graph. Solairaju and Ambika [6] have showed that the connected graph $P_n \ast kC_4$ and Mirror image of $P_n \ast kC_4$ are graceful. Also they [7] obtained a result that Copies of $E_n$-Tree and $nV_3$ are graceful graph.

**Preliminaries**

**Definition 2.1**
The graceful labeling of a graph $G$ with $q$ edges is a function $f : V(G) \rightarrow \{0, 1, 2, \ldots, q\}$ such that distinct vertices receive distinct numbers and $\{ |f(u) - f(v)| : uv \in E(G)\} = \{1, 2, 3, \ldots, q\}$. A graph is graceful if it has a graceful labeling.

**Definition 2.2**
A cycle $C$ in a graph is a connected subgraph in which the degree of every vertex in $C$ is two. A cycle with $n$ vertices is denoted by $C_n$.

**Definition 2.3**
A path in a graph is a sequence of vertices such that from each of its vertices there is an edge to the next vertex in the sequence. A path with $n$ vertices is denoted by $P_n$.

**Definition 2.4**
The graph $kC_4 \uplus P_{2n}$ is defined as a connected simple graph, which contains the vertex set $V = \{v_1, v_2, v_3, \ldots, v_{4k}\}$ with $4k$ vertices and edges $q = (4k + n)$ such that $v_i$ is adjacent to $v_{i+1}$ and $v_{i+4}$ for $i = 1, 2, 3, \ldots, 4k-3$ and $v_{4i}$ is adjacent to $v_{4i-1}$ and $v_{4i+4}$ for $i = 1, 2, 3, \ldots, 4k$ and $k$ varies on $1, 2, 3, \ldots$ Also $v_1$ and $v_4$ are connected. The arbitrary labeling of the graph $kC_4 \uplus P_{2n}$ is shown in the following figure 1.

![Figure 1: The graph $kC_4 \uplus P_{2n}$](image)
Definition 2.5
The graph $P_2 * nC_3$ is defined as a connected simple graph, which contains the vertex set $V = \{v_0, v_1, v_2, \ldots, v_{n+1}\}$ with $(n+2)$ vertices and edges $q = 2n + 1$ such that $v_0$ and $v_{n+1}$ are adjacent to $v_1, v_2, v_3, \ldots, v_{n}$. The arbitrary labeling of the graph $P_2 * nC_3$ is shown in the following figure 2.

![Figure 2: The graph $P_2 * nC_3$](image)

Definition 2.6
Let $T$ be a tree and $u$ and $v$ be two adjacent vertices in $T$. Let there be two pendant vertices $x$ and $y$ in $T$ such that the length of $u - x$ path is equal to the length of $v - y$ path. If the edge $uv$ is deleted from $T$ and $x$, $y$ are joined by an edge $xy$, then such a transformation of $T$ is called an elementary parallel transformation (or an ept) and the edge $uv$ is called a transformable edge. If by a sequence of ept’s $T$ can be reduced to a path then $T$ is called a TP-tree (transformed tree). A TP-tree is shown in figure 3.

![Figure 3: TP-Tree](image)

Gracefulness of $kC_4 \lor P_{2n}$ and $P_2 * nC_3$

Theorem 3.1
The connected graph $kC_4 \lor P_{2n}$ is graceful.
Proof:
Gracefulness of the graph $kC_4 \lor P_{2n}$ with $(4k + n)$ edges such that the vertex set $V = \{v_1, v_2, \ldots, v_{4k}\}$ is labeled in the following manner which is shown in figure 1.
To be graceful of above graph, define a map $f: V \rightarrow \{0, 1, 2, \ldots, q\}$ where $q = (4k + n)$ by

\[
\begin{align*}
    f(v_{8i - 7}) &= 5i - 5 \text{ for } i = 1, 2, 3, \ldots \\
    f(v_{8i - 6}) &= q - 6i + 6 \text{ for } i = 1 \& 2 \\
    f(v_{8i - 5}) &= 5i - 4 \\
    f(v_{8i - 4}) &= q - 5i + 3 \\
    f(v_{8i - 3}) &= q - 5i + 1 \\
    f(v_{8i - 2}) &= 5i - 1 \\
    f(v_{8i - 1}) &= q - 5i + 2 \\
    f(v_{8i}) &= 5i - 2 \\
    f(v_{8i + 2}) &= q - 5i - 1 \text{ for } i = 1, 2, 3, \ldots
\end{align*}
\]

Also $f(v_i, v_j)$ is the absolute difference of $f(v_i)$ and $f(v_j)$. The graph $kC_4 \lor P_{2n}$ is graceful.
Hence this labeling is graceful.

Example 3.1 The connected graph $3C_4 \lor P_{2(2)}$ is graceful.
Gracefulness of the graph $3C_4 \lor P_{2(2)}$ with 14 edges such that the vertex set $V = \{v_1, v_2, \ldots, v_{12}\}$ is labeled in the following manner which is shown in figure 4.
The vertex labels are calculated as follows.
When $n = 2$ and $k = 3$, $q = 4k + n = 14$
For $i = 1$; $v_1 = 0, \quad v_2 = 14, \quad v_3 = 1, \quad v_4 = 12, \quad v_5 = 10, \quad v_6 = 4, \quad v_7 = 11, \quad v_8 = 3,$
For $i = 2$; $v_9 = 5, \quad v_{10} = 8, \quad v_{11} = 6, \quad v_{12} = 7$

Figure 4: The graph $3C_4 \lor P_{2(2)}$
Theorem 3.2
The connected graph $P_2 \ast nC_3$ is graceful.

Proof:
Gracefulness of the graph $P_2 \ast nC_3$ with $(2n+1)$ edges such that the vertex set $V = \{ v_0, v_1, v_2, \ldots, v_{n+1} \}$ is labeled in the following manner which is shown in figure 2. To be graceful of above graph, define a map $f: V \to \{0,1,2,\ldots,q\}$ where $q = (2n+1)$ by $f(v_0) = 0$ ; $f(v_{n+1}) = q$ and $f(v_i) = i$ for $i=1,2,3,\ldots,n$.

Also $f(v_i,v_j)$ is the absolute difference of $f(v_i)$ and $f(v_j)$. The graph $P_2 \ast nC_3$ is graceful. Hence this labeling is graceful.

Example: 3.2
The connected graph $P_2 \ast 3C_3$ is graceful.
Gracefulness of the graph $P_2 \ast 3C_3$ with 7 edges such that the vertex set $V = \{v_0,v_1,v_2,v_3,v_4\}$ is labeled in the following manner which is shown in figure 5.
When $n =3$ and $q =7$, the vertex labels are calculated as follows.
$v_0 = 0$ ; $v_1 = 1$ ; $v_2 =2$ ; $v_3 = 3$ ; $v_4 = 7$

Example 3.2: The connected graph $P_2 \ast 3C_3$ is graceful.

Theorem 3.3
The connected TP-tree is graceful.

Proof:
Gracefulness of the TP-tree with $(4n+3)$ edges such that the vertex set $V = \{v_1,v_2,\ldots,v_{4n+4}\}$ is labeled in the following manner which is shown in figure 6.
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**Figure 6: TP-Tree with ordinary labeling**

To be graceful of graph, define a map \( f: V \rightarrow \{0,1,2,\ldots,q\} \) where \( q = (4n+3) \) by

\[
\begin{align*}
    f(v_8i – 7) & = q–4i+4 ; \\
    f(v_8i – 6) & = 4i–4 ; \\
    f(v_8i – 5) & = q–4i+3 ; \\
    f(v_8i – 4) & = 4i–3 \\
    f(v_8i – 3) & = 4i–1  ; \\
    f(v_8i – 2) & = q–4+1 ; \\
    f(v_8i ) & = q–4i+2 \\
\end{align*}
\]

Also \( f(v_iv_j) \) is the absolute difference of \( f(v_i) \) and \( f(v_j) \). The graph TP-tree is graceful. Hence this labeling is graceful.

**Example: 3.3** The connected TP-tree with 16 vertices and 15 edges is graceful.

Gracefulness of the graph with 15 edges such that the vertex set \( V= \{v_1,v_2,v_3,\ldots,v_{16}\} \) is labeled in the following manner which is shown in figure 7

**Figure 7: TP-Tree with 16 vertices**
References
