Homomorphism in Intuitionistic Fuzzy Automata

A. Uma and M. Rajasekar

Mathematics Section,
Faculty of Engineering and Technology,
Annamalai University, Annamalainagar,
Chidambaram, Tamil Nadu, India - 608 002.
E-mail: uma.feat@yahoo.com, mrajdiv@yahoo.com

Abstract

In this paper we introduce some properties of homomorphism in Intuitionistic Fuzzy Automata.

AMS subject classification: 18B20.
Keywords: Intuitionistic Fuzzy Automata.

1. Introduction

The concept of intuitionistic fuzzy set was introduced by K.T. Atanassov [1], as a generalization of the notion of fuzzy set. Using the notion of intuitionistic fuzzy sets [1], it is possible to obtain intuitionistic fuzzy language [3]. J.B. Jun [2] defined a homomorphism in intuitionistic fuzzy automata. We introduce some properties of an intuitionistic fuzzy automata with homomorphism.

2. Preliminaries

2.1. Fuzzy Set [1]

Let a set ‘E’ be fixed. A Fuzzy set ‘A’ in ‘E’ is an object having the form \( A = \{< x, \mu_A(x) > | x \in E \} \) where, the function \( \mu_A(x) : E \to [0, 1] \) define the degree of membership of the element \( x \in E \) to the set ‘A’ and for every \( x \in E \), \( 0 \leq \mu_A(x) \leq 1 \).
2.2. Automata [4]

A non-deterministic finite Automaton is a triple \( A = (Q, X, \delta) \) where \( Q \) is a finite set (the set of states), \( X \) is an alphabet and \( \delta \) is a subset of \( Q \times X \times Q \), called the set of transitions. Two transitions \((p, a, q)\) and \((p', a', q')\) are consecutive if \( q = p' \).

Consider a word \( a_0, a_1, \ldots, a_{n-1} \) with \( a_i \in X \). A run \( \alpha \) in \( A \) is sequence of states \( q_0 a_0 \rightarrow q_1 a_1 \rightarrow q_2, \ldots, q_{n-1} a_{n-1} \rightarrow q_n \).

2.3. Fuzzy Automata [3]

A Fuzzy Automaton is a triple \( A = (Q, X, \delta) \), where \( Q \) is a nonempty set of states of \( A, X \) is a monoid (the input monoid of \( M \)), with identity \( e \), and \( \delta \) is a Fuzzy subset of \( Q \times X \times Q \), i.e., a map \( \delta : Q \times X \times Q \rightarrow [0, 1] \), such that \( \forall q, p \in Q, \forall x, y \in X. \)

\[
\delta(q, e, p) = \begin{cases} 
1 & \text{if } q = p \\
0 & \text{if } q \neq p 
\end{cases}
\]

and

\[
\delta(q, xy, p) = \vee \{\delta(q, x, r) \wedge \delta(r, y, p) : r \in Q\}
\]

2.4. Intuitionistic Fuzzy Set [1]

Let a set ‘E’ be fixed. An Intuitionistic Fuzzy set ‘A’ in ‘E’ is an object having the form \( A = \{< x, \mu_A(x), \gamma_A(x) > | x \in E \} \) where, the functions \( \mu_A(x) : E \rightarrow [0, 1] \) and \( \gamma_A(x) : E \rightarrow [0, 1] \) define the degree of membership and the degree of nonmembership of the element \( x \in E \) to the set ‘A’, the subset of ‘E’ respectively, and for every \( x \in E, 0 \leq \mu_A(x) + \gamma_A(x) \leq 1 \).

2.5. Intuitionistic Fuzzy Automata [2]

An Intuitionistic Fuzzy Automaton is a triple \( A = (Q, X, \delta) \), where \( Q \) is a set of states of \( A, X \) is a monoid (the input monoid of \( M \) with identity \( e \)), and \( \delta \) is an Intuitionistic Fuzzy subset of \( Q \times X \times Q \), such that \( \forall q, p \in Q, \forall x, y \in X. \)

\[
\delta_1(q, e, p) = \begin{cases} 
1 & \text{if } q = p \\
0 & \text{if } q \neq p 
\end{cases}
\]

\[
\delta_2(q, e, p) = \begin{cases} 
1 & \text{if } q \neq p \\
0 & \text{if } q = p 
\end{cases}
\]

\[
\delta_1(q, xy, p) = \vee \{\delta_1(q, x, r) \wedge \delta_1(r, y, p) : r \in Q\},
\]

and

\[
\delta_2(q, xy, p) = \wedge \{\delta_2(q, x, r) \vee \delta_2(r, y, p) : r \in Q\}.
\]
2.6. Homomorphism between Automata

Let $A_1 = (Q_1, X_1, \delta_1)$ and $A_2 = (Q_2, X_2, \delta_2)$ be two finite automata. A pair $(\alpha, \beta)$ of mappings, $\alpha : Q_1 \to Q_2$ and $\beta : X_1 \to X_2$ is called a homomorphism, written $(\alpha, \beta) : A_1 \to A_2$, if

$$\alpha(\delta_1(q_1, a)) = \delta_2(\alpha(q_1), \beta(a))$$

$$\forall q_1 \in Q_1 \text{ and } \forall a \in X_1.$$

2.7. Homomorphism between Fuzzy Automata [3]

Let $A_1 = (Q_1, X_1, \mu_1)$ and $A_2 = (Q_2, X_2, \mu_2)$ be ffsms. A pair $(\alpha, \beta)$ of mappings, $\alpha : Q_1 \to Q_2$ and $\beta : X_1 \to X_2$ is called a homomorphism, written $(\alpha, \beta) : A_1 \to A_2$, if

$$\mu_1(q, x, p) \leq \mu_2(\alpha(q), \beta(x), \alpha(p)) \forall q, p \in Q_1 \text{ and } \forall x \in X_1.
$$

The pair $(\alpha, \beta)$ is called a strong homomorphism if

$$\mu_2(\alpha(q), \beta(x), \alpha(p)) = \vee \{\mu_1(q, x, t)|t \in Q_1, \alpha(t) = \alpha(p)\}$$

$$\forall q, p \in Q_1 \text{ and } \forall x \in X_1.$$


Let $A_1 = (Q_1, X_1, \mu_1, \gamma_1)$ and $A_2 = (Q_2, X_2, \mu_2, \gamma_2)$ be iffsms. A pair $(\alpha, \beta)$ of mappings, $\alpha : Q_1 \to Q_2$ and $\beta : X_1 \to X_2$ is called an intuitionistic fuzzy homomorphism, written $(\alpha, \beta) : A_1 \to A_2$, if

$$\mu_1(q, x, p) \leq \mu_2(\alpha(q), \beta(x), \alpha(p))$$

$$\gamma_1(q, x, p) \geq \gamma_2(\alpha(q), \beta(x), \alpha(p)) \forall p, q \in Q_1$$

and $\forall x \in X_1$.

The pair $(\alpha, \beta)$ is called a strong intuitionistic fuzzy homomorphism if

$$\mu_2(\alpha(q), \beta(x), \alpha(p)) = \vee \{\mu_1(q, x, t)|t \in Q, \alpha(t) = \alpha(p)\}$$

$$\gamma_2(\alpha(q), \beta(x), \alpha(p)) = \wedge \{\gamma_1(q, x, t)|t \in Q, \alpha(t) = \alpha(p)\}$$

$$\forall q, p \in Q_1 \text{ and } \forall x \in X_1.$$

3. Some properties of Homomorphism in Intuitionistic Fuzzy Automata

**Lemma 3.1.** Let $A_1 = (Q_1, X_1, \mu_1, \gamma_1)$ and $A_2 = (Q_2, X_2, \mu_2, \gamma_2)$ be two iffsms. Let $(\alpha, \beta) : A_1 \to A_2$ be a strong intuitionistic homomorphism. Then $\forall q, r, x \in Q_1, \forall x \in X_1$, if

$$\mu_2(\alpha(q), \beta(x), \alpha(r)) > 0$$

$$\gamma_2(\alpha(q), \beta(x), \alpha(r)) < 1,$$
Theorem 3.3. Let \( \mu_1(q, x, t) \geq 0, \gamma_1(q, x, t) < 1 \) and \( \alpha(t) = \alpha(r) \). Furthermore, \( \forall p \in Q \) if \( \alpha(p) = \alpha(q) \), then
\[
\mu_1(q, x, t) \geq \mu_1(p, x, r)
\]
and
\[
\gamma_1(q, x, t) \leq \gamma_1(p, x, r).
\]

Proof. Let \( p, q, r \in Q_1, x \in X_1 \), and
\[
\mu_2(\alpha(q), \beta(x), \alpha(r)) > 0
\]
\[
\gamma_2(\alpha(q), \beta(x), \alpha(r)) < 1.
\]
Then
\[
\forall \{\mu_1(q, x, s)|s \in Q_1, \alpha(s) = \alpha(r)\} > 0
\]
\[
\land \{\gamma_1(q, x, s)|s \in Q_1, \alpha(s) = \alpha(r)\} < 1.
\]
Since \( Q_1 \) is finite, \( \exists t \in Q_1 \) such that \( \alpha(t) = \alpha(r) \) and
\[
\mu_1(q, x, t) = \forall \{\mu_1(q, x, s)|s \in Q_1, \alpha(s) = \alpha(r)\} > 0
\]
\[
\gamma_1(q, x, t) = \land \{\gamma_1(q, x, s)|s \in Q_1, \alpha(s) = \alpha(r)\} < 1
\]
suppose \( \alpha(p) = \alpha(q) \). Then
\[
\mu_1(q, x, t) = \mu_2(\alpha(q), \beta(x), \alpha(r))
\]
\[
= \mu_2(\alpha(p), \beta(x), \alpha(r))
\]
\[
\geq \mu_1(p, x, r)
\]
\[
\gamma_1(q, x, t) = \gamma_2(\alpha(q), \beta(x), \alpha(r))
\]
\[
= \gamma_2(\alpha(p), \beta(x), \alpha(r))
\]
\[
\leq \gamma_1(p, x, r)
\]

Definition 3.2. Let \( A_1 = (Q_1, X_1, \mu_1, \gamma_1) \) and \( A_2 = (Q_2, X_2, \mu_2, \gamma_2) \) be two iffsms. Let \( (\alpha, \beta) : A_1 \rightarrow A_2 \) be an Intuitionistic fuzzy homomorphism. Define \( \beta^* : X_1^* \rightarrow X_2^* \) by \( \beta^*(\lambda) = \lambda \) and \( \beta^*(ua) = \beta^*(u)\beta(a) \forall u \in X_1^*, a \in X_1 \).

Theorem 3.3. Let \( A_1 = (Q_1, X_1, \mu_1, \gamma_1) \) and \( A_2 = (Q_2, X_2, \mu_2, \gamma_2) \) be two iffsms. Let \( (\alpha, \beta) : A_1 \rightarrow A_2 \) be an Intuitionistic fuzzy homomorphism. Then
\[
\mu_1^*(q, x, p) \leq \mu_2^*(\alpha(q), \beta^*(x), \alpha(p))
\]
\[
\gamma_1^*(q, x, p) \geq \mu_2^*(\alpha(q), \beta^*(x), \alpha(p))
\]
\[
\forall q, p \in Q_1 \text{ and } x \in X_1^*.
\]
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**Proof.** Let \( q, p \in Q_1 \) and \( x \in X^*_1 \). We prove the result by induction on \( |x| = n \). If \( n = 0 \), then \( x = \lambda \) and \( \beta^*(x) = \beta^*(\lambda) = \lambda \).

Now if \( q = p \), then

\[
\begin{align*}
\mu^*_1(q, \lambda, p) &= 1 = \mu^*_2(\alpha(q), \lambda, \alpha(p)) \\
\gamma^*_1(q, \lambda, p) &= 0 = \gamma^*_2(\alpha(q), \lambda, \alpha(p))
\end{align*}
\]

If \( q \neq p \), then

\[
\begin{align*}
\mu^*_1(q, \lambda, p) &= 0 \leq \mu^*_2(\alpha(q), \lambda, \alpha(p)) \\
\gamma^*_1(q, \lambda, p) &= 1 \geq \gamma^*_2(\alpha(q), \lambda, \alpha(p)).
\end{align*}
\]

Suppose now the result is true \( \forall y \in X^* \) such that \( |y| = n - 1, n > 0 \). Let \( x = ya \) and \( y \in X^*_1, a \in X_1 \) and \( |y| = n - 1 \).

Now

\[
\begin{align*}
\mu^*_1(q, x, p) &= \mu^*_1(q, ya, p) \\
&= \vee\{\mu^*_1(q, y, r) \wedge \mu^*_1(r, a, p) | r \in Q_1\} \\
&\leq \vee\{\mu^*_2(\alpha(q), \beta^*(y), \alpha(r)) \\
&\wedge \mu^*_2(\alpha(r), \beta(a), \alpha(p)) | r \in Q_1\} \\
&\leq \vee\{\mu^*_2(\alpha(q), \beta^*(y), r') \\
&\wedge \mu^*_2(r', \beta^*(a), \alpha(p)) | r' \in Q_2\} \\
&= \mu^*_2(\alpha(q), \beta^*(y) \beta(a), \alpha(p)) \\
&= \mu^*_2(\alpha(q), \beta^*(ya), \alpha(p)) \\
&= \mu^*_2(\alpha(q), \beta^*(x), \alpha(p)).
\end{align*}
\]

and

\[
\begin{align*}
\gamma^*_1(q, x, p) &= \mu^*_1(q, ya, p) \\
&= \wedge\{\gamma^*_1(q, y, r) \vee \mu^*_1(r, a, p) | r \in Q_1\} \\
&\geq \wedge\{\gamma^*_2(\alpha(q), \beta^*(y), \alpha(r)) \\
\wedge \gamma^*_2(\alpha(r), \beta(a), \alpha(p)) | r \in Q_1\} \\
&\geq \wedge\{\gamma^*_2(\alpha(q), \beta^*(y), r') \\
\wedge \gamma^*_2(r', \beta^*(a), \alpha(p)) | r' \in Q_2\} \\
&= \gamma^*_2(\alpha(q), \beta^*(y) \beta(a), \alpha(p)) \\
&= \gamma^*_2(\alpha(q), \beta^*(ya), \alpha(p)) \\
&= \gamma^*_2(\alpha(q), \beta^*(x), \alpha(p)).
\end{align*}
\]

■
**Theorem 3.4.** Let $A_1 = (Q_1, X_1, \mu_1, \gamma_1)$ and $A_2 = (Q_2, X_2, \mu_2, \gamma_2)$ be two ifsms. Let $(\alpha, \beta) : A_1 \rightarrow A_2$ be an strong homomorphism. Then $\alpha$ is one-one if and only if

\[
\mu_2^*(q, x, p) = \mu_2^*(\alpha(q), \beta^*(x), \alpha(p))
\]

\[
\gamma_1 * (q, x, p) = \gamma_2 * (\alpha(q), \beta^*(x), \alpha(p))
\]

\[
\forall q, p \in Q_1 \text{ and } x \in X_1^*.
\]

**Proof.** Suppose $\alpha$ is one-one. Let $p, q \in Q_1$ and $x \in X_1^*$. Let $|x| = n$. We prove the result by induction on $n$. Let $n = 0$. Then $x = \lambda$ and $\beta^*(\lambda) = \lambda$.

Now $\alpha(q) = \alpha(p)$ iff $q = p$. Hence

\[
\mu_1^*(q, \lambda, p) = 1, \quad \gamma_1^*(q, \lambda, p) = 0
\]

iff

\[
\mu_2^*(\alpha(q), \beta^*(\lambda), \alpha(p)) = 1,
\]

\[
\gamma_2^*(\alpha(q), \beta^*(\lambda), \alpha(p)) = 0.
\]

Suppose the result is true $\forall y \in X_1^*, |y| = n - 1, n > 0$.

Let $x = ya$, $|y| = n - 1, y \in X^*_1, a \in X_1$.

Then

\[
\mu_2^*(\alpha(q), \beta^*(x), \alpha(p)) = \mu_2^*(\alpha(q), \beta(ya), \alpha(p))
\]

\[
= \mu_2^*(\alpha(q), \beta^*(y)\beta(a), \alpha(p))
\]

\[
= \land \{\mu_2^*(\alpha(q), \beta^*(y), \alpha(r))
\]

\[
\land \mu_2^*(\alpha(r), \beta(a), \alpha(p)) | r \in Q_1\}
\]

\[
= \land \{\mu_2^*((q,y,r) \land \mu_1(r,a,p)) | r \in Q_1\}
\]

\[
= \mu_1^*(q, ya, p)
\]

\[
= \mu_1^*(q, x, p).
\]

and

\[
\gamma_2^*(\alpha(q), \beta^*(x), \alpha(p)) = \gamma_2^*(\alpha(q), \beta(ya), \alpha(p))
\]

\[
= \gamma_2^*(\alpha(q), \beta^*(y)\beta(a), \alpha(p))
\]

\[
= \land \{\gamma_2^*(\alpha(q), \beta^*(y), \alpha(r))
\]

\[
\land \gamma_2 * (\alpha(r), \beta(a), \alpha(p)) | r \in Q_1\}
\]

\[
= \land \{\gamma_1^*((q,y,r) \land \gamma_1(r,a,p)) | r \in Q_1\}
\]

\[
= \gamma_1^*(q, ya, p)
\]

\[
= \gamma_1^*(q, x, p).
\]

Conversely, let $q, p \in Q_1$ and let $\alpha(q) = \alpha(p)$. Then

\[
1 = \mu_2^*(\alpha(q), \lambda, \alpha(p)) = \mu_1^*(q, \lambda, p)
\]

\[
0 = \gamma_2^*(\alpha(q), \lambda, \alpha(p)) = \gamma_1^*(q, \lambda, p).
\]

Hence $q = p$, i.e., $\alpha$ is one-one.
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References