On Q-Fuzzy N-Subgroup and Q-Fuzzy Ideal of An N-Group

Gopi Kanta Barthakur and Jugal Khargharia

Department of Mathematics, G. K. B. College, Morigaon, Assam, India
gopik2003@gmail.com
Department of Mathematics, S. P. P. college, Sivasagar, Assam, India

Abstract

In this paper, we shall study Q-fuzzy N-subgroup, Q-fuzzy normal subgroup and Q-fuzzy ideal of a near-ring group and investigate some of there properties.

Keywords: Near-ring, N-group, Q-fuzzy set, Q-fuzzy N-subgroup, Q-fuzzy ideal.

1. Introduction


2. Preliminaries:

2.1. Definition: Let N be a near-ring and E an additive group. Then E is said to be a near-ring group or an left N-group if there exist a mapping $N \times E \rightarrow E$, $(n, e) \rightarrow ne$

Such that

(i) $(n + m)e = ne + me$
(ii) $(nm)e = n(me)$
(iii) $1.e = e$ for all $n, m \in N$ and $e \in E$. 
Unless otherwise stated we denote the zero element of E by 0. Note that N can be considered as an N-group denoted by \(N^N\).

Throughout our discussion by an N-group we mean left N-group.

A subset S of N-group E is called an N-subgroup of E if S is a subgroup of (E, +) and \(NS \subseteq S\). A subgroup S of E is called normal subgroup if \(x + y - x \in S\) for all \(y \in S\), \(x \in E\) and a normal subgroup S of E is called an ideal of E if for \(n \in N\), \(x \in S\), a\( \in E\), \(n(a + x) - na \in S\).

2.2. Definition:[2] Let X be a non-empty set. A function \(\mu : X \rightarrow [0,1]\) is called a fuzzy subset of X.

2.3. Definition: If \(\mu\) is a fuzzy subset of a set X then the set \(\{x \in X : \mu(x) > 0\}\) is said to be support of \(\mu\) and is denoted by \(\mu^*\), and the set \(\{x \in X : \mu(x) \geq t, t \in [0,1]\}\), denoted by \(\mu_t\), is called level subset of X.

2.4. Definition: [7] A fuzzy subset \(\mu\) of an N-group E is said to be a fuzzy subgroup of E if for all \(x, y \in E\), the following holds:
(i) \(\mu(x + y) \geq \mu(x) \land \mu(y)\)
(ii) \(\mu(-x) \geq \mu(x)\)

2.5. Definition:[7] A fuzzy subgroup \(\mu\) of an N-group E is said to be a fuzzy N-subgroup of E if for all \(n \in N\), \(x \in E\) the following hold:
\(\mu(nx) \geq \mu(x)\)

2.6. Definition:[7] A fuzzy subset \(\mu\) of an N-group E is said to be a fuzzy ideal of E if for all \(n \in N\), \(a, x, y \in E\) the following holds:
(i) \(\mu(x - y) \geq \mu(x) \land \mu(y)\)
(ii) \(\mu(x) = \mu(y + x - y)\)
(iii) \(\mu(n(a + x) - na) \geq \mu(x)\)

2.7. Definition: Let \(\mu\) and \(\theta\) be two fuzzy subsets of E. We define fuzzy subset \(\mu + \theta\) of E as follows:
\((\mu + \theta)(x) = \bigvee_{a+b=x} \{ \mu(a) \land \theta(b) : a, b \in E\}\)

2.8. Definition: Let E and F be two N-groups. Then a mapping \(f : E \rightarrow F\) is called an N-homomorphism if
(i) \(f(x + y) = f(x) + f(y)\)
(ii) \(f(nx) = nf(x)\)
for all \(n \in N\) and \(x, y \in E\).
An $N$-homomorphism $f$ is called a $N$-monomorphism if $f$ is onto. An $N$-homomorphism $f$ is called an $N$-epimorphism if $f$ is surjective.

2.9.**Definition:** Let $Q$ and $N$ denote a set and a near-ring respectively. A mapping $\mu: N \times Q \rightarrow [0,1]$ is called a $Q$-fuzzy set in $N$.

2.10. **Definition:** If $\mu$ is a $Q$-fuzzy set in $N$, then the set $\{ x \in N, q \in Q : \mu(x, q) > 0 \}$ is said to be $Q$-support of $\mu$ and is denoted by $\mu^*$.

2.11. **Definition:** If $\mu$ is a $Q$-fuzzy set in $N$, then the set $\{ x \in N, q \in Q : \mu(x, q) \geq t, t \in [0,1] \}$ is said to be $Q$-level subset of $\mu$ and denoted by $\mu_t$.

2.12. **Definition:** A $Q$-fuzzy set $\mu$ in a near-ring $N$ is called $Q$-fuzzy sub near-ring of $N$ if

(i) $\mu(x - y, q) \geq \mu(x, q) \land \mu(y, q)$

(ii) $\mu(xy, q) \geq \mu(x, q) \land \mu(y, q)$  for all $x, y \in N$ and $q \in Q$.

2.13. **Definition:** A $Q$-fuzzy set $\mu$ in a near-ring $N$ is called $Q$-fuzzy ideal of $N$ if

(i) $\mu(x - y, q) \geq \mu(x, q) \land \mu(y, q)$

(ii) $\mu(xn, q) \geq \mu(x, q)$

(iii) $\mu(y+x-y, q) \geq \mu(x, q)$

(iv) $\mu(n(y+x) - ny, q) \geq \mu(x, q)$  for all $x, y, n \in N$ and $q \in Q$.

2.14. **Definition:** Let $f$ be a mapping from a set $S$ to a set $R$. Let $\mu$ and $\theta$ be $Q$-fuzzy set in $S$ and $R$ respectively. Then $f(\mu)$ the image of $\mu$ is a $Q$-fuzzy set in $R$ defined as

$f(\mu)(y, q) = \sup_{x=f^{-1}(y)} \mu(x, q) : f^{-1}(y) \neq \emptyset$

$= 0$ otherwise.

Also the pre-image of $\theta$ under $f$ is denoted by $f^{-1}(\theta)$ is a $Q$-fuzzy set of $S$ and defined as

$[f^{-1}(\theta))(x, q) = \theta(f(x), q)$, for all $x \in S, y \in R$ and $q \in Q$.

3. **$Q$-fuzzy ideal of an $N$-group.**

3.1. **Definition:** Let $Q$ and $E$ denote a set and an $N$-group respectively. A mapping $\mu: E \times Q \rightarrow [0,1]$ is called a $Q$-fuzzy set in $E$.

3.2. **Definition:** Let $\mu$ be a $Q$-fuzzy set of an $N$-group $E$. Then $\mu$ is called $Q$-fuzzy $N$-subgroup of $E$ if

(i) $\mu(x + y, q) \geq \mu(x, q) \land \mu(y, q)$

(ii) $\mu(x, q) = \mu(-x, q)$
(iii) \( \mu (nx, q) \geq \mu (x, q) \) for all \( x, y \in E, n \in N, q \in Q \).

Conditions (i) and (ii) are equivalent to \( \mu (x - y, q) \geq \mu (x, q) \land \mu (y, q) \).

3.3. **Remarks:** If \( \mu \) is a Q-fuzzy N-subgroup of N-group \( E \). Then for all \( x \in E, q \in Q \) the following are equivalent:

(i) \( \mu (-x, q) \geq \mu (x, q) \)

(ii) \( \mu (-x, q) \leq \mu (x, q) \)

(iii) \( \mu (x, q) = \mu (-x, q) \)

3.4. **Lemma:** If \( \mu \) is a Q-fuzzy N-subgroup of N-group \( E \). Then for all \( x \in E, q \in Q \) the following are equivalent:

(i) \( \mu (-x, q) \geq \mu (x, q) \)

(ii) \( \mu (-x, q) \leq \mu (x, q) \)

(iii) \( \mu (x, q) = \mu (-x, q) \)

3.5. **Theorem:** A Q-fuzzy set \( \mu \) of \( E \) is a Q-fuzzy N-subgroup of \( E \) if and only if \( \mu_t \), \( t \in [0,1] \) is an N-subgroup of \( E \).

**Proof:** Straightforward.

3.6. **Theorem:** A Q-fuzzy set \( \mu \) of an N-group \( E \) is a Q-fuzzy N-subgroup of \( E \) if and only if

(i) \( \mu (0, q) \geq \mu (x, q) \)

(ii) \( \mu (mx + ny, q) \geq \mu (x, q) \land \mu (y, q) \), for all \( m, n \in N \) and \( x, y \in E, q \in Q \).

**Proof:** Let \( \mu \) be a Q-fuzzy N-subgroup of \( E \).

Clearly \( \mu (0, q) = \mu (x - x, q) \geq \mu (x, q) \land \mu (-x, q) = \mu (x, q) \), for all \( x \in E, q \in Q \).

Also \( \mu (mx + ny, q) \geq \mu (mx, q) \land \mu (ny, q) \).
On Q-Fuzzy N-Subgroup and Q-Fuzzy Ideal of An N-Group

\[ \geq \mu(x, q) \land \mu(y, q), \text{ for all } m, n N \text{ and } x, y \in E. \]

Conversely, we assume conditions (i) and (ii).
Now \( \mu(x + y, q) = \mu(1x + 1y, q) \geq \mu(x, q) \land \mu(y, q) \)
And \( \mu(-x, q) = \mu(-1x + 0, q) \geq \mu(x, q) \land \mu(0, q) \)
\[ = \mu(x, q) \]
Also \( \mu(nx, q) = \mu(nx + m0, q) \geq \mu(x, q) \land \mu(0, q) \)
\[ = \mu(x, q) \]
Thus \( \mu \) is a Q-fuzzy N-subgroup of E.

3.7. Theorem: Let E and F be two N-groups and \( f : E \rightarrow F \) be an N-epimorphism. Let \( \mu \) be a Q-fuzzy N-subgroup of E. Then \( f(\mu) \) is a Q-fuzzy N-subgroup of F.

Proof: Given \( \mu \) is a Q-fuzzy N-subgroup of E.
Let \( u, v \in F \) and \( q \in Q \). Then there exist \( x, y \in E \) such that \( f(x) = u \) and \( f(y) = v \)
Now \[ f(\mu)(u + v, q) = \bigvee_{f(w) = u + v} \mu(w, q) \]
\[ = \bigvee_{f(x + y) = u + v} \mu(x + y, q) \]
\[ = \bigvee_{f(x) = u, f(y) = v} \mu(x + y, q) \]
\[ \geq \bigvee_{f(x) = u, f(y) = v} [\mu(x, q) \land \mu(y, q)] \]
\[ \geq [\bigvee_{f(x) = u} \mu(x, q)] \land [\bigvee_{f(y) = v} \mu(y, q)] \]
\[ \geq [f(\mu)](u, q) \land [f(\mu)](v, q) \]
Also it is clear that \( f(\mu)(-x, q) \geq f(\mu)(x, q) \)
Now let \( y \in F \), \( n \in N \) and \( q \in Q \).

Then there exist \( z \in E \) such that \( f(z) = y \) and hence \( f(nz) = ny \)
Now \[ f(\mu)(ny, q) = \{ \bigvee_{f(x) = ny} \mu(x, q) : x \in E, q \in Q \} \]
\[ = \{ \bigvee_{f(nx) = ny} \mu(nx, q) : nx \in E, q \in Q \} \]
\[ = \{ \bigvee_{nf(z) = nz} \mu(nz, q) : nz \in E, q \in Q \} \]
\[ \geq \{ \bigvee_{f(z) = y} \mu(z, q) : z \in E, q \in Q \} \]
\[ = f(\mu)(y, q) \]
Thus \( f(\mu) \) is a Q-fuzzy N-subgroup of F.
3.8. **Theorem**: Let $E$ and $F$ be two $N$-groups and $f : E \rightarrow F$ be an $N$-homomorphism. Let $\mu$ be a $Q$-fuzzy $N$-subgroup of $F$. Then $f^{-1}(\mu)$ is a $Q$-fuzzy $N$-subgroup of $E$.

**Proof**: Let $\mu$ be a $Q$-fuzzy $N$-subgroup of $F$. Let $x, y \in E$ and $q \in Q$. Then

\[
[f^{-1}(\mu)](x - y, q) = \mu(f(x - y), q) = \mu(f(x) - f(y), q) \geq \mu(f(x), q) \wedge \mu(f(y), q) \geq f^{-1}(\mu)(x, q) \wedge f^{-1}(\mu)(y, q)
\]

Also $f^{-1}(\mu)(nx, q) = \mu(f(nx), q) = \mu(nf(x), q) \geq \mu(f(x), q) = f^{-1}(\mu)(x, q)$

Therefore $f^{-1}(\mu)$ is a $Q$-fuzzy $N$-subgroup of $E$.

3.9. **Theorem**: The intersection of a non-empty family of $Q$-fuzzy $N$-subgroup of an $N$-group is again a $Q$-fuzzy $N$-subgroup of $E$.

**Proof**: Straightforward.

3.10. **Definition**: Let $\mu$ be a $Q$-fuzzy set of an $N$-group $E$. Then $\mu$ is called a $Q$-fuzzy normal subgroup of $E$. If

(i) $\mu(x - y, q) \geq \mu(x, q) \wedge \mu(y, q)$.

(ii) $\mu(y + x - y, q) \geq \mu(x, q)$, for all $x, y \in E$, $q \in Q$.

3.11. **Definition**: Let $\mu$ be a $Q$-fuzzy set of an $N$-subgroup $E$. Then $\mu$ is called a $Q$-fuzzy ideal of $E$. If

(i) $\mu$ is a $Q$-fuzzy normal subgroup of $E$.

(ii) $\mu[n(a + x) - na, q] \geq \mu(x, q)$, for all $x, a \in E$, $n \in N$, $q \in Q$.

3.12. **Theorem**: A $Q$-fuzzy normal subgroup $\mu$ of an group $E$ is a $Q$-fuzzy ideal of $E$ if

(i) $\mu[-na + n(a + x), q] \geq \mu(x, q)$

(ii) $\mu[-na + n(x + a), q] \geq \mu(x, q)$ for all $x, a \in E$, $n \in N$, $q \in Q$.

**Proof**: Straightforward.

3.13. **Lemma**: Let $\mu$ be a $Q$-fuzzy ideal of $E$. Then following holds:

(i) $\mu(0, q) \geq \mu(x, q)$, for all $x \in E$, $q \in Q$. 

(ii) \( \mu( x, q) = \mu(-x, q) \), for all \( x \in E \).

(iii) \( \mu^{*} \) is an ideal of \( E \).

**Proof**: Straightforward.

3.14.**Theorem**: The intersection of non empty family of Q-fuzzy ideal of \( E \) is again a Q-fuzzy ideal of \( E \).

3.15.**Theorem**: A Q-fuzzy set \( \mu \) of \( E \) is a Q-fuzzy ideal of \( E \) if and only if \( \mu_{t} \), for all \( t \in [0,1] \), is an ideal of \( E \).

**Proof**: Straightforward.

3.16.**Theorem**: Let \( E \) and \( F \) be two N-groups. Let \( f : E \rightarrow F \) be an N-homomorphism. Let \( \mu \) and \( \theta \) be Q-fuzzy ideal of \( E \) and \( F \) respectively. Then

(i) \( f(\mu) \) is a Q-fuzzy ideal of \( F \), if \( f \) is onto.

(ii) \( f^{-1}(\theta) \) is a Q-fuzzy ideal of \( E \).

**Proof**: (i) let \( x, y \in F \). Then there exist \( a, b \in E \) such that \( f(a) = x \), \( f(b) = y \) and \( q \in Q \). Then by theorem 3.7., \( f(\mu) \) is a Q-fuzzy subgroup of \( E \).

Now \( f(\mu)(y + x - y, q) = \bigvee_{f(a) = x} \mu(a, q) \)
\( \geq \mu(b + a - b, q) \), as \( y + x - y = f(b + a - b) \)
\( \geq \mu(a, q) \), whenever \( f(a) = x \)
\( f(\mu)(y + x - y, q) \geq \bigvee_{f(a) = x} \mu(a, q) \)
\( = f(\mu)(x, q) \)

Also let \( n \in N \) and \( x, y \in F \). since \( f \) is onto ,there exist \( a, b \in E \) such that \( f(a) = x \), \( f(b) = y \)

Now \( f(\mu)(n(x + y) - nx, q) = \bigvee_{f(a) = n(x + y) - nx} \mu(a, q) \)
\( \geq \mu(n(a + b) - na, q) \), as \( n(x + y) - nx = f[n(a + b) - na] \)

Thus \( f(\mu)(n(x + y) - nx, q) \geq \mu(b, q) \) whenever \( f(b) = y \)
\( \geq \bigvee_{f(b) = y} \mu(b, q) \)
\( = f(\mu)(y, q) \)

\( \therefore f(\mu) \) is a Q-fuzzy ideal of \( F \).

(ii) By theorem 3.8., \( f^{-1}(\theta) \) is a Q-fuzzy subgroup of \( E \).

Let \( x, y \in E \) and \( q \in Q \). Then
\[ f^{-1}(\theta)(y + x - y, q) = \theta(f(y) + f(x) - f(y), q) \]
\[ \geq \theta(f(x), q) \]
\[ = f^{-1}(\theta)(x, q) \]

Again let \( x, a \in E \), \( n \in \mathbb{N} \) and \( q \in \mathbb{Q} \). Then
\[ f^{-1}(\theta)(n(a + x) - na, q) = \theta[nf(a + x) - nf(a), q] \]
\[ \geq \theta(f(x), q) \]
\[ = f^{-1}(\theta)(x, q) \]

\( f^{-1}(\theta) \) is a \( Q \)-fuzzy ideal of \( E \).

**3.17. Theorem:** If \( \mu \) is a \( Q \)-fuzzy ideal of \( E \) and \( \theta \) is a \( Q \)-fuzzy \( N \)-subgroup of \( E \). Then \( \mu + \theta \) is a \( Q \)-fuzzy \( N \)-subgroup of \( E \).

**Proof:** Let \( x, y \in E \), \( q \in \mathbb{Q} \). Then
\[ (\mu + \theta)(x + y, q) = \vee_{x+y=a+b} \{ \mu(a, q) \land \theta(b, q) : a, b \in E, q \in \mathbb{Q} \} \]
\[ = \vee_{x+y=(u+v)+(r+s)} \{ \mu(u + v, q) \land \theta(r + s, q) : u, v, r, s \in E, q \in \mathbb{Q} \} \]
\[ = \vee_{x+y=r+v, y=v+s} \{ \mu(u + v + r - v, q), \land \theta(v + s, q) \} \]
\[ \geq \vee_{x+y=r+v, y=v+s} \{ \mu(u, q) \land \mu(v + r - v, q) \} \land \{ \theta(v, q) \land \theta(s, q) \} \]

Next let \( (\mu + \delta)(x, q) = t \) and \( \epsilon > 0 \)

Then \( t - \epsilon < \mu(c, q) \land \theta(d, q) \) for some \( x = c + d, c, d \in E \) and \( q \in \mathbb{Q} \)

\[ = \mu(x - d, q) \land \theta(-d, q) \]
\[ = \mu(-d + x, q) \land \theta(-d, q) \]
\[ = \mu(-(-x + d), q) \land \theta(-d, q) \]
\[ = \mu((-x + d), q) \land \theta(-d, q) \]
\[ \leq \vee_{-x+d} \{ \mu(a, q) \land \theta(b, q) : a, b \in E \} \]
\[ = (\mu + \delta)(-x, q) \]

\[ \therefore (\mu + \theta)(-x, q) \geq (\mu + \theta)(x, q) \]
On $Q$-Fuzzy $N$-Subgroup and $Q$-Fuzzy Ideal of An $N$-Group

Again suppose $(\mu + \theta)(x, q) = m$. So for $\epsilon > 0$ there exist $a + b \in E$ with $x = a + b \in Q$ such that

$$m - \epsilon < \mu(a, q) \wedge \theta(b, q)$$

$$\leq \mu(n(b + a) - nb, q) \wedge \theta(nb, q)$$

$$= \mu(n(a + b) - nb, q) \wedge \theta(nb, q)$$

$$= \mu(r, q) \wedge \theta(s, q)$$

where $n(a + b) = r + s$

Hence $(\mu + \theta)(n x, q) \geq (\mu + \theta)(x, q)$

Thus $\mu + \theta$ is a $Q$-fuzzy $N$-subgroup of $E$.

Reference:
